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GENERALIZED GRADIENTS, LIPSCHITZ BEHAVIOR AND DIRECTIONAL DERIVATIVES

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1. Introduction. In the study of optimization problems it is necessary to consider functions that are not differentiable. This has led to the consideration of generalized gradients and a corresponding calculus for certain classes of functions. Rockafellar [**16**] and others have developed a very strong and elegant theory of subgradients for convex functions. This convex theory gives point-wise criteria for the existence of extrema in optimization problems.

There are however many optimization problems that involve functions which are neither differentiable nor convex. Such functions arise in many settings including optimal value functions [**15**]. In order to deal with such problems Clarke [**3**] defined a type of subgradient for nonconvex functions. This definition was initially for Lipschitz functions on $\mathbf{R}^n$. Clarke extended this definition to include lower semicontinuous (l.s.c.) functions on Banach spaces through the use of a directional derivative, the distance function from a closed set and tangent and normal cones to closed sets. These generalized gradients have found many uses; see for example Clarke [**3**, **4**, **5**, **6**], Aubin [**1**], Hiriarty-Urruty [**9**, **10**] and Rockafellar [**15**].

Rockafellar [**14**] has given a more direct characterization of Clarke's subgradients $\partial f(x)$ to a l.s.c. function f on a Banach space E by way of the upper subderivative

$$f^{\uparrow}(x; y) = \sup_{Y \in \mathcal{N}(y)} \inf_{\substack{x \in \mathcal{N}(x) \\ \delta > 0 \\ \lambda > 0}} \sup_{\substack{t \in (0,\lambda) \\ x' \in X \\ f(x') \leqq f(x) + \delta}} \inf_{y' \in Y} \frac{f(x' + ty') - f(x')}{t}.$$

This is a l.s.c. convex function of y for each x. The subgradients of f at x are given by

$$\partial f(x) = \{v^* \in E^* : \langle v^*, y \rangle \leqq f^{\uparrow}(x; y), y \in E\}.$$

We will not use the first of the formulas directly but will require the second.

Rules for estimating $\partial f(x)$ are important for analyzing when $0 \in \partial f(x)$

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since $0 \in \partial f(x)$ is the subgradient optimality condition. Two often considered cases are when $f = f_1 + f_2$ and when $f = g \cdot F$ where $F:E_1 \to E$ is a continuously Gateaux differentiable mapping with derivative $A:E_1 \to E$. The desired rules for estimation are

$$(1.1) \quad \partial f(x) \subset \partial f_1(x) + \partial f_2(x)$$

and

$$(1.2) \quad \partial(g \cdot F)(z) \subset A^*(g(x)) \cdot \partial g(F(z))$$

where A^* is the adjoint mapping of A.

Conditions must be put on f_1, f_2 and g to make (1.1) and (1.2) valid. These conditions involve two types of Lipschitz behavior. A function $h(x)$ is *Lipschitz* on a neighborhood X of x if for some L,

$$(1.3) \quad |h(x^1) - h(x^2)| \leqq L\|x^1 - x^2\|$$

for any x^1 and x^2 in X. If (1.3) holds in some neighborhood of x then h is *locally Lipschitz* at x. The result in this case is

THEOREM 1. **[5]** *Assume that f_1 and g are locally Lipschitz at x and $F(z)$, respectively, and assume that f_2 is* l.s.c. *Then* (1.1) *and* (1.2) *are valid.*

The other condition used in this setting is that f_1 and g are directionally Lipschitz at x. A l.s.c. function $h(x)$ is *directionally Lipschitz* at x with respect to y if

$$\limsup_{\substack{x' \to x \\ y' \to y,\, t \searrow 0}} \frac{h(x' + ty') - h(x')}{t} < \infty.$$

If any such y exists we simply say that h is directionally Lipschitz at x. Rockafellar **[14]** has noted that h is locally Lipschitz at x if and only if h is directionally Lipschitz at x with respect to 0. Under the assumptions that f_1 is directionally Lipschitz at $F(z)$ and some technical conditions on the upper subderivatives of f_1, f_2 and g, (1.1) and (1.2) hold **[12]**.

