A SIMPLE ALGORITHM FOR FINDING MAXIMAL NETWORK FLOWS AND AN APPLICATION TO THE HITCHCOCK PROBLEM

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Introduction. The network-flow problem, originally posed by T. Harris of the Rand Corporation, has been discussed from various viewpoints in (**1**; **2**; **7**; **16**). The problem arises naturally in the study of transportation networks; it may be stated in the following way. One is given a network of directed arcs and nodes with two distinguished nodes, called *source* and *sink*, respectively.[1] All other nodes are called *intermediate*. Each directed arc in the network has associated with it a nonnegative integer, its *flow capacity*. Source arcs may be assumed to be directed away from the source, sink arcs into the sink. Subject to the conditions that the flow in an arc is in the direction of the arc and does not exceed its capacity, and that the total flow into any intermediate node is equal to the flow out of it, it is desired to find a maximal flow from source to sink in the network, i.e., a flow which maximizes the sum of the flows in source (or sink) arcs.

Thus, if we let P_1 be the source, P_n the sink, we are required to find x_{ij} $(i, j = 1, \ldots, n)$ which maximize

$$\sum_{i=2}^{n} x_{1j} \tag{1}$$

subject to

$$\sum_{j} (x_{ij} - x_{ji}) = 0 \qquad (i = 2, \ldots, n-1),\ 0 \leqslant x_{ij} \leqslant c_{ij}, \tag{2}$$

where the c_{ij} are given nonnegative integers.

This is, of course, a linear programming problem, and hence may be solved by Dantzig's simplex algorithm (**3**). In fact, the simplex computation for a problem of this kind is particularly efficient, since it can be shown that the sets of equations one solves in the process are always triangular (**1**). However, for the flow problem, we shall describe what appears to be a considerably more efficient algorithm.

Of some theoretical interest is the fact that the procedure assures one of obtaining a strict increase in the total flow at each step (in contrast with the simplex method). In addition, the Hitchcock transportation problem (**4**; **12**; **14**) can be solved via the flow algorithm in a way which naturally generalizes

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[1]A problem in which there are several sources and sinks, with flows permitted from any source to any sink, is reducible to a single-source, single-sink problem.

the combinatorial method recently proposed by Kuhn **(15)** for the optimal-assignment problem. Informal tests by hand indicate that this way of solving the Hitchcock problem is extremely efficient.

1. The minimal cut theorem. A nonconstructive proof of the minimal cut theorem, which asserts equality of maximal flow value and minimal cut value,[2] has been given by the present writers in **(7)**. Subsequently a constructive proof based on the simplex criterion of linear programming was developed **(1)**. The algorithm which we describe in the next section also provides a constructive proof of the theorem in case the c_{ij} are integral (or rational),[3] as we have assumed for this paper. Like the simplex algorithm, it not only produces a maximal flow but a minimal cut as well. This will be important for our application to the Hitchcock problem.

It should perhaps be pointed out that an *undirected problem*, by which we mean that the directions of flow in intermediate arcs are not specified, so that the capacity constraints on these arcs are of the form

$$x_{ij} + x_{ji} \leqslant c_{ij} \qquad (i, j = 2, \ldots, n-1;\ i < j), \tag{3}$$

presents nothing new, since we may replace each undirected arc by a pair of oppositely directed arcs, each with capacity equal to that of the original arc; i.e., replace (3) by

$$x_{ij} \leqslant c_{ij}, \quad x_{ji} \leqslant c_{ij}. \tag{4}$$

For, given $X' = (x_{ij}')$ satisfying (4) and the conservation equations at intermediate nodes, setting

$$x_{ij} = \max(x_{ij}' - x_{ji}', 0) \tag{5}$$

yields an equivalent flow $X = (x_{ij})$ satisfying these equations and (3).

The minimal cut theorem is true for undirected networks as well as for directed networks, or more generally, for mixed networks in which some intermediate arcs are directed, others not, the obvious changes in definitions of cuts and chains having been made. This follows from the comments above and the fact, easily proved, that the minimal cut value is the same for a mixed network and its equivalent directed network. The theorem is still valid when capacity constraints on nodes are admitted, where a cut now, of course, includes nodes as well as arcs. This may be proved by splitting each node into two nodes as suggested in Fig. 1, and placing the capacity c of the old node on the new directed arc joining the two new nodes, thus obtaining an equivalent network with capacity constraints on arcs only. For a direct proof of the theorem in this general form, see **(1)**. It is also shown there (and follows

[2]A *cut* is a collection of arcs which meets every directed chain from source to sink; the *value* of a cut is the sum of capacities of its members. This definition of a cut corresponds to that of *disconnecting set* given in **(7)**.

