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GAUSSIAN HOLOMORPHIC SECTIONS ON NONCOMPACT COMPLEX MANIFOLDS

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Abstract We provide two constructions of Gaussian random holomorphic sections of a Hermitian holomorphic line bundle (L,h_L) on a Hermitian complex manifold (X,Θ), that are particularly interesting in the case where the space of $\mathcal{L}^2$-holomorphic sections $H^0_{(2)}(X,L)$ is infinite dimensional. We first provide a general construction of Gaussian random holomorphic sections of L, which, if $H^0_{(2)}(X,L)$ is infinite dimensional, are almost never $\mathcal{L}^2$-integrable on X. The second construction combines the abstract Wiener space theory with the Berezin–Toeplitz quantization and yields a Gaussian ensemble of random $\mathcal{L}^2$-holomorphic sections. Furthermore, we study their random zeros in the context of semiclassical limits, including their distributions, large deviation estimates, local fluctuations and hole probabilities.

Contents

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1. Introduction

The origins of the study of random polynomials as well as their generalizations in geometry trace back to value distribution theory of analytic functions and quantum chaos, the former being a classic topic. In order to understand the typical distribution of zeros, one forms ensembles of analytic functions by defining the coefficients of their representation in a given basis to be independent random variables with a prescribed distribution and then studies their statistical properties. Importantly, the eigenfunctions of quantum chaotic Hamiltonians are well modeled by random polynomials in terms of the distribution of their zeros, critical points, sup-norms and quantum expectation values (cf. [11, 62, 79, 68]).

A natural geometric generalization of polynomials are the holomorphic sections of a holomorphic line bundle. In this paper we study the construction of Gaussian ensembles of $\mathcal{L}^2$-holomorphic sections of a line bundle, focusing on the case when this space is infinite dimensional. We consider a connected complex n-dimensional Hermitian manifold (X,Θ) without boundary, where Θ is a smooth Hermitian form compatible with the complex structure of X. We denote by L a holomorphic line bundle over X, and let h_L be a smooth Hermitian metric on L. We denote by $H^0(X,L)$ the space of holomorphic sections of L and by $H^0_{(2)}(X,L)$ the space holomorphic sections which are $\mathcal{L}^2$-integrable with respect to the $\mathcal{L}^2$-norm (2.2) induced by Θ and h_L. If the dimension $d := \dim_{\mathbb{C}} H^0_{(2)}(X,L)$ is finite, one can construct a Gaussian probability measure on $H^0_{(2)}(X,L)$ induced by the $\mathcal{L}^2$-metric. If, however, $H^0_{(2)}(X,L)$ is infinite dimensional, then this construction is bound to fail.

To tackle this difficulty, we will provide two different approaches for constructing a random holomorphic section from the infinite-dimensional $H^0_{(2)}(X,L)$. Both of them are natural as extensions of the finite-dimensional case.

The first approach is a direct generalization of the study of random holomorphic functions on $\mathbb{C}^n$ to the context of complex geometry. The random holomorphic functions given by power series on $\mathbb{C}$ as well as the distributions of their zeros (or other values) have been studied by Kac [45], Littlewood–Offord [52, 53], Offord [63, 64, 65] and by Edelman–Kostlan [29, 30], etc. For Gaussian random holomorphic functions, the results have further been extended by Sodin [71], Sodin–Tsirelson [72, 73, 74], and then, on $\mathbb{C}^n$, by Zrebiec [82]. In particular, the general Gaussian random holomorphic functions on the domains in $\mathbb{C}$ (also known under the name Gaussian analytic functions, GAFs) have been investigated vastly (cf. [40]) from probabilistic perspectives, serving as examples of point processes on $\mathbb{C}$.

We construct a Gaussian random section in terms of an orthonormal basis of $H^0_{(2)}(X,L)$ such that its distribution is independent of the choice of such basis (cf. Proposition 2.3). More concretely, if $\{S_j\}_{j=1}^d$ is an orthonormal (Hilbert) basis of $H^0_{(2)}(X,L)$ and if $\{\eta_j\}_{j=1}^d$ denotes a sequence of independent and identically distributed (i.i.d.) standard complex Gaussian variables, we can define a random holomorphic section of L via

$$\psi^S_\eta := \sum_{j=1}^{d} \eta_j S_j \tag{1.1}$$

by using properties of the Bergman kernel associated with $H^0_{(2)}(X,L)$ (cf. Proposition 2.1). We will call ψ^S_η a standard Gaussian random holomorphic section of L. It turns out that when $d=\infty$, then ψ^S_η as constructed in Equation (1.1) is almost surely non-$\mathcal{L}^2$-integrable over X (cf. Lemma 2.5). In the case of the Bargmann–Fock space on $\mathbb{C}^n$ (cf. Example 2.14), ψ^S_η is just a Gaussian holomorphic function on $\mathbb{C}^n$ as mentioned before. If $d<\infty$ the above construction is equivalent to endowing $H^0_{(2)}(X,L)$ with the standard Gaussian probability measure associated to the $\mathcal{L}^2$ inner product.

An interesting question which arises is what are the possibilities to randomize $\mathcal{L}^2$-holomorphic sections in a natural manner, or equivalently, how to construct Gaussian probability measures on $H^0_{(2)}(X,L)$ *in a geometric way*. The starting point is the simple observation that for $\mathbf{a} = (a_j)_{j=1}^d \in \ell^2(\mathbb{C})$, the section

$$\psi^S_{\mathbf{a},\eta} := \sum_{j=1}^{d} \eta_j a_j S_j \tag{1.2}$$

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is almost surely an $\mathcal{L}^2$-integrable holomorphic section since $\mathbb{P}(\sum_j |a_j|^2|\eta_j|^2 < \infty) = 1$. We introduce the geometric input by taking $(a_j)_{j=1}^d$ to be the spectrum of a Toeplitz operator on $H^0_{(2)}(X,L)$. For any bounded function $f \in \mathscr{C}^\infty(X)$, the Toeplitz operator with symbol f is the endomorphism of $H^0_{(2)}(X,L)$ given by $T_f = PM_f$, where M_f is the pointwise multiplication with f and P is the orthogonal projection from $\mathcal{L}^2(X,L)$ onto $H^0_{(2)}(X,L)$. For our purpose, we consider an injective Hilbert–Schmidt Toeplitz operator T_f on $H^0_{(2)}(X,L)$, which defines a measurable norm $\|T_f \cdot\|$ on $H^0_{(2)}(X,L)$ (cf. Definition 4.1). Taking advantage of the theory of abstract Wiener spaces of Gross [36], we can construct in a unique way a Gaussian probability measure $\mathbb{P}_f$ on $H^0_{(2)}(X,L)$ associated with T_f. A random $\mathcal{L}^2$-holomorphic section following the probability law $\mathbb{P}_f$ is exactly given as in Equation (1.2), where each $a_j > 0$ is an eigenvalue of T_f and the orthonormal basis $\{S_j\}_{j=1}^d$ is such that $T_f S_j = a_j S_j$. For a brief introduction to Gross' abstract Wiener spaces, we refer to [42, Example 1.25].