In finite dimensions the relationship between local Lipschitz behavior and Clarke's subgradients has been known for some time.

THEOREM 2. **[13]** *Let $f:\mathbf{R}^n \to \mathbf{R}$ be a* l.s.c. *function and x a point of $\mathbf{R}^n$ where f is finite. Then f is Lipschitz on a neighborhood of x if and only if $\partial f(x)$ is a bounded nonempty set.*

The goal of this paper is to prove a theorem similar to Theorem 2 for directionally Lipschitz functions on Banach spaces. In order to achieve this we first discuss the relationship between the upper subderivative and the lower Hadamard derivative. Next we prove a theorem similar to Theorem 1 for $f:E \to \mathbf{R}$ where E is an arbitrary Banach space. Finally an analogous result for directionally Lipschitz functions is obtained.

2. Relations between derivatives. Throughout the rest of this paper E and E_1 will be Banach spaces and all functions will be l.s.c.

Before proceeding to the main results of this work we need to discuss the relationship between the *lower Hadamard derivative*

$$f^{\#}(x; y) = \liminf_{\substack{y' \to y \\ t \searrow 0}} \frac{f(x + ty') - f(x)}{t}$$

and the upper subderivative, $f^{\uparrow}(x; y)$. Each of these directional derivatives corresponds to a tangent cone to the *epigraph* of f,

$$\text{epi}\, f: = \{ (x, \alpha) \in E \times R : \alpha \geqq f(x) \}.$$

The definitions of these tangent cones are:

Definition 1. Let C be a closed subset of a Banach space E_1 and $x \in C$. The *contingent cone* to C at x, $K(C, x)$, is the set of all y such that for any $\epsilon, \lambda > 0$ the truncated cone

$$x + (0, \lambda) \cdot B(y, \epsilon)$$

intersects C.

Here $B(y, \epsilon)$ is the open ball centered at y with radius ϵ and

$$B \cdot A = \{\beta \cdot z : \beta \in B \subset R \text{ and } z \in A \subset E_1\}.$$

Definition 2. [**17**] Let C be a closed subset of a Banach space E_1 and $x \in C$. *Clarke's tangent cone* (the *tangent cone*) to C at x, $T(C, x)$, is the set of y such that for all $\epsilon > 0$,

$$\exists \delta > 0 \text{ such that } \forall x' \in B(x, \delta) \cap C, \lambda > 0,$$

$$[x' + (0, \lambda) \cdot B(y, \epsilon)] \cap C \neq \emptyset.$$

This is not Clarke's original definition but is equivalent to it. The properties of these cones and their relationship are discussed in a number of articles (see [7] and [**11**]).

From the definitions of the tangent and contingent cones it follows that

$$T(C, x) \subset K(C, x).$$

Combining this with Theorem 3.1 of [**17**] we have

$$\liminf_{\substack{x' \to x \\ C}} K(C, x) \subset T(C, x) \subset K(C, x) \tag{2.1}$$

where for a multifunction $F : E_1 \to E_2$

$$\liminf_{\substack{x' \to x \\ C}} F(x') = \bigcap_{\epsilon > 0} \bigcup_{X \in \mathcal{N}(x)} \bigcap_{x' \in X \cap C} [F(x') + \bar{B}(0, \epsilon)]. \tag{2.2}$$

Here $\mathcal{N}(x)$ denotes the family of neighborhoods of x.

The following facts along with (2.1) will yield the desired result. The epigraph of the lower Hadamard derivative of f at x is the contingent cone of epi f at $(x, f(x))$ and the epigraph of the upper subderivative of f at x is the tangent cone of epi f at $(x, f(x))$. The first of these statements follows easily from the definitions. For a discussion of the second see [**14**].

THEOREM 3. *Let E be a Banach space, f a* l.s.c. *function on E and x a point where f is finite. For all $y \in E$,*

$$f^{\#}(x; y) \leqq f^{\uparrow}(x; y) \leqq \limsup_{\substack{x' \to x \\ f}} f^{\#}(x'; y).$$

Proof. The first inequality is a direct consequence of (2.1). Therefore we concentrate on the second inequality.