[3]No restriction of this kind is made in **(1, 7)**.

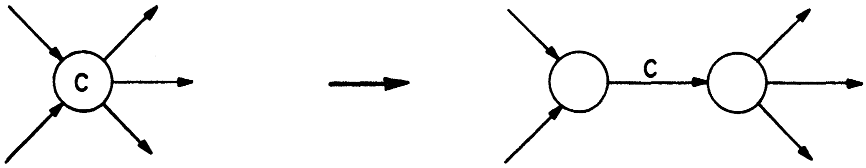

Fig. 1

from the algorithm of the next section) that for integral c_{ij}, an integral maximizing flow exists.

There are several well-known combinatorial theorems which can be viewed as rather direct consequences of the minimal cut theorem and the existence of an integral maximizing flow. Among these we mention Menger's theorem on linear graphs **(13)**, Dilworth's chain decomposition theorem for partially ordered sets **(5)**, and Hall's theorem concerning set representatives **(11)**. For proofs, see **(1; 8; 9)**.

2. The algorithm. We start the computation from any convenient integral flow whatsoever. The initial flow is used to define a starting matrix $A = (a_{ij})$ by letting a_{ij} be the capacity of the arc from P_i to P_j diminished by the flow from P_i to P_j and increased by the flow from P_j to P_i. If no other flow is readily available, one may start with the zero flow, corresponding to $A = C$, where c_{ij} is the original capacity from P_i to P_j.

For certain values of $j = 1, \ldots, n$ we shall define labels v_j, μ_j recursively as follows.

Let $v_1 = \infty$, $\mu_1 = 0$. For those j such that $a_{1j} > 0$, define $v_j = a_{1j}$, $\mu_j = 1$. In general, from those i which have received labels v_i, μ_i but which have not previously been examined, select an i and scan for all j such that $a_{ij} > 0$ and v_j, μ_j have not been defined. For these j, define

$$v_j = \min (v_i, a_{ij}), \quad \mu_j = i. \tag{6}$$

Continue this process until v_n, μ_n have been defined, or until no further definitions may be made and v_n, μ_n have not been defined. In the latter case the computation ends. In the former case, proceed to obtain a new a_{ij} matrix as follows.

Replace $a_{\mu_n n}$ by $a_{\mu_n n} - v_n$ and $a_{n\mu_n}$ by $a_{n\mu_n} + v_n$. In general, replace $a_{\mu_j j}$ by $a_{\mu_j j} - v_n$ and $a_{j\mu_j}$ by $a_{j\mu_j} + v_n$, where each j is the $\mu_{j'}$ of the preceding j' in the backward replacement. This replacement continues until $\mu_j = 1$ has been completed. The labels v_j, μ_j are then recomputed on the basis of the new A matrix and the process is repeated.

Notice that v_n is a positive integer, and hence the process terminates. Upon termination, the maximal flow X is given by defining

$$x_{ij} = \max(c_{ij} - a_{ij}, 0), \tag{7}$$

as we now prove.

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LEMMA 1. *X is a flow.*

Proof. Clearly $0 \leqslant x_{ij} \leqslant c_{ij}$, since $a_{ij} \geqslant 0$. It remains to show that X satisfies $\sum_j (x_{ij} - x_{ji}) = 0$ $(i = 2, \ldots, n-1)$. Now the process ensures that $c_{ij} + c_{ji} = a_{ij} + a_{ji}$. It follows from this and the definition of X that

$$\sum_j (x_{ij} - x_{ji}) = \sum_j (c_{ij} - a_{ij}).$$

It therefore suffices to prove that $\sum_j a_{ij}$ $(i = 2, \ldots, n-1)$ is invariant under the computation, as certainly $\sum_j (c_{ij} - a_{ij}) = 0$ for the starting matrix A obtained from a flow X. But if i $(\neq 1, n)$ is the μ_l of some l, then there is a $k = \mu_i$. Thus, for this i, the new a_{ij}' are either equal to the old a_{ij} or are given by

$$a_{ij}' = \begin{cases} a_{ij} - v_n & \text{for } j = l, \\ a_{ij} + v_n & \text{for } j = k, \\ a_{ij} & \text{otherwise}; \end{cases}$$

hence $\sum_j a_{ij}' = \sum_j a_{ij}$.