On top of the constructions of random holomorphic sections outlined above, we aim to study the distribution of their zeros as (1,1)-currents on X in the framework of semiclassical limits, that is, considering random holomorphic sections of the sequence of high tensor powers $(L^p,h_p) := (L^{\otimes p},h_L^{\otimes p})$, $p \in \mathbb{N}$, of a given positive Hermitian line bundle (L,h_L). As $p \to \infty$, the parameter $h := 1/p$, playing the role of the Planck constant, tends to 0. For this purpose, we need to make further assumptions on (X,Θ) and (L,h_L); see Condition 1.2. For a sequence of random sections $\psi_\eta^{S_p}$ constructed as in Equation (1.1) using an orthonormal basis of the Hilbert spaces $H^0_{(2)}(X,L^p)$, the distributions of the normalized zeros of these sections are expected to converge towards the first Chern form $c_1(L,h_L)$ as $p \to \infty$. Thus, zeros tend to concentrate in regions of high curvature.

For compact Kähler manifolds, the asymptotic distribution as $p \to \infty$ of zeros of sections of $H^0(X,L^p)$ for a positive line bundle L was extensively studied by Shiffman and Zelditch [68, 81]. Pioneering works in the physics literature were Bogomolny, Bohigas and Leboeuf [11] as well as Nonnenmacher–Voros [62] in the context of quantum chaotic dynamics. A key ingredient in their approach is the asymptotic expansion of the associated Bergman kernel (cf. [56, 76, 80] and the references therein).

Using ideas from complex dynamics, a new method to study the distribution of zeros was introduced by Dinh–Sibony [26], which also provides bounds for the convergence speed. Subsequently, Dinh, Marinescu and Schmidt [25] extended such results to the noncompact setting, where they assumed that the dimension $d_p := \dim_{\mathbb{C}} H^0_{(2)}(X,L^p) \in \mathbb{N} \cup \{\infty\}$ satisfies $d_p = \mathcal{O}(p^n)$ for $p \gg 0$ (which is the same growth as in the compact case). Along these lines, there are also plenty of generalizations to different geometric (singular Hermitian metrics or singular space X) or probabilistic settings (general probability measures), cf. [6, 10, 15, 13, 16, 17, 24, 61]. We refer to the survey papers [5, 81] for more details and references on this topic.

For compact Kähler manifolds, Shiffman, Zelditch and Zrebiec [70] established large deviation estimates for the random zeros of Gaussian holomorphic sections as p grows to infinity. As a consequence, they obtained the exponential decay of the hole probabilities, that is, the probabilities that the Gaussian random holomorphic sections do not vanish on a given domain in X. In our previous paper [27] (see also [54]), we generalized their results to the noncompact setting, especially for the case of Riemann surfaces with cusps, under the assumption $d_p = \mathcal{O}(p^n)$ for $p \gg 0$. In this paper, we show that these results also extend to the case of Gaussian random holomorphic sections $\psi_\eta^{S_p}$ without assuming $d_p < \infty$.

In this semiclassical setting, the use of Toeplitz operators in our construction of the random $\mathcal{L}^2$-holomorphic sections is natural. The family of Toeplitz operators $T_{f,p} = P_p M_f$ on $H^0_{(2)}(X,L^p)$, $p \in \mathbb{N}$, is called Berezin–Toeplitz quantization of a given real smooth function $f : X \to \mathbb{R}$. The function f can be regarded as a classical observable on the phase space (X,Θ) and $T_{f,p}$ as the corresponding quantum observable on the quantum space $H^0_{(2)}(X,L^p)$ (cf. [12]). Such operators are central in the study of geometric quantization on Kähler or, more generally, on symplectic manifolds. For more details, we refer to [12, 55, 57, 58, 59] and [56, Chapter 7]. In our context, the geometry of (X,Θ) and (L,h_L) enters the picture through the sequence of Bergman projectors P_p on $H^0_{(2)}(X,L^p)$ and the multiplication with f will have the effect localizing the Gaussian probability measure $\mathbb{P}_f$ on the support of f.

Here, we introduce a class of functions f on X such that $T_{f,p}$ is Hilbert–Schmidt for all $p \gg 0$. Associated to a nonnegative smooth function f in this class, we construct canonically a sequence of probability spaces $(H^0_{(2)}(X,L^p),\mathbb{P}_{f,p})$, $p \gg 0$. We are then interested in investigating the asymptotic behavior of the zeros of random $\mathcal{L}^2$-holomorphic sections as $p \to \infty$. Their limit as $(1,1)$-currents will be given by $c_1(L,h_L)$ on the support of f (where f has zeros of at most second order). One goal of geometric quantization is to recover information about the classical observable f from the action of the quantum observable $T_{f,p}$ in the semiclassical limit $p \to \infty$. We show here that we can recover the complex Laplacian $\partial\overline{\partial}\log f^2$ of the logarithm of f^2 (see also Ancona–Le Floch [1] for the compact case in a slightly different setting).

The next level of results concerns the universality of the statistics of zeros of sections of large degree p on small length scales of order the Planck scale $1/\sqrt{p}$ (cf. also [8, 37, 75]). Note that the Bergman and Toeplitz kernels exhibit their universality at this scale (see Theorem 3.14 and [56, Theorems 4.2.1 & 6.1.1]). By zooming in a small ball $B(x,R/\sqrt{p})$ around an arbitrary point x, one loses track of the special features of the geometrical setting and obtains universal limiting fluctuations of random zeros. We observe two different behaviors depending on whether $f(x) = 0$ (with vanishing order 2) or $f(x) \neq 0$; see Equation (1.25). A further interesting question is to describe the asymptotic distribution of random zeros outside the support of f.

Our approach to the above results relies on the asymptotic expansion of the on-diagonal Schwartz kernel of the operator $T^2_{f,p} = T_{f,p} \circ T_{f,p}$, as $p \to \infty$, whose first terms are computed explicitly in [56, Chapter 7] and in [59], and which also holds on noncompact manifolds cf. [56, 57, 58, 59]. In particular, we can apply them to the case considered in [56, Section 7.5] and the case of bounded geometry discussed in [60] and [33].

A comprehensive study of zeros of Gaussian $\mathcal{L}^2$-holomorphic sections by means of Toeplitz operators associated with a nonsmooth symbol f is given in our follow-up paper [28].

In the next four sections, we provide the setting and formulate our main results.

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1.1. Zeros of Gaussian random holomorphic sections

Let us start with a (paracompact) connected complex manifold (X,J,Θ), where J denotes the complex structure of X and Θ is a J-compatible smooth Hermitian form on X. Then $g^{TX}(\cdot,\cdot)=\Theta(\cdot,J\cdot)$ is the associated Riemannian metric on TX. In this case, we refer to the triplet (X,J,Θ) simply as a complex Hermitian manifold. Let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L. Assume that we have $d=\dim_{\mathbb{C}} H^0_{(2)}(X,L)\geq 1$.

The zero-sets of holomorphic sections are complex analytic sets, and they will be studied through the current of integration they define. We refer to Section 2.3 for basic definitions concerning the currents of integration along zero-divisors of a holomorphic section.