The epigraph of $\limsup_{\substack{x' \to x \\ f}} f^{\#}(x; y)$ is

$$\bigcup_{\substack{X \in \mathcal{N}(x) \\ \delta > 0}} \bigcap_{\substack{x' \in X \\ f(x') \leqq f(x) + \delta}} K(\operatorname{epi} f, (x', f(x'))). \tag{2.3}$$

Since $K(\operatorname{epi} f, (x', \alpha)) \supset K(\operatorname{epi} f, (x', f(x')))$ for any $\alpha \geqq f(x')$, combining (2.1), (2.2) and (2.3) yields the second inequality.

3. The Lipschitz case. The statement of Theorem 2 gives a simple characterization of local Lipschitz behavior around a point x. Unfortunately the same statement is not true in infinite dimensional spaces.

Example. Let C be a closed convex subset with a boundary point $\bar{x}$ where there are no supporting hyperplanes (see [**11**]). Since C is convex, the tangent cone is the same as the convex tangent cone, and thus $T(C, \bar{x}) = E$. If f is the indicator function of C then f is not locally Lipschitz at $\bar{x}$ however $\partial f(\bar{x}) = \{0\}$.

A result slightly weaker than Theorem 2 does hold in Banach spaces. The following preliminary results will help clarify this situation.

LEMMA 1. *Let E be a Banach space, f a* l.s.c. *function on E and x a point where f is finite. If there is a neighborhood N of x on which $\partial f(x')$ is bounded ($\cup_{x' \in N} \partial f(x')$ is bounded set) and nonempty then f is locally Lipschitz at x.*

Proof. We proceed by contradiction. Assume that $\partial f(x')$ is bounded and nonempty on an open convex neighborhood N of x and assume that f is not Lipschitz on N.

For any $L > 0$ there are $z \in N$ and $y \in E \setminus \{0\}$ such that $z + y \in N$ and

$$f(z + y) > f(z) + L\|y\|.$$

Since f is l.s.c. there is a neighborhood

$$\bar{B}((z + y, f(z) + L\|y\|), \beta) \subset E \times \mathbf{R}$$

of $(z + y, f(z) + L\|y\|)$ that does not intersect epi f. Take β so that

$$B(z + y, \beta) \subset N.$$

Thus we have the following situation; there is a closed subset $C = \operatorname{epi} f$ of a Banach space $E_1 = E \times \mathbf{R}$, a point $\bar{z} = (y, L\|y\|)$ such that

$$C \cap [\bar{z} + Y] = \emptyset$$

where Y is a ball around w with radius less than $\|w\|$.

The technique of Bishop and Phelps [**2**] as applied in the proof of Lemma 2.1 of [**17**] can be employed here to find an $x' \in N$ and a $\lambda > 0$ so that $f(x') < \infty$ and

$$\operatorname{epi} f \cap [(x', f(x')) + (0, \lambda) \cdot \bar{B}((y, L\|y\|), \beta)] = \emptyset.$$

This implies that $f^{\#}(x'; y) > L\|y\|$. Therefore $f^{\uparrow}(x'; y) > L\|y\|$.

Hence either $\partial f(x')$ is empty or $\partial f(x')$ contains an element of norm greater than L.

What we want to have is Lemma 1 without the added condition that $\partial f(x')$ is nonempty.

LEMMA 2. *Let E be a Banach space, f a* l.s.c. *function on E and x a point where f is finite. If, in every neighborhood of x, there is a point x' such that $\partial f(x') = \emptyset$ then ∂f is unbounded on any neighborhood of x.*

Proof. Let N be any open convex neighborhood of x. Since f is l.s.c. we may assume $f(x') > -\infty$ for all $x' \in N$. Let $L > 0$ be arbitrary.

There are two situations to consider: when $f(x') = \infty$ for some $x' \in N$ and when $\partial f(x') = \emptyset$ and $f(x') < \infty$ for some $x' \in N$. In the first case the technique used in Lemma 1 can be applied to find a point $z \in N$ and a $y \in E \setminus \{0\}$ such that $f(z) < \infty$ and

$$f^{\uparrow}(z; y) > L\|y\|.$$

If $\partial f(z)$ is nonempty we are done. Otherwise the problem reduces to the second case.