LEMMA 2. *X is a maximal flow.*

Proof. At the point where termination occurs, we have defined a set S of nodes, consisting of those nodes P_i for which v_i, μ_i have been defined; and further, $P_1 \in S$, $P_n \notin S$. Consider the set Γ of directed arcs P_iP_j such[4] that $P_i \in S$, $P_j \notin S$. Clearly $a_{ij} = 0$ for such pairs i, j, as otherwise we would have defined v_j, μ_j.

We will show that Γ is a cut whose value is equal to $\sum_j x_{1j}$, thus proving that X is a maximal flow and Γ a minimal cut.

That Γ is a cut is clear. For if there were a chain

$$P_1P_{i_1} \ldots P_{i_k}P_n, \text{ with the arcs } P_1P_{i_1} \notin \Gamma, \ldots, P_{i_k}P_n \notin \Gamma,$$

then we could deduce successively (since $P_1 \in S$) that

$$P_{i_1} \in S, \ldots, P_n \in S,$$

which is a contradiction. To see that Γ is equal to the flow value, notice that

$$\sum_j (c_{ij} - a_{ij}) = \begin{cases} 0 & (1 < i < n), \\ \sum_j x_{1j} & (i = 1). \end{cases}$$

Now sum these equations over those i for which $P_i \in S$. On the left side, if P_i and P_j are both in S, then $c_{ij} - a_{ij}$ and $c_{ji} - a_{ji}$ are both in the summation and are negatives of each other. All that remain are terms of the form $c_{ij} - a_{ij}$, where $P_i \in S$, $P_j \notin S$. For these, as we pointed out above, $a_{ij} = 0$. Thus the sum on the left reduces precisely to the sum of capacities of members of Γ, as was to be shown.

[4]The set Γ is actually the set of "left arcs" defined in (7).

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3. The Hitchcock problem. The Hitchcock transportation problem arises frequently in applications of linear programming. We shall propose another computation for the problem which will amount to solving a sequence of flow problems of a particularly simple kind. The basic idea, which stems from a proof given by Egerváry **(6)** for a theorem of König **(13**, p. 232**)** on linear graphs, has been used by Kuhn **(15)** to develop a very efficient combinatorial algorithm for the optimal assignment problem, a special case of the Hitchcock problem. Our method differs only in details from the Kuhn algorithm in this case.

We take the Hitchcock problem in the following form: Given a matrix $D = (d_{ij})$ of nonnegative integers, and two sets of nonnegative integers $(a_1, \ldots, a_m)$, $(b_1, \ldots, b_n)$, with

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,$$

it is desired to find a matrix $X = (x_{ij})$ satisfying the constraints

$$x_{ij} \geqslant 0, \quad \sum_j x_{ij} = a_i, \quad \sum_i x_{ij} = b_j \tag{8}$$

which minimizes the linear form

$$\sum_{i,j} d_{ij} x_{ij}. \tag{9}$$

A physical interpretation of the problem is that there are m originating points for a commodity, the ith point having a_i units, and n destinations, the jth one requiring b_j units. If d_{ij} is the cost, per unit of commodity, of shipping from origin i to destination j, find a shipping programme of minimal cost.

The dual (see **(10)** for linear programming duality theorems) of the Hitchcock problem is: Find α_i, β_j satisfying the constraints

$$\alpha_i + \beta_j \leqslant d_{ij} \qquad (i = 1, \ldots, m; \quad j = 1, \ldots, n) \tag{10}$$

which maximize

$$\sum_i a_i \alpha_i + \sum_j b_j \beta_j. \tag{11}$$

The proof that the algorithm to be described yields a solution to the Hitchcock problem will be based on the fact that the dual form (11) increases by at least one unit with the solution of each successive flow problem. Since (11) is bounded above by (9), this suffices.

Each of the flow problems will be of the following form. Find $X = (x_{ij})$ satisfying

$$\begin{aligned} &x_{ij} \geqslant 0, \ \sum_j x_{ij} \leqslant a_i, \ \sum_i x_{ij} \leqslant b_j, \\ &x_{ij} = 0 \text{ for a given set } \Omega \text{ of pairs } i, j, \end{aligned} \tag{12}$$

which maximizes

$$\sum_{i,j} x_{ij}. \tag{13}$$

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This is actually a special Hitchcock problem. To see that it is also a flow problem, set up the directed network of $m + n + 2$ nodes shown in Fig. 2, where the capacity on the directed arc P_iQ_j is zero if $i, j \in \Omega$, large otherwise, and the capacities of source and sink arcs are the a_i and b_j, as shown, and interpret x_{ij} as the flow from P_i to Q_j.

