LETTERS TO THE EDITOR

ON ARNOLD'S TREATMENT OF MORAN'S BOUNDS

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Abstract

We prove a conjecture of Arnold (1968) which simplifies the determination of an optimal bound on absorption probability originally due to Moran (1960).

ABSORPTION PROBABILITY; MARKOV-CHAIN MODEL WITH SELECTION

In a problem concerning calculation of an optimal bound on absorption probability in Wright's Markov-chain model with selection (for the context refer to Moran (1960), Arnold (1968)), it is required to find

\[ \theta_N = \sup \{ \theta \mid \pi_i(\theta) \leq p_i, i = 0, 1, 2, \ldots, 2N \}, \]

where for \( \theta \in (0, \infty) \),

\[ \pi_i(\theta) = \frac{1 - \exp(-\theta i/N)}{1 - \exp(-2\theta)}, \quad p_i = \frac{(1 + \sigma)i}{2N + \sigma i}, \quad i = 0, \ldots, 2N. \]

Here \( N \) (integer, \( \geq 1 \)), and \( \sigma \) (>0) are assumed known. Since for \( i = 1, 2, 3, \ldots, 2N-1 \), \( \pi_i(\theta) \) is an increasing function of \( \theta > 0 \), and \( \lim_{\theta \downarrow 0^+} \pi_i(\theta) = i/2N + p_i \), \( \lim_{\theta \uparrow \infty} \pi_i(\theta) = 1 > p_i \), it follows that there exists a unique \( \theta^*_i > 0 \) such that \( \pi_1(\theta^*_i) = p_i, \) with \( \pi_i(\theta) < p_i \) for \( \theta < \theta^*_i \), \( \pi_i(\theta) > p_i \) for \( \theta > \theta^*_i \). Clearly

\[ \theta_N = \min \{ \theta^*_1, \theta^*_2, \ldots, \theta^*_{2N-1} \} \]

where for each \( i \), \( \theta^*_i \) is the unique root in \((0, \infty)\) of the equation \( f_\theta(i/2N) = 0 \), where for \( 0 \leq x \leq 1 \),

\[ f_\theta(x) = (1 + \sigma)x(1 - \exp(-2\theta)) - (1 + \sigma x)(1 - \exp(-2\theta x)). \]

Thus for fixed \( i = 1, \ldots, 2N-1 \), \( f_\theta(i/2N) > 0 \) for \( \theta < \theta^*_i \); <0 for \( \theta > \theta^*_i \). Using a different notation, Arnold (1968) arrived at this result; and conjectured that

\[ \theta^*_1 > \theta^*_2 > \cdots > \theta^*_{2N-1}. \]

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We prove this conjecture, whence $\theta_N = \theta^*_{2N-1}$, resulting in considerable saving in computational labour as perceived by Arnold, in determining $\theta_N$.

Suppose the conjecture is false: then there exist $i_1, i_2$, $1 \leq i_1 < i_2 \leq 2N - 1$, such that $\theta^*_{i_1} \leq \theta^*_{i_2}$. Taking any henceforth fixed $\theta$ satisfying $\theta^*_{i_1} \leq \theta \leq \theta^*_{i_2}$, it follows that $f_\theta(i_2/2N) = f_{\theta^*_{i_2}}(i_2/2N) = f_{\theta^*_{i_1}}(i_2/2N)$. By the mean-value theorem, since $f'(0) = 0$ there is a $\xi_1$, $0 < \xi_1 < i_1/2N$, such that $f'_\theta(\xi_1) \leq 0$; and, since $f_\theta(1) = 0$, $\xi_2$, $i_2/2N < \xi_2 < 1$, such that $f'_\theta(\xi_2) \geq 0$. By applying the mean-value theorem again, there is a $\xi_3$, $1/2N < \xi_3 < i_2/2N$, such that $f'_\theta(\xi_3) \geq 0$. Now applying the mean-value theorem to the function $f_\theta(x)$, there exist numbers $\xi_1$, $(\xi_1 < \xi_1 < \xi_2)$ and $\xi_3$, $(\xi_3 < \xi_2 < 1/2N)$ such that $f'_\theta(\xi_1) \geq 0$, $f'_\theta(\xi_2) \geq 0$.

Since $f'_\theta(x) = 4\theta \exp(-2\theta x)(\theta - \sigma + \theta \alpha x)$, there is a unique $x = x_0$ such that $f'_\theta(x_0) = 0$, and for $x > x_0$, $f'_\theta(x) > 0$, while for $x < x_0$, $f'_\theta(x) < 0$. Since we have $\xi_1 < \xi_2$ with $f'_\theta(\xi_1) \geq 0$, $f'_\theta(\xi_2) \geq 0$, a contradiction results, completing the proof.

It follows from (1) that

\[ \bar{\theta}_N = \inf \{ \theta \mid \pi_i(\theta) \geq p_i, i = 0, 1, 2, \cdots, 2N \} \]

= max $\{ \theta^*_{i_1}, \theta^*_{i_2}, \cdots, \theta^*_{2N-1} \} = \theta^*_N$.

It is of interest to find quantities such as $\bar{\theta} = \sup \{ \theta; f_\theta(x) \geq 0, 0 \leq x \leq 1 \}$, so $\bar{\theta} \leq \bar{\theta}_N$, and $\bar{\theta}$ defined analogously (so $\bar{\theta} \leq \bar{\theta}_N$), which will lead to bounds at least as tight as those of Moran (1960) and likewise valid for all $N$. It is readily seen by a contradiction argument similar to the above that $f_\theta(x) \geq 0$ for all $0 \leq x \leq 1$ if and only if $f'_\theta(1) \leq 0$, which leads to $\bar{\theta}$ as the unique root in $(0, \infty)$ of $\exp 2\theta - 1 - (1 + \sigma)2\theta$, while $\bar{\theta}$ is the unique root of $1 - \exp -2\theta = 2\theta/(1 + \sigma)$, being the smallest $\theta$ in $(0, \infty)$ for which $f'_\theta(0) \leq 0$. Note (without digression as to causes) that $2\theta/(1 + \sigma)$ is the survival probability of a Galton–Watson process with offspring p.g.f. $f(s) = \exp(1 + \sigma)(s - 1)$ and $e^{-2\theta}$ is the extinction probability. The argument used to prove (1) can again be used to prove e.g. that $\theta_N > \theta_N^{*+1}$, whence as $N \to \infty \theta_N \uparrow \bar{\theta}$; and similarly $\bar{\theta}_N \uparrow \bar{\theta}$. Note also that Moran’s (1960) explicit simple bounding interval, $[\sigma/(1 + \sigma), \sigma]$, containing $[\theta, \bar{\theta}]$, in particular leads to simple explicit bounds on the above survival probability.

References