Assume that $f(x') < \infty$ and $\partial f(x') = \emptyset$ for $x' \in N$. Since $\partial f(x') = \emptyset$, the line $(-\infty, \infty) \cdot (0, -1)$ is in $T(\operatorname{epi} f, (x', f(x')))$. By Definition 2 the interior of every truncated cone of the form

$$(x', f(x')) + (0, \lambda) \cdot B((0, -1), \epsilon)$$

intersects epi f. Here $\epsilon, \lambda > 0$ are arbitrary.

The fact that f is l.s.c. guarantees that for some λ_0, ϵ_0, we have

$$\lambda_0 \cdot B(x', \epsilon_0) \subset N \quad \text{and}$$

$$\{ (x', f(x')) + \lambda_0 \cdot B((0, -1), \epsilon_0) \} \cap \operatorname{epi} f = \emptyset.$$

This implies that for any $\epsilon \in (0, \epsilon_0)$,

$$\{ (x', f(x')) + \lambda_0 \cdot B((0, -1), \epsilon) \} \cap \operatorname{epi} f = \emptyset.$$

Fix $\epsilon \in (0, \epsilon_0)$ with $\epsilon < 1/L$ and define a function $F:E \to \mathbf{R}$ by

$$F(z) = \begin{cases} f(z) - f(x') + \dfrac{1}{\epsilon}\|z - x'\| & \text{if } \|z - x'\| \leqq \lambda_0\epsilon \\ +\infty & \text{otherwise.} \end{cases}$$

This definition insures that F is l.s.c., $F(x') = 0$, $\inf F < 0$ and if $\|z - x'\| \leqq \lambda_0\epsilon$ then $F(z) > 0$.

We can apply Ekeland's variational principle [**8**] to F to get the desired result. The version needed is the following:

THEOREM 4. *Let E be a Banach space and F a* l.s.c. *function from E into* $\mathbf{R}$ *that is bounded from below. For any $\delta > 0$ there is a $z \in E$ such that*

$$F(z) \leqq \inf F + \delta \quad \textit{and}$$

$$\forall z' \in E \quad F(z') \geqq F(z) - \delta\|z - z'\|.$$

Applying this theorem with $\delta < \min\left\{\dfrac{1}{\epsilon} - L, -\inf F\right\}$ yields a $z \in E$ such that

$$\|z - x'\| < \lambda_0\epsilon, \ z \neq x' \quad \text{and}$$

$$\forall z' \in E \quad F(z') \geqq F(z) - \delta\|z' - z\|.$$

Thus

$$f(z') + \frac{1}{\epsilon}\|z' - x\| \geqq f(z) + \frac{1}{\epsilon}\|z - x\| - \delta\|z' - z\|$$

or

$$f(z') \geqq f(z) + \frac{1}{\epsilon}(\|z - x\| - \|z' - x\|) - \delta\|z' - z\|.$$

The function

$$G(z') = f(z) - \frac{1}{\epsilon}(\|z' - x\| - \|z - x\|) - \delta\|z' - z\|$$

has lower Hadamard derivatives

$$G^{\#}\left(z; \frac{z - x}{\|z - x\|}\right) = -\frac{1}{\epsilon} - \delta \quad \text{and}$$

$$G^{\#}\left(z; \frac{x - z}{\|z - x\|}\right) = \frac{1}{\epsilon} - \delta.$$

Since $G(z) = f(z)$, $G(z') \leqq f(z')$ for all z' and $\frac{1}{\epsilon} - \delta > L$, we have that f is supported below by a Lipschitz function G on a neighborhood of z,

$$f^{\#}\left(z; \frac{z - x}{\|z - x\|}\right) > -\infty \quad \text{and}$$

$$f^{\#}\left(z; \frac{x - z}{\|z - x\|}\right) > L.$$

Therefore $\partial f(z) \neq \emptyset$ and $\partial f(z)$ contains an element of norm greater than L.

We can now state and prove the main theorem of this section.