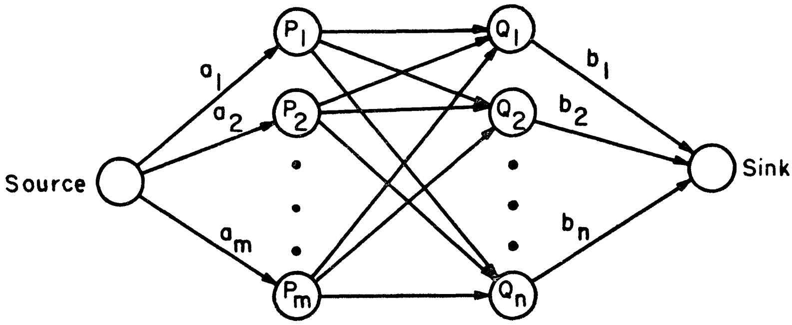

Fig. 2

To solve such a problem we may of course use the computation of §2 involving, in this case, an $m + n + 2$ by $m + n + 2$ matrix. It is possible (and computationally convenient) to describe the process in terms of an m by n array. The verification that the two descriptions are the same for this particular class of problems will be left to the reader.

Let X be an integral solution of the constraints (12). Corresponding to certain of the rows $i = 1, \ldots, m$ of X we will define integers v_i, μ_i; similarly, for certain of the columns $j = 1, \ldots, n$ we will define integers w_j, λ_j. These definitions will be made recursively; first a set of v_i, μ_i will be defined, from these a set of w_j, λ_j will be defined, and so forth, alternating between the rows and columns.

For those i such that $\sum_j x_{ij} < a_i$, define

$$v_i = a_i - \sum_j x_{ij}, \mu_i = 0. \tag{14}$$

Next select an i which has been labelled and scan for all (unlabelled) j such that $i, j \notin \Omega$; for these j, define

$$w_j = v_i, \lambda_j = i. \tag{15}$$

Repeat until the previously labelled i's are exhausted. We then select a labelled j and scan for unlabelled i such that $x_{ij} > 0$; for these i, define

$$v_i = \min(x_{ij}, w_j), \ \mu_i = j. \tag{16}$$

Repeat until the previously labelled j's are exhausted. Again select one of the i's just labelled, look for unlabelled j such that $i, j \notin \Omega$, and define w_j, λ_j by (15). Continue in this fashion, using (15) and (16) alternately until either

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w_j, λ_j have been defined for some j with $\sum_i x_{ij} < b_j$, or until no further definitions may be made. In the latter case X is maximal by Lemma 2; in the former we can get an improvement as follows. Let

$$v = \min(w_j, b_j - \sum_i x_{ij}). \tag{17}$$

Alternately add and subtract v from the sequence

$$x_{\lambda_j j}, x_{\lambda_j j_1}, x_{i_1 j_1}, x_{i_1 j_2}, \ldots, x_{i_{k-1} j_k}, x_{i_k j_k}, \tag{18}$$

where

$$j_1 = \mu_{\lambda_j}, i_1 = \lambda_{j_1}, j_2 = \mu_{i_1}, i_2 = \lambda_{j_2}, \ldots, j_k = \mu_{i_{k-1}}, i_k = \lambda_{j_k},$$

and $\mu_{i_k} = 0$.

i \ j	1	2	3	4	5	6	7	8	9	10	a_i	v_i	μ_i
1			①	①							2	1	3
2							②		①		3		
3		①								○	1	1	2
4	①				③						4	1	1
5		○			③		○	○		○	3		
6		○	○					②			3	1	0
7	○	○							①		1	1	9
8		○		①		○					1	1	4
9								○	○	③	5	2	0
10				⑥		②					8	1	4
b_j	1	1	1	8	9	2	2	2	2	3			
w_j	1	1	1	1	1	1		2	2	2			
λ_j	7	6	6	1	4	10		9	9	9			

Fig. 3

Example. In Fig. 3, cells containing a circle comprise $\bar{\Omega}$ (the complement of Ω). The entries within the circles (zeros elsewhere) constitute an X satisfying the constraints (12). The defining process terminates with $w_5 = 1$, $\lambda_5 = 4$, since column 5 is a column in which the sum of the entries x_{i5} is less than b_5; i.e., $3 + 3 < 9$. The sequence along which an improvement of

$$\min(w_5, b_5 - \sum_i x_{i5}) = 1$$

can be made is $x_{45} = 3$, $x_{41} = 1$, $x_{71} = 0$, $x_{79} = 1$, $x_{99} = 0$, as is easily read off using the λ's and μ's alternately.