Our first result concerns the expectation of the currents of integration on the zero-divisors of the Gaussian random holomorphic section ψ^S_η defined in Equation (1.1), as a current on X, that is, of the random (1,1)-current $[\mathrm{Div}(\psi^S_\eta)]$. For any test form $\varphi\in\Omega_0^{n-1,n-1}(X)$, the random variable $\langle[\mathrm{Div}(\psi^S_\eta)],\varphi\rangle$ is measurable (cf. [15, proof of Proposition 4.2]). If the random variable $\langle[\mathrm{Div}(\psi^S_\eta)],\varphi\rangle$ is integrable for any test form φ, then the linear map

$$\varphi\mapsto\mathbb{E}\left[\langle[\mathrm{Div}(\psi^S_\eta)],\varphi\rangle\right],\quad \varphi\in\Omega_0^{n-1,n-1}(X),$$

defines a (1,1)-current on X, which we refer to as the expectation of $[\mathrm{Div}(\psi^S_\eta)]$, denoted by $\mathbb{E}[[\mathrm{Div}(\psi^S_\eta)]]$. Next, we define the *Fubini–Study current* $\gamma(L,h_L)$ on X. For this purpose, let

$$P:\mathcal{L}^2(X,L)\to H^0_{(2)}(X,L) \tag{1.3}$$

be the $\mathcal{L}^2$-orthogonal projection, called the *Bergman projection*. It has a smooth Schwartz kernel $P(x,y)$, called the *Bergman kernel*, cf. Subsection 2.1. The Bergman kernel function $X\ni x\mapsto P(x,x)$ is a nonnegative smooth function on X, and the function $\log P(x,x)$ is locally integrable on X. We set

$$\gamma(L,h_L)=c_1(L,h_L)+\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log P(x,x), \tag{1.4}$$

where $c_1(L,h_L)$ is the Chern form of (L,h_L). See Remark 2.7 for their geometric interpretation.

Theorem 1.1. *Assume that $d\geq 1$. Then the expectation of the random variable $[\mathrm{Div}(\psi^S_\eta)]$ exists as a (1,1)-current on X and we have in the sense of currents,*

$$\mathbb{E}[[\mathrm{Div}(\psi^S_\eta)]]=\gamma(L,h_L)\,. \tag{1.5}$$

In the case $d<\infty$, Equation (1.5) was already known for line bundles with empty base locus (cf. [68, Lemma 3.1], [56, Theorem 5.3.1]) and in several situations when the metric h_L or the base X are singular (see, e.g., [15, Proposition 4.2], [14, Theorem 1.4]). When $d=\infty$ analogs of this result are known in the context of random holomorphic functions on $\mathbb{C}^m$; for instance, Edelman and Kostlan [29, Sections 7 & 8] studied the expectations of complex zeros of random power series (in their paper, they mainly aimed to study the distribution of real zeros). Other interesting examples from complex geometry, where

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our Theorem 1.1 applies, are given in Subsection 2.5. In particular, Theorem 2.13 gives a more general version of Theorem 1.1 where we allow the Hermitian metric h_L to be singular. The equidistribution results for the case of singular Hermitian metric h_L with finite-dimensional holomorphic sections were studied in [13, 14, 15, 16, 17, 24].

1.2. High tensor powers of L: equidistribution and large deviations

We are interested in the semiclassical limit of the zeros of Gaussian holomorphic sections when we replace L by its high tensor powers. For this purpose, we need to make further assumptions on the complex Hermitian manifold (X,J,Θ) and on (L,h_L) as follows.

Condition 1.2. The Riemannian metric g^{TX} induced by Θ is complete and there exist $C,C_0,\varepsilon>0$ such that on X,

$$\sqrt{-1}R^L \geq \varepsilon\Theta,\ \sqrt{-1}R^{\det} \geq -C_0\Theta,\ |\partial\Theta|_{g^{TX}} \leq C, \tag{1.6}$$

where $R^{\det}$ be the curvature of the holomorphic connection $\nabla^{\det}$ on $K_X^* = \det(T^{(1,0)}(X))$.

If Condition 1.2 holds, the Bergman kernel functions $P_p(x,x)$ have an asymptotic expansion in the tensor power p, which is uniform on any given compact subset of X, by [56, Chapter 6]. As a consequence, we have the convergence of currents

$$\frac{1}{p}\gamma(L^p,h_p) \to c_1(L,h_L) \quad \text{as } p\to\infty. \tag{1.7}$$

In the following, we denote by $\psi_\eta^{S_p}$ the Gaussian random holomorphic section (as in Equation (1.1)) constructed from an orthonormal basis $S_p = \{S_j^p\}_{j=1}^{d_p}$ of $H^0_{(2)}(X,L^p)$. As is natural, before formulating our concentration estimates, we begin with stating findings for the limit of the expectations $\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]]$. While the results are novel in our specific setting and formulated precisely in Theorems 3.1 and 3.7 below, we roughly speaking prove the following results.

Theorem 1.3. *Under the above settings, we have, as $p\to\infty$,*

$$\frac{1}{p}\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]] \to c_1(L,h_L).$$

Moreover, for each $\varphi\in\Omega_0^{n-1,n-1}(X)$, we have that

$$\mathbb{P}\Big(\lim_{p\to\infty}\frac{1}{p}\langle[\mathrm{Div}(\psi_\eta^{S_p})],\varphi\rangle = \langle c_1(L,h_L),\varphi\rangle\Big) = 1. \tag{1.8}$$

It is clear that the first point is a consequence of Theorem 1.1 in combination with the limit (1.7). Note that in Theorem 3.1, we actually prove that, on any given compact subset of X, $\frac{1}{p}\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]]$ admits an asymptotic expansion as $p\to\infty$. By our proof of equality (1.8) (see Theorem 3.7), the same conclusion holds for any twice differentiable $(n-1,n-1)$-form φ with compact support, that is, the limit above holds in the sense of currents of order 2.

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A deeper question on the distribution of zeros is whether sequences of individual sections tend to have equidistributed zeros, that is, if for a random sequence of sections $\psi_\eta^{S_p}$, the normalized currents of integration $p^{-1}[\mathrm{Div}(\psi_\eta^{S_p})]$ converge to $c_1(L,h_L)$ as $p\to\infty$. We show in Corollary 3.8, that under natural finiteness hypotheses (of the mass of the current $c_1(L,h_L)$ or of the volume of X in the metric given by $c_1(L,h_L)$) we can improve the equality (1.8) to

$$\mathbb{P}\Big(\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(\psi_\eta^{S_p})]=c_1(L,h_L)\Big)=1, \tag{1.9}$$

where the limit is taken with respect to the weak convergence of (1,1)-currents on X. In fact, by Remark 3.9 (3), the limit in the equality (1.9) can be taken in the sense of weak convergence of currents of order 0, that is, continuous linear functionals on the space of continuous $(n-1,n-1)$-forms with compact support.

With these distribution results about zeros of random sections at our disposal, a natural next step is to investigate the speed of convergence in terms of large deviation estimates as in [27, 70] by allowing the possibility that $d_p=\infty$.

Theorem 1.4. *Let (X,J,Θ) be a connected complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L such that Condition 1.2 holds. For any $\delta>0$ and $\varphi\in\Omega_0^{n-1,n-1}(X)$, there exists a constant $c=c(\delta,\varphi)>0$ such that for $p\in\mathbb{N}$, we have*

$$\mathbb{P}\Big(\ \Big|\Big\langle\frac{1}{p}[\mathrm{Div}(\psi_\eta^{S_p})]-c_1(L,h_L),\varphi\Big\rangle\Big|>\delta\ \Big)\leq\exp(-cp^{n+1}). \tag{1.10}$$

Estimates like (1.10) are called *large deviation estimates* or *concentration of measure estimates* [50]. Moreover, the proof of the above theorem is independent of equality (1.8); in fact, we can prove the equality (1.8) by using the estimates (1.10) (cf. [27, Section 3.6]).

A natural follow-up question is whether a central limit theorem for the distribution of zeros of $\psi_\eta^{S_p}$ as $p\to\infty$ holds true; we will touch upon this question in Remark 3.17.