THEOREM 5. *Let E be a Banach space, f a* l.s.c. *function on E and x a point where f is finite. Under these hypotheses the following are equivalent*:

(i) *f is locally Lipschitz at x,*

(ii) *$\cup_{x' \in N} \partial f(x')$ is a bounded subset of E^* for some neighborhood N of x,*

(iii) *for some $L > 0$ and some neighborhood N of x,*

$$|f^{\#}(x'; y)| < L \quad \textit{for all } x' \in N \textit{ and } y \in \bar{B}(0, 1),$$

(iv) *there exists a neighborhood N of x and an $M > 0$ such that*

$$f^{\#}(x'; y) \leqq M \quad \textit{for all } x' \in N \textit{ and } y \in \bar{B}(0, 1).$$

Proof. We will prove the following implications: (i) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (i) and (ii) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (iii). Assume f is Lipschitz on an open convex neighborhood N of x with Lipschitz constant L. Since f is Lipschitz on N, at any point $x' \in N$ and for any $y \in E$ the lower Hadamard derivative and lower directional derivative coincide. For any x' in N and $y \in E$ and some $t_0 > 0$,

$$\left|\frac{f(x' + ty) - f(x')}{t}\right| \leqq L\|y\|.$$

Thus

$$|f^{\#}(x':y)| \leqq L\|y\|$$

for any $x' \in N$ and $y \in E$.

(iii) $\Rightarrow$ (ii). Let the neighborhood N of x be an open set. Since

$$|f^{\#}(x'; y)| \leqq L\|y\|$$

for all $x' \in N$ and $y \in \bar{B}(0, 1)$, Theorem 3 shows that

$$|f^{\uparrow}(x'; y)| \leqq L$$

for all $x' \in N$ and $y \in \bar{B}(0, 1)$. Thus $\cup_{x' \in N} \partial f(x')$ contains no elements of norm greater than L.

(ii) $\Rightarrow$ (i). Assume N is an open neighborhood of x. Since $\cup_{x' \in N} \partial f(x')$ is a bounded set, Lemmas 1 and 2 imply f is Lipschitz on N.

(iii) $\Rightarrow$ (iv). This follows directly from the statements.

(iv) $\Rightarrow$ (iii). Without loss of generality let N be an open convex neighborhood of x. We proceed by contradiction. Assume that (iv) holds on N and that (iii) doesn't hold on N. Then there exist $x' \in N$ and $y \in \bar{B}(0, 1)$ such that

$$f^{\#}(x'; y) < -2M.$$

This implies that for some $t > 0$ and $y' \in B(0, 2)$

$$\frac{f(x' + ty) - f(x')}{t} < -2M$$

where $x'' = x' + ty' \in N$.

Rewriting, this becomes

$$\frac{f(x'' - ty') - f(x'')}{t} > 2M.$$

Applying the argument used in Lemma 1 yields a point $z \in N$ such that

$$f^{\#}(z; -y') > 2M$$

or

$$f^{\#}\left(z, -\frac{y'}{2}\right) > M$$

where $-y'/2 \in \bar{B}(0, 1)$. This contradicts our assumptions. Therefore (iv) $\Rightarrow$ (iii).

Note. Using the same techniques as those in the proof that (iii) is equivalent to (iv) one can show that f being locally Lipschitz is equivalent to (iii) or (iv) where the lower Hadamard derivative is replaced by any of the following three directional derivatives; the standard lower directional derivative, the upper Hadamard derivative

$$\limsup_{\substack{y' \to y \\ t \searrow 0}} \frac{f(x' + ty') - f(x')}{t},$$

or the upper directional derivative

$$\limsup_{t \searrow 0} \frac{f(x' + ty) - f(x') - f(x')}{t}.$$

4. The directionally Lipschitz case. Theorem 5 of the previous section gives the relation between Lipschitz behavior and Clarke's subgradients on Banach spaces. There has not been any such characterization of directionally Lipschitz behavior. The main results along this line concern conditions implying directional Lipschitz behavior on $\mathbf{R}^n$ [Theorem 1 of **12**].