We are now in a position to describe a general routine for the Hitchcock problem. As a starting point, form the difference matrix

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(19) $$d_{ij} - \alpha_i - \beta_j,$$

where

$$\alpha_i = \min_j d_{ij},\ \beta_j = \min_i (d_{ij} - \alpha_i).$$

Thus $d_{ij} - \alpha_i - \beta_j \geqslant 0$; i.e., α_i, β_j satisfy the dual constraints (10). The next step is to solve the flow problem with

(20) $$\Omega = \{ij | d_{ij} - \alpha_i - \beta_j > 0\}.$$

If the maximizing flow X satisfies

$$\sum_{i,j} x_{ij} = \sum_i a_i,$$

then X is a minimizing solution to the original Hitchcock problem.

This is easily deduced as follows. Observe first of all that for any α_i, β_j, the two Hitchcock problems with cost matrices d_{ij} and $d_{ij} - \alpha_i - \beta_j$ are equivalent; for $\sum_j x_{ij} = a_i$, $\sum_i x_{ij} = b_j$ imply that

$$\sum_{i,j} (d_{ij} - \alpha_i - \beta_j)\, x_{ij} = \sum_{i,j} d_{ij} x_{ij} - \sum_i a_i \alpha_i - \sum_j b_j \beta_j,$$

and the last two sums on the right are independent of x_{ij}. Thus if we have α_i, β_j with $d_{ij} - \alpha_i - \beta_j \geqslant 0$, and are able to find an X satisfying (8) and $x_{ij} = 0$ for $ij \in \Omega$, then clearly X minimizes $\sum_{i,j} (d_{ij} - \alpha_i - \beta_j) x_{ij}$, hence minimizes $\sum_{i,j} d_{ij} x_{ij}$.

If, on the other hand, $\sum_{i,j} x_{ij} < \sum_i a_i$, let I be the index set of the labelled rows of X, J the index set of the labelled columns, and define new dual variables by

(21) $$\alpha_i' = \begin{cases} \alpha_i + k & (i \in I), \\ \alpha_i & (i \notin I), \end{cases} \qquad \beta_j' = \begin{cases} \beta_j - k & (j \in J), \\ \beta_j & (j \notin J), \end{cases}$$

where $k = \min(d_{ij} - \alpha_i - \beta_j),\ i \in I, j \notin J$. Notice that $k > 0$, since pairs i, j with $i \in I$, $j \notin J$ are contained in Ω.

LEMMA 3. $\sum_i a_i \alpha_i' + \sum_j b_j \beta_j' > \sum_i a_i \alpha_i + \sum_j b_j \beta_j$.

Proof. The fact that X is a maximal flow implies that $\sum_i x_{ij} = b_j$ for $j \in J$. Also, the labelling process ensures that $x_{ij} = 0$ for $i \notin I$, $j \in J$, and, as we have mentioned earlier, $x_{ij} = 0$ for $i \in I$, $j \notin J$. Since all i with $\sum_j x_{ij} < a_i$ are in I, it follows that

$$\sum_{i \in I} a_i > \sum_{i \in I} \sum_{j \in J} x_{ij} = \sum_{j \in J} b_j,$$

and hence the new dual form has been increased by the amount

$$k\Big(\sum_{i \in I} a_i - \sum_{j \in J} b_j\Big) > 0.$$

Another way to see this is to note that the minimal cut in the associated network (Fig. 2) has value $\sum_{i \in I} a_i + \sum_{j \in J} b_j < \sum_i a_i$ and hence

$$\sum_{i \in I} a_i > \sum_{j \in J} b_j.$$

Now form the new difference matrix $d_{ij} - \alpha_i - \beta_j \geqslant 0$ by subtracting k from the I-rows of the previous difference matrix, and adding k to the J-columns. We may now take the maximal flow X of the previous flow problem as a starting point in the new flow problem and proceed as before.

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