Since $c_1(L,h_L)$ is positive, $\frac{1}{n!}c_1(L,h_L)^n$ also defines a positive volume element on X, and for $U\subset X$ open we set

$$\mathrm{Vol}_{2n}^L(U)=\int_U\frac{1}{n!}c_1(L,h_L)^n. \tag{1.11}$$

Then for $s_p\in H^0(X,L^p)\setminus\{0\}$ we define the $(2n-2)$-dimensional volume with respect to $c_1(L,h_L)$ of the divisor $\mathrm{Div}(s_p)$ (cf. (2.27)) in an open subset $U\subset X$ as

$$\mathrm{Vol}_{2n-2}^L\left(\mathrm{Div}(s_p)\cap U\right)=\sum_{V\subset Z(s_p)}\mathrm{ord}_V(s_p)\int_{V\cap U}\frac{c_1(L,h_L)^{n-1}}{(n-1)!}. \tag{1.12}$$

If we use this volume to measure the size of the zeros of s_p in U, then Theorem 1.4 leads to the following result.

Theorem 1.5. *Let (X,J,Θ) be a connected complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L. We assume that Condition 1.2 holds. If U is a nonempty relatively compact open subset of X such that*

∂U has zero measure in X, then for any $\delta > 0$, there exists a constant $c_{U,\delta} > 0$ such that for all p large enough, we have

$$\mathbb{P}\Big(\Big|\frac{1}{p}\,\mathrm{Vol}^L_{2n-2}(\mathrm{Div}(\psi^{S_p}_\eta) \cap U) - n\,\mathrm{Vol}^L_{2n}(U)\Big| > \delta \Big) \le \exp(-c_{U,\delta}\,p^{n+1}). \tag{1.13}$$

In addition, there exists a constant $C_U > 0$ such that for $p > 0$,

$$\mathbb{P}\big(\mathrm{Div}(\psi^{S_p}_\eta) \cap U = \varnothing\big) \le \exp(-C_U p^{n+1}). \tag{1.14}$$

The proofs of the above two theorems will be provided in Subsection 3.2. One essential ingredient for these proofs is Proposition 3.11, for which we need a more refined investigation of the local sup-norms of holomorphic sections on X (cf. Subsection 3.3).

The probability in estimate (1.14) is referred to as the *hole probability* of the random section $\psi^{S_p}_\eta$ on the subset U. This estimate then provides us with an upper bound for the hole probabilities for $p > 0$. In [70, Theorem 1.4] and [27, Proposition 1.7], under additional assumptions on U, a lower bound of the form $\exp(-C'_U p^{n+1})$ for the hole probabilities was proved. In general, however, such a lower bound remains unclear in the case $d_p = \infty$.

Let us mention here that in the special case of the Bargmann–Fock space, for the standard Gaussian random holomorphic function on $\mathbb{C}^n$ (cf. Equation (2.44)), the two-sided bound on the hole probabilities when $U = \mathbb{B}(0,r)$ as $r \to \infty$ was proved by Sodin–Tsirelson (for $\mathbb{C}$, [73, Theorem 1]) and by Zrebiec (for $\mathbb{C}^n$, [82, Theorem 1.2]). In Subsection 3.4, we will explain how to recover their findings from our general results by specializing to the scaled Bargmann–Fock spaces.

1.3. Random $\mathcal{L}^2$-holomorphic sections and Toeplitz operators

In the setting of Section 1.1, we introduce for a bounded function f on X the associated Toeplitz operator T_f defined by $T_f : H^0_{(2)}(X,L) \ni S \mapsto P(fS) \in H^0_{(2)}(X,L)$, where P is the Bergman projection (1.3) (see Definition 4.4 for further details). If f is smooth and also satisfies

$$\int_X |f(x)|P(x,x)\mathrm{dV}(x) < \infty, \tag{1.15}$$

then the operator T_f is Hilbert–Schmidt (cf. Proposition 4.7). If in addition f is nonnegative and not identically zero, then T_f is injective. For such a function f we get a Hilbert metric $\langle T_f\,\cdot, T_f\,\cdot\rangle_{\mathcal{L}^2(X,L)}$ on $H^0_{(2)}(X,L)$, which is a measurable norm in the sense of Gross (cf. [36]). Let $\mathcal{B}_f(X,L)$ be the completion of $H^0_{(2)}(X,L)$ under this measurable norm. The theory of abstract Wiener spaces implies that for f given as above, there exists a unique Gaussian probability measure $\mathcal{P}_f$ on $\mathcal{B}_f(X,L)$ such that it extends the Gaussian probability measure on any finite-dimensional subspace of $\mathrm{Im}(T_f)$ associated with the standard $\mathcal{L}^2$-metric. The injective operator T_f extends to an isometry of Hilbert spaces

$$\widehat{T}_f : \big(\mathcal{B}_f(X,L), \|T_f\cdot\|\big) \to \big(H^0_{(2)}(X,L), \|\cdot\|_{\mathcal{L}^2(X,L)}\big). \tag{1.16}$$

The pushforward $\mathbb{P}_f$ of $\mathcal{P}_f$ by $\widehat{T}_f$ is a Gaussian probability measure on $H^0_{(2)}(X,L)$. We consider the random variable on $(H^0_{(2)}(X,L),\mathbb{P}_f)$ given by $\boldsymbol{s} \mapsto [\mathrm{Div}(\boldsymbol{s})]$. Its expectation

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$\mathbb{E}^{\mathbb{P}_f}[[\mathrm{Div}(\boldsymbol{s})]]$ is a (1,1)-current given for $\varphi \in \Omega_0^{n-1,n-1}(X)$ by

$$\left\langle \mathbb{E}^{\mathbb{P}_f}[[\mathrm{Div}(\boldsymbol{s})]], \varphi \right\rangle = \int_{H^0_{(2)}(X,L)} \left(\int_{\mathrm{Div}(\boldsymbol{s})} \varphi \right) \mathrm{d}\mathbb{P}_f(\boldsymbol{s}).$$

The on-diagonal restriction $T_f^2(x,x)$ of the Schwartz kernel of $T_f^2 = T_f \circ T_f$ is locally integrable on X (cf. Lemma 4.14). Analogously to Equation (1.4), we define the Fubini–Study current associated to *f*, by

$$\gamma_f(L,h_L) = c_1(L,h_L) + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log T_f^2(x,x). \tag{1.17}$$

It is a closed positive (1,1)-current on X; see Remark 4.16 for their geometric interpretation. In Subsection 4.4, we prove the following result for the expectation of the random zeros of $\mathcal{L}^2$-holomorphic section.

Theorem 1.6. *Let (X,J,Θ) be a connected complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L. The expectation of the current valued random variable $[\mathrm{Div}(\boldsymbol{s})]$, where $\boldsymbol{s}$ runs in the Gaussian ensemble $(H^0_{(2)}(X,L),\mathbb{P}_f)$ constructed via Equation (1.16), exists and is given by the Fubini–Study current (1.17), that is,*

$$\mathbb{E}^{\mathbb{P}_f}[[\mathrm{Div}(\boldsymbol{s})]] = \gamma_f(L,h_L). \tag{1.18}$$

