A REMARK ON THE DIVISOR FUNCTION $d(n)$

by R. C. VAUGHAN

(Received 31 May, 1971)

Let $d(n)$ denote the number of positive divisors of $n$. A long time ago, Erdös and Mirsky [1] raised the question whether the equation $d(n) = d(n+1)$ holds for infinitely many $n$. It does not seem easy to settle this problem, and in the present note we give a partial result.

**PROPOSITION.** At least one of the following two statements is valid. (i) For infinitely many primes $p$, $8p + 1$ is the product of at most two distinct primes. (ii) For infinitely many $n$, $d(n) = d(n+1)$.

**Proof.** Let $P_3$ be the set of natural numbers that are products of at most three, not necessarily distinct, primes. Denote by $\rho(n)$ the least prime divisor of $n$. From the work of Richert [2, Theorem 7], it is known that there exist positive numbers $\delta_1, \delta_2$ such that, for all sufficiently large $N$,

$$
\sum \left\{ \begin{array}{l} 8p + 1 \leq N, \\
8p + 1 \in P_3, \\
\rho(8p + 1) \geq N^{\delta_2} \end{array} \right\} 1 \geq \frac{\delta_1 N}{\log^2 N}.
$$

(Actually, the condition $\rho(8p + 1) \geq N^{\delta_1}$ is not stated in Richert's Theorem 7, but follows immediately from his appeal to Theorem 2.)

The sum on the left-hand side of (1) is equal to $\Sigma_1 + \Sigma_2$, where $\mu(8p + 1) \neq 0$ in $\Sigma_1$ and $\mu(8p + 1) = 0$ in $\Sigma_2$. We have

$$
\Sigma_2 \leq \sum \left\{ \begin{array}{l} 8p + 1 \leq N, \\
8p + 1 \equiv 0 \pmod{p' \delta_2}, \\
p' \geq N^{\delta_2} \end{array} \right\} 1 \leq \sum \left\{ \begin{array}{l} n \leq N, \\
m \equiv 0 \pmod{p' \delta_2}, \\
p' \geq N^{\delta_2} \end{array} \right\} 1 = O(N^{1-\delta_2})
$$

and therefore, by (1),

$$
\sum \left\{ \begin{array}{l} 8p + 1 \leq N, \\
8p + 1 \in P_3, \\
\mu(8p + 1) \neq 0 \end{array} \right\} 1 \geq \frac{\delta_1 N}{\log^2 N} - \Sigma_2 \geq \frac{\delta_1 N}{\log^2 N}.
$$

Hence, at least one of the following statements is valid. (i) For infinitely many $p$, $8p + 1$ is the product of at most two distinct primes. (ii) For infinitely many $p$, $8p + 1$ is the product of three distinct primes, and in that case $d(8p + 1) = 8 = d(8p)$. 

https://www.cambridge.org/core/terms. https://doi.org/10.1017/S0017089500001725
A REMARK ON THE DIVISOR FUNCTION $d(n)$

REFERENCES


UNIVERSITY OF SHEFFIELD