In order to use the concept of directionally Lipschitz functions on Banach spaces a theorem similar to Theorem 5 is required. The central theorem of this section along with its corollaries help fill this void.

THEOREM 6. *Let f be a* l.s.c. *function from E into* $\mathbf{R}$ *and let x be a point where f is finite. Under these assumptions the following are equivalent*:

(i) *f is directionally Lipschitz at x with respect to y,*

(ii) *for some neighborhoods Y of y and X of x and $\delta > 0$,*

$$\{f^{\#}(x', y'):x' \in X, y' \in Y, f(x') < f(x) + \delta\}$$

is bounded from above

(iii) *for some neighborhoods Y of y and X of x and some $\delta > 0$, f is directionally Lipschitz at all $x' \in X$ such that $f(x') < f(x) + \delta$ with respect to all $y' \in Y$.*

(iv) *for some neighborhoods Y of y and X of x and some $\delta > 0$,*

$$\{ \langle v^*, y'\rangle :\forall v^* \in \partial f(x'), x' \in X, y' \in y, f(x') \leqq f(x) + \delta\}$$

is bounded from above.

Proof. Rockafellar [**14**] states that f being directionally Lipschitz at x with respect to 0 is equivalent to f being locally Lipschitz around x. With minor modifications the proof of Theorem 5 works for Theorem 6 with $y = 0$. Hence we only consider the case when $y \neq 0$.

Without loss of generality assume that $||y|| = 1$ and that all neighborhoods of y are convex. The implications to the proven are (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (i) and (ii) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (ii). This follows directly from the definitions.

(ii) $\Rightarrow$ (iii). We proceed by contradiction. Assume that for all neighborhoods Y of y and X of x and for all $\delta > 0$, f is not directionally Lipschitz at some $x' \in X$ with respect to some $y' \in Y$ where $f(x') < f(x) + \delta$. This implies that for any $M > 0$ there exist x'', y'' and $t > 0$ with x'' as close to x' as desired, $f(x'') < f(x) + \delta$, y'' as close to y' as desired and t as small as wanted such that

$$\frac{f(x'' + ty'') - f(x'')}{t} > M.$$

By taking y'' close enough to y' and t small enough one can make

$$y'' \in Y, x'' + ty'' \in X \quad \text{and}$$
$$f(x'') + Mt||y''|| < f(x) + \delta.$$

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The argument used in Lemma 1 can be applied to find a $z \in X$ with $f(z) < f(x) + \delta$ and

$$f^{\#}(z; y'') \geqq M.$$

This proves the contrapositive of (ii) $\Rightarrow$ (iii), or (iii) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (ii). This is clear.

(ii) $\Leftrightarrow$ (iv). This follows from Theorem 3 and the duality between $\partial f(x)$ and $f^{\uparrow}(x; y)$.

The following two results give more conditions that allow one to detect directionally Lipschitz functions. They may be obtained from either Theorem 6 or the definition of a directionally Lipschitz function using the techniques in this paper.

COROLLARY 1. *Let f be a* l.s.c. *function from E into $\mathbf{R}^n$ and x a point where f is finite. Then f is directionally Lipschitz at x if and only if there exist a y, neighborhoods Y of y and X of x, an $L > 0$ and t, $\delta > 0$ such that*

$$f(x' + z) - f(x') < L\|z\|$$

for all $x' \in X$ with $f(x') < f(x) + \delta$ and all z in the truncated cone $(0, t) \cdot Y$.

This corollary along with the next result show how closely directionally Lipschitz functions are related to Lipschitz functions. Before stating the final corollary a definition is needed.

Definition 3. **[13]** A closed set $C \subset E$, is said to be *epi-Lipschitzian at x* if there is some neighborhood X of x, a $\lambda > 0$ and a non-empty open set Y such that

$$x' + ty' \in C \quad \text{for } x \in C \cap X, y' \in Y, t \in (0, \lambda).$$

This is equivalent to saying that C is the epigraph of a Lipschitz function **[13]**.

COROLLARY 2. **[13]** *A* l.s.c. *function f on a Banach space E is directionally Lipschitz at x if and only if* epi f *is* epi-*Lipschitzian at* $(x, f(x))$.

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