Remark 1.7. The equality (1.18) shows that the expectation of the zeros of $\widehat{T}_f \widetilde{\boldsymbol{s}}$ for $\widetilde{\boldsymbol{s}}$ running in the Gaussian ensemble $(\mathcal{B}_f(X,L),\mathcal{P}_f)$, equals the Fubini–Study current (1.17). All the results pertaining to $\mathbb{P}_f$, such as Theorems 1.8 and 1.9, can be viewed as concerning the zeros of $\widehat{T}_f \widetilde{\boldsymbol{s}}$. During the writing of this paper, we became aware of the work of Ancona and Le Floch [1] on zeros of images $T_f \boldsymbol{s}$ of random sections $\boldsymbol{s}$ by a Toeplitz operator T_f, in the case of a compact Kähler manifold X. For compact X, one has $d < \infty$ (we assume that $d > 0$), and in this case, the random section $\boldsymbol{s}$ in $H^0(X,L)$ with respect to the probability measure $\mathbb{P}_f$ defined above has the same distribution as the random section $T_f \widetilde{\boldsymbol{s}}$ considered by Ancona and Le Floch, where $\widetilde{\boldsymbol{s}}$ is a random section in the standard Gaussian ensemble $(H^0(X,L),\mathbb{P}_{\mathrm{st}})$ defined by the inner product $\langle \cdot,\cdot \rangle_{\mathcal{L}^2}$. In this case, we have $\mathcal{P}_f = \mathbb{P}_{\mathrm{st}}$, so $\mathbb{P}_f$ is the pushforward of $\mathbb{P}_{\mathrm{st}}$ by T_f, and Equation (1.18) reads

$$\mathbb{E}[[\mathrm{Div}(T_f \widetilde{\boldsymbol{s}})]] = \int_{H^0(X,L)} [\mathrm{Div}(T_f \widetilde{\boldsymbol{s}})] \mathrm{d}\mathbb{P}_{\mathrm{st}}(\widetilde{\boldsymbol{s}}) = \gamma_f(L,h_L); \tag{1.19}$$

that is, the expectation of the zeros of $T_f \widetilde{\boldsymbol{s}}$ for $\widetilde{\boldsymbol{s}}$ in the Gaussian ensemble $(H^0(X,L),\mathbb{P}_{\mathrm{st}})$ equals the Fubini–Study current (1.17).

1.4. High tensor powers of *L*: equidistribution on the support of *f*

To consider the semiclassical limit in the noncompact setting, we need to impose the same assumptions as in Subsection 1.2. For simplicity, in this subsection we only consider a nontrivial nonnegative smooth function f on X with compact support. Note that our results hold for a general class of nonnegative smooth functions f that are not required to have compact support (cf. Subsections 5.1 and 5.2).

Since f has compact support, condition (1.15) is satisfied for the line bundle L^p for each p. This way, we can construct a sequence of probability spaces $(H^0_{(2)}(X,L^p),\mathbb{P}_{f,p})$ using the corresponding Toeplitz operator $T_{f,p}$. We denote by $\boldsymbol{s}_{f,p}$ the identity map on the canonical probability space $(H^0_{(2)}(X,L^p),\mathbb{P}_{f,p})$.

In Theorems 5.7, 5.9 and 5.11, we prove the general version of the following results.

Theorem 1.8. *Let (X,J,Θ) be a connected complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L such that Condition 1.2 holds. Let U be a (relatively compact) open subset of X such that $f>0$ on $\overline{U}$. Then there exists $p_U\in\mathbb{N}$ such that for all $p\geq p_U$, $\gamma_f(L^p,h_p)$ is smooth on U, and we have the following expansion of smooth forms on U in any $\mathscr{C}^\ell$-norm for a given $\ell\in\mathbb{N}$ as $p\to\infty$,*

$$\gamma_f(L^p,h_p)=pc_1(L,h_L)+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\frac{c_1(L,h_L)^n}{\Theta^n}f^2\right)+\mathcal{O}\left(p^{-1}\right). \tag{1.20}$$

We have the convergence of smooth forms in any $\mathscr{C}^\ell$-norm (induced by g^{TX} and the associated Levi–Civita connection) on U given by

$$\frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U]=\frac{1}{p}\gamma_f(L^p,h_p)|_U\to c_1(L,h_L)|_U\,,\ p\to\infty. \tag{1.21}$$

Moreover, for any $\varphi\in\Omega_0^{n-1,n-1}(U)$, we have

$$\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}\langle[\mathrm{Div}(\boldsymbol{S}_{f,p})],\varphi\rangle=\langle c_1(L,h_L),\varphi\rangle\right)=1. \tag{1.22}$$

If Θ is a Kähler form and $\int_X c_1(L,h_L)\wedge\Theta^{n-1}<\infty$, or if $\int_X c_1(L,h_L)^n<\infty$, then

$$\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U=c_1(L,h_L)|_U\right)=1, \tag{1.23}$$

where the limit is taken with respect to the weak convergence of $(1,1)$-currents on U.

If we allow f to vanish at some points in $\operatorname{supp} f$, we observe that the smallest vanishing order of f at a zero is two since $f\geq 0$. In this situation, we have the following result.

Theorem 1.9. *We assume the same conditions for (X,J,Θ) and (L,h_L) as in Theorem 1.8 and in addition we assume that the prequantum condition $\Theta=c_1(L,h_L)$ holds. Fix a nontrivial nonnegative smooth function f on X as above. Let U be an open subset of $\operatorname{supp} f$ such that f only vanishes up to second order in U with nonzero Δf at the zeros. Then the convergence (1.21) holds in the sense of $(1,1)$-currents on U, and the conclusions (1.22), (1.23) of Theorem 1.8 hold.*

A general version of the above theorem is provided in Theorem 5.14. One important ingredient in the proofs of the above results is the following identity from Theorem 1.6,

$$\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{p,f})]]-pc_1(L,h_L)=\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(T^2_{f,p}(x,x)). \tag{1.24}$$

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We consider next the *fluctuations of zeros* on a small geodesic ball $B(x,R/\sqrt{p})$ centered at x, via pairing with a test form $\varphi \in \Omega_0^{n-1,n-1}(X)$. In Subsection 5.3 and under the prequantum condition, our computations (especially by Theorem 5.22 for the case $f(x) = 0$, $\Delta f(x) < 0$) show that

$$\left\langle \mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]] - pc_1(L,h_L), \chi_{B\left(x,\frac{R}{\sqrt{p}}\right)}\varphi \right\rangle = \begin{cases} \mathcal{O}(p^{-n}), & \text{if } f(x) > 0, \\ \mathcal{O}(p^{-n+1}), & \text{if } f(x) = 0, \Delta f(x) < 0, \end{cases} \tag{1.25}$$

where $\chi_{B(x,R/\sqrt{p})}$ is the indicator function of the set $B(x,R/\sqrt{p})$, and the first case $f(x) > 0$ follows from an analog of [1, Theorem 1.7] (cf. (1.20)). The coefficients of p^{-n} and of p^{-n+1} in the above estimates can be worked out explicitly. The different powers in Equation (1.25) show that, in the Planck scale, zeros of random sections have higher fluctuations near a zero of f of order 2 than near points where f does not vanish.

At last, in Subsection 5.4, we consider a not necessarily nonnegative real smooth function f satisfying condition (1.15) for L^p, $p \gg 0$. In this case, $T_{f,p}$ might not be injective and with suitable conditions on the vanishing points of f, we can still extend Theorem 1.9 to this case.

The next four sections of this paper correspond exactly to the above four subsections describing the main results: The first two sections deal with Gaussian random holomorphic sections, and the last two sections deal with random $\mathcal{L}^2$-holomorphic sections using the Toeplitz operators.

2. Gaussian random holomorphic sections

In this section, we define Gaussian ensembles of holomorphic sections of L and study their zeros as a (1,1)-currents on X. While some results proved in this section are not new in the special case of random functions or power series, we were not able to locate these results for holomorphic sections with $d = \infty$ in the literature.

2.1. Holomorphic line bundles and Bergman kernels

Let (X,J,Θ) be a connected complex Hermitian manifold (without boundary) of complex dimension $n \geq 1$. To Θ one can associate a J-invariant smooth Riemannian metric $g^{TX}(\cdot,\cdot) = \Theta(\cdot,J\cdot)$. Let L be a holomorphic line bundle over X, and let h_L be a smooth Hermitian metric on L. We denote the corresponding Chern curvature form of L by R^L, and the first Chern form of (L,h_L) is denoted by

$$c_1(L,h_L) = \frac{\sqrt{-1}}{2\pi} R^L. \tag{2.1}$$

Let $\mathscr{C}_0^\infty(X,L)$ denote the space of compactly supported smooth sections of L on X. Associated with the metrics g^{TX} and h_L, we define the $\mathcal{L}^2$-inner product as follows, for $s_1,s_2 \in \mathscr{C}_0^\infty(X,L)$,

$$\langle s_1,s_2\rangle_{\mathcal{L}^2(X,L)} := \int_X \langle s_1(x),s_2(x)\rangle_{h_L} \mathrm{dV}(x), \tag{2.2}$$

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where $\mathrm{dV} = \frac{1}{n!}\Theta^n$ is the volume form induced by Θ. We also let $\mathcal{L}^2(X,L)$ be the separable Hilbert space obtained by completing $\mathscr{C}_0^\infty(X,L)$ with respect to the norm $\|\cdot\|_{\mathcal{L}^2(X,L)}$ induced by Equation (2.2). Let $H^0(X,L)$ denote the vector space of holomorphic sections of L over X. Set

$$H^0_{(2)}(X,L) := \mathcal{L}^2(X,L) \cap H^0(X,L). \tag{2.3}$$

It follows from the Cauchy estimates for holomorphic functions that for every compact set $K \subset X$ there exists $C_K > 0$ such that

$$\sup_{x\in K} |s(x)|_{h_L} \leq C_K \|s\|_{\mathcal{L}^2(X,L)} \quad \text{for } s \in H^0_{(2)}(X,L), \tag{2.4}$$

which in turn implies that $H^0_{(2)}(X,L)$ is a closed subspace of $\mathcal{L}^2(X,L)$. Moreover, $H^0_{(2)}(X,L)$ is a separable Hilbert space with induced $\mathcal{L}^2$-metric (cf. [78, p. 60]).

The evaluation functional $H^0_{(2)}(X,L) \ni S \mapsto S(x)$ is continuous by estimate (2.4), so by Riesz representation theorem for each $x \in X$ there exists $P(x,\cdot) \in \mathcal{L}^2(X, L_x \otimes L^*)$ such that

$$s(x) = \int_X P(x,y)s(y)\,\mathrm{dV}(y), \quad \text{for all } s \in H^0_{(2)}(X,L).$$

Set

$$d = \dim H^0_{(2)}(X,L) \in \mathbb{N} \cup \{\infty\}. \tag{2.5}$$

If X is compact, then $d < \infty$. If $d \geq 1$, consider an orthonormal basis $\{S_j\}_{j=1}^d$ of $H^0_{(2)}(X,L)$. Then the series $\sum_{j=1}^d S_j(x) \otimes (S_j(y))^*$ converges uniformly on every compact together with all its derivatives (cf. [2, Proposition 2.4], [56, Remark 1.4.3], [78, p. 63]). In particular, $P(x,y)$ is smooth on $X \times X$. It follows that

$$P(x,y) = \sum_{j=1}^d S_j(x) \otimes (S_j(y))^*. \tag{2.6}$$

We obtain thus for the Bergman projection (1.3),

$$(Ps)(x) = \int_X P(x,y)s(y)\,\mathrm{dV}(y),$$

that is, $P(x,y)$ is the integral kernel of the Bergman projection. Recall that the line bundle $L \boxtimes L^*$ on $X \times X$ has fibres $(L \boxtimes L^*)_{(x,y)} := L_x \otimes L_y^*$ for $(x,y) \in X \times X$. The section $P(\cdot,\cdot)$ of $L \boxtimes L^* \to X \times X$ is called *Bergman kernel*.

The canonical identification $L_x \otimes L_x^* = \mathrm{End}(L_x) = \mathbb{C}$, $s \otimes s^* \mapsto s^*(s) = |s|^2_{h_L}$ allows to identify $P(x,x)$ as the smooth function

$$P(x,x) = \sum_{j=1}^d |S_j(x)|^2_{h_L}, \tag{2.7}$$

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called the *Bergman kernel function*. We deduce that $d = \int_X P(x,x)\,\mathrm{dV}(x) \in \mathbb{N} \cup \{\infty\}$. Hence, the Bergman kernel function is the dimensional density of $H^0_{(2)}(X,L)$. If $d = 0$, then the above considerations are trivially true.

2.2. Gaussian random holomorphic sections

The results proved in this subsection are extensions of the well-known results for random power series or random analytic functions on $\mathbb{C}^n$ (cf. [46] or [29, Section 3]) to the complex geometric setting. We include details of the proofs for the sake of completeness.

Let $\eta = \{\eta_j\}_{j\in\mathbb{N}}$ be a sequence of i.i.d. centered real or complex Gaussian random variables, and denote by $\mathbb{P}$ and $\mathbb{E}$ the underlying probability measure and its expectation.

For $d \geq 1$, let $S = \{S_j\}_{j=1}^d$ be an orthonormal basis of $H^0_{(2)}(X,L)$. Define

$$\psi^S_\eta(x) = \sum_{j=1}^{d} \eta_j S_j(x). \tag{2.8}$$

If $d = 0$, we simply set $\psi^S_\eta \equiv 0$.

Proposition 2.1. *The section ψ^S_η is almost surely a holomorphic section of L on X.*

Proof. If d is finite, the claim is clearly true. Hence, it remains to prove it for the case $d = \infty$. In this case, X is noncompact. Let $\{K_i\}_{i\in\mathbb{N}}$ be an increasing sequence of compact subsets of X such that $X = \cup_{i\in\mathbb{N}} K_i$. We can take each K_i to be the closure of a relatively compact open subset U_i of X. Then to prove this proposition, we only need to show that for each i, ψ^S_η is almost surely a holomorphic section of L on U_i.

Let K be a compact subset of X, and let U be an open relatively compact neighborhood of K. Similarly to estimate (2.4), there exists a constant $C_U > 0$ such that for $s \in H^0_{(2)}(X,L)$,

$$\sup_{x\in K} |s(x)|_{h_L} \leq C_U \|s\|_{\mathcal{L}^2(\overline{U},L)}. \tag{2.9}$$

By Equation (2.7), we have

$$\sum_{j=1}^{d} \|S_j\|^2_{\mathcal{L}^2(\overline{U},L)} = \int_{x\in\overline{U}} P(x,x)\mathrm{dV}(x) < +\infty. \tag{2.10}$$

For $j \in \mathbb{N}_{>0}$, $x \in X$ we consider the L_x-valued random variable $X_j(x) = \eta_j S_j(x)$. Since η_j is centered, we infer

$$\mathbb{E}[X_j(x)] = 0 \in L_x. \tag{2.11}$$

It is then consistent to define the variance as

$$\mathrm{Var}(X_j(x)) = \mathbb{E}[|X_j(x)|^2_{h_L}],$$

and we can compute

$$\mathrm{Var}(X_j(x)) = \mathbb{E}[|X_j(x)|^2_{h_L}] = \mathrm{Var}(\eta_j)|S_j(x)|^2_{h_L} = \mathrm{Var}(\eta_1)|S_j(x)|^2_{h_L}. \tag{2.12}$$

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We next prove that for any $k \in \mathbb{N}, N \in \mathbb{N}_{>0}$ and for $r > 0$, we have

$$\mathbb{P}\Big(\sup_{\ell=1,\ldots,N}\sup_{x\in K}\big|\sum_{j=1}^{\ell} X_{k+j}(x)\big|_{h_L} > r\Big) < \frac{C_U^2 \operatorname{Var}(\eta_1)}{r^2}\sum_{j=1}^{N}\|S_{k+j}\|^2_{\mathcal{L}^2(\overline{U},L)}. \tag{2.13}$$

For this purpose, define the stochastic process

$$Y_\ell = \big\|\sum_{j=1}^{\ell} X_{k+j}\big\|^2_{\mathcal{L}^2(\overline{U},L)}, \quad \ell = 1,\ldots,N, \tag{2.14}$$

and observe that by virtue of Equation (2.9), we have

$$\sup_{x\in K}\big|\sum_{j=1}^{\ell} X_{k+j}(x)\big|_{h_L} \le C_U Y_\ell^{\frac{1}{2}}. \tag{2.15}$$

As a consequence, we have

$$\mathbb{P}\Big(\sup_{\ell=1,\ldots,N}\sup_{x\in K}\big|\sum_{j=1}^{\ell} X_{k+j}(x)\big|_{h_L} > r\Big) \le \mathbb{P}\Big(\sup_{\ell=1,\ldots,N} C_U^2 Y_\ell > r^2\Big). \tag{2.16}$$

Now, the process (Y_ℓ), $\ell = 1,\ldots,N$, is a submartingale with respect to the filtration $(\mathcal{F}_\ell)$, where $\mathcal{F}_\ell = \sigma\big(\langle X_{k+i}, X_{k+j}\rangle_{\mathcal{L}^2(\overline{U},L)},\ i,j = 1,\ldots,\ell\big)$. Therefore, Doob's submartingale inequality (see, e.g., [47, Lemma 11.1]) yields

$$\mathbb{P}\Big(\sup_{\ell=1,\ldots,N} Y_\ell > \frac{r^2}{C_U^2}\Big) \le C_U^2\frac{\mathbb{E}[Y_N]}{r^2}, \tag{2.17}$$

which immediately entails Equation (2.13). Now, letting $N \to +\infty$ in Equation (2.13), we get

$$\mathbb{P}\Big(\sup_{\ell\in\mathbb{N}_{>0}}\sup_{x\in K}\big|\sum_{j=1}^{\ell} X_{k+j}(x)\big|_{h_L} > r\Big) \le \frac{C_U^2 \operatorname{Var}(\eta_1)}{r^2}\sum_{j=1}^{+\infty}\|S_{k+j}\|^2_{\mathcal{L}^2(\overline{U},L)}. \tag{2.18}$$

Then taking the limit of Equation (2.18) as $k \to \infty$, and using Equation (2.10), we infer

$$\mathbb{P}\Big(\limsup_{k\to+\infty}\sup_{\ell\in\mathbb{N}_{>0}}\sup_{x\in K}\big|\sum_{j=1}^{\ell} X_{k+j}(x)\big|_{h_L} > r\Big) = 0. \tag{2.19}$$

Therefore, a union bound along the sequence of $r = \frac{1}{N}$ immediately supplies us with

$$\mathbb{P}\Big(\limsup_{k\to+\infty}\sup_{\ell\in\mathbb{N}_{>0}}\sup_{x\in K}\big|\sum_{j=1}^{\ell} X_{k+j}(x)\big|_{h_L} > 0\Big) = 0. \tag{2.20}$$

If we take V to be a relatively compact open subset of X, and take $K = \overline{V}$, then by Equation (2.20), the sum $\sum_{j=1}^{\infty} X_j$ is almost surely uniformly convergent on K so that it almost surely defines a holomorphic section on V. This completes the proof of our proposition. □

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For the purpose of the following definition, we note at this point that a standard complex Gaussian is a random variable having the distribution $\frac{1}{\sqrt{2}}(X+\sqrt{-1}Y)$, where X and Y are standard real Gaussian variables.

Definition 2.2. The random section ψ_η^S defined by the equality (2.8) is called a standard Gaussian random holomorphic section of L over X if $\eta=\{\eta_j\}_{j\in\mathbb{N}}$ is a sequence of i.i.d. standard complex Gaussian random variables.

Now, we prove that the distribution of a standard Gaussian random holomorphic section ψ_η^S does not depend on the choice of the orthonormal basis.

Proposition 2.3. *Assume that $d\geq 1$, and assume that $\eta=\{\eta_j\}_{j=1}^d$ is a sequence of i.i.d. standard complex Gaussian random variables. If $S'=\{S'_j\}_{j=1}^d$ is another choice of orthonormal basis of $H^0_{(2)}(X,L)$, then $\psi_\eta^{S'}$ and ψ_η^S have the same distribution as random holomorphic sections.*

Proof. It is sufficient to find a sequence $\eta'=\{\eta'_j\}_{j=1}^d$ of i.i.d. standard complex Gaussian random variables such that a.s. $\psi_\eta^{S'}=\psi_{\eta'}^S$. Let $\ell^2(\mathbb{C})$ denote the Hilbert space of ℓ^2-summable complex sequences. If $u=(u_j)_{j\in\mathbb{N}}\in\ell^2(\mathbb{C})$, set

$$(\eta,u)_{\ell^2}=\sum_{j\in\mathbb{N}}\eta_j\bar{u}_j. \tag{2.21}$$

By Kolmogorov's three-series theorem (cf. [77]), the series in definition (2.21) is almost surely convergent so that $(\eta,u)_{\ell^2}$ is a well-defined random variable. By the property of Gaussian random variables, we conclude that $(\eta,u)_{\ell^2}$ is a centered complex Gaussian random variable with variance $|u|^2_{\ell^2}$. In particular, if $|u|_{\ell^2}=1$, then $(\eta,u)_{\ell^2}$ has the same distribution as η_1. Moreover, if nonzero $u,v\in\ell^2$ is such that $(u,v)_{\ell^2}=0$, then $(\eta,u)_{\ell^2}$ and $(\eta,v)_{\ell^2}$ are independent. Take $(a_{ij}\in\mathbb{C})_{i,j\in\mathbb{N}}$ such that for each i,

$$S'_i=\sum_{j\in\mathbb{N}}a_{ij}S_j. \tag{2.22}$$

For $j\in\mathbb{N}$, set $b_j=(\bar{a}_{ij})_{i\in\mathbb{N}}$. Then $b_j\in\ell^2(\mathbb{C})$ is with norm 1, moreover, if $j\neq j'$, then $(b_j,b_{j'})_{\ell^2}=0$. Now, define

$$\eta'_j=(\eta,b_j)_{\ell^2}. \tag{2.23}$$

Then $\eta'=(\eta'_j)_{j\in\mathbb{N}}$ is a sequence of i.i.d. centered Gaussian random variables with the same distribution as η. By definition, we get that almost surely, $\psi_\eta^{S'}=\psi_{\eta'}^S$. Therefore, $\psi_\eta^{S'}$ and ψ_η^S have the same distribution. □

Remark 2.4. (a) When $d=\infty$, note that by taking a sequence of compact subset $\{K_i\}_{i\in\mathbb{N}}$ as in the proof of Proposition 2.1, we can define a sequence of seminorms for $H^0(X,L)$, hence a Fréchet distance so that $H^0(X,L)$ is a Fréchet space. In Proposition 2.1, we actually prove that ψ_η^S is a random variable taking values in the Fréchet space $H^0(X,L)$.

(b) When $d\geq 1$, we have

$$\mathbb{P}(\psi_\eta^S\equiv 0)=0. \tag{2.24}$$

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Indeed, since $\{S_j\}_{j=1}^d$ is a basis of $H^0_{(2)}(X,L)$, we have

$$\mathbb{P}(\psi_\eta^S \equiv 0) = \mathbb{P}(\eta_j = 0, j = 1, \cdots) = 0. \tag{2.25}$$

(c) In the proof of Proposition 2.1, we do not use the Gaussianity of the η_j in an essential way. Hence, we can work with any sequence η of pairwise uncorrelated centered random variables with uniformly bounded variance. In that case, however, the distribution of the random section ψ_η^S might depend on the choice of the basis S. Generally, one needs suitable moment conditions on η to obtain more results, such as the universality of the zeros of ψ_η^S, and we refer to [6, 27, 44] for the related details.

Lemma 2.5. *If $d = \infty$, then with probability one, ψ_η^S is not $\mathcal{L}^2$-integrable on X.*

Proof. The event that ψ_η^S is $\mathcal{L}^2$-integrable is equivalent to the event $\{\sum_{j=1}^\infty |\eta_j|^2 < \infty\}$. But, for example, by the law of large numbers, we infer

$$\mathbb{P}\Big(\sum_{j=1}^{\infty} |\eta_j|^2 < \infty\Big) = 0, \tag{2.26}$$

and the statement of the lemma follows. □

2.3. Currents and the Lelong–Poincaré formula

The zero-set of a holomorphic section is a complex analytic set which is in general singular. The analytic tool used to deal with singularities in complex geometry is the theory of currents, introduced by de Rham [18] (see [22, 35] and especially [31] for complete expositions).

Let X be a complex manifold of dimension n, and let E be a complex vector bundle on X. The space of smooth sections of E is denoted by $\mathscr{C}^\infty(X,E)$ and is endowed with the $\mathscr{C}^\infty$-topology of uniform convergence of all derivatives on compact sets. The space of smooth sections of E with compact support is denoted by $\mathscr{C}^\infty_0(X,E)$ and is endowed with the topology of inductive limit of spaces of smooth sections with support on a given compact set. In particular, we denote by $\Omega_0^{n-1,n-1}(X)$ the space of of smooth $(n-1,n-1)$-forms with compact support.

The space of (1,1)-currents on X is the topological dual of the space $\Omega_0^{n-1,n-1}(X)$ (called test forms in this context). In the sequel, we let $\langle T,\varphi\rangle$ be the pairing between a (1,1)-current T and a test form $\varphi \in \Omega_0^{n-1,n-1}(X)$. A (1,1)-current is called of order $k \in \mathbb{N}_0$ if it is continuous in the $\mathscr{C}^k$-topology; equivalently, it extends as a linear continuous functional to the space of $(n-1,n-1)$-forms of class $\mathscr{C}^k$ with compact support.

For any analytic hypersurface $V \subset X$, we define the current of integration $[V]$ on V by

$$\varphi \mapsto \int_V \varphi := \int_{V_{\mathrm{reg}}} \varphi, \quad \varphi \in \Omega_0^{n-1,n-1}(X),$$

where V_{reg} is the regular set of V (a complex submanifold of codimension 1). By a theorem of Lelong ([35, p. 32] [22, III-2.7]) the current of integration on V is a closed positive (1,1)-current. It is clear that $[V]$ is a current of order 0 on X.

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Let L be a holomorphic line bundle on X. For a holomorphic section $s \in H^0(X,L)\setminus\{0\}$, the divisor $\mathrm{Div}(s)$ of s is defined as the formal sum

$$\mathrm{Div}(s) = \sum_{V\subset Z(s)} \mathrm{ord}_V(s)V, \tag{2.27}$$

where V runs over all the irreducible analytic hypersurfaces contained in $Z(s)$, and $\mathrm{ord}_V(s)$ denotes the vanishing order of s along V. Let $Z(s)$ denote the set of zeros of s, which is a purely 1-codimensional analytic subset of X. The current of integration (with multiplicities) on the divisor $\mathrm{Div}(s)$ is defined by

$$[\mathrm{Div}(s)] = \sum_{V\subset Z(s)} \mathrm{ord}_V(s)[V], \quad \big\langle [\mathrm{Div}(s)],\varphi\big\rangle = \int_{\mathrm{Div}(s)} \varphi := \sum_{V\subset Z(s)} \mathrm{ord}_V(s)\int_V \varphi. \tag{2.28}$$

Assume that L is endowed with a smooth Hermitian metric h_L. By the Lelong–Poincaré formula [56, Theorem 2.3.3], we have

$$[\mathrm{Div}(s)] = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|s|^2_{h_L} + c_1(L,h_L), \quad \text{for } s\in H^0(X,L). \tag{2.29}$$

This important formula is crucial for our purposes. It links the zero-divisor to the curvature and to the logarithm of the pointwise norm of a section, which is analytically easier to tackle and allows the introduction of the Bergman kernel into the picture.

2.4. Expectation of random zeros: proof of Theorem 1.1

In the sequel, we always assume $d = \dim H^0_{(2)}(X,L) \geq 1$. We start with some considerations about the Fubini–Study currents.

Lemma 2.6. *Assume that $d\geq 1$. Then the function $X \ni x \mapsto \log P(x,x) \in \{-\infty\}\cup\mathbb{R}$ is locally $\mathcal{L}^1$-integrable on X with respect to dV. Thus, $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log P(x,x)$ defines a (1,1)-current on X.*

Proof. Let $e_L : U \to L$ be a local holomorphic frame of L. Let $\varphi \in \mathscr{C}^\infty(U)$ be the local weight of h_L with respect to e_L, that is, $|e_L|^2_{h_L} = e^{-2\varphi}$ on U. We consider an orthonormal basis $\{S_j\}_{j=1}^d$ of $H^0_{(2)}(X,L)$ and write

$$S_j(x) = f_j(x)e_L(x),\ x\in U,\ f_j \in \mathscr{O}(U), \tag{2.30}$$

where f_j are nontrivial holomorphic functions on U. Then $P(x,x) = \sum_{j=1}^d |S_j|^2_{h_L} = \sum_{j=1}^d |f_j|^2 e^{-2\varphi}$ on U, hence

$$\log P(x,x) = \log\Big(\sum_{j=1}^d |f_j|^2\Big) - 2\varphi. \tag{2.31}$$

The series $\sum_{j=1}^d |f_j|^2$ converges locally uniformly on U, thus $\log\big(\sum_{j=1}^d |f_j|^2\big)$ is a plurisubharmonic function that is not identically $-\infty$, hence locally integrable. □

Lemma 2.6 shows that the Fubini–Study currents (1.4) are well defined. Note that $c_1(L,h_L)|_U = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi$. By applying $\partial\overline{\partial}$ to both sides of equality (2.31) and taking into account formula (1.4), we see that

$$\gamma(L,h_L)\big|_U = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\Big(\sum_{j=1}^{d}|f_j|^2\Big), \tag{2.32}$$

thus $\gamma(L,h_L)$ is a closed positive (1,1)-current. The base locus of $H^0_{(2)}(X,L)$ is the proper analytic set

$$\mathrm{Bl}(X,L) := \big\{x\in X \mid s(x)=0 \text{ for all } s\in H^0_{(2)}(X,L)\big\}. \tag{2.33}$$

Thus, $\{x\in X : P(x,x)=0\} = \mathrm{Bl}(X,L)$. Hence, $\gamma(L,h_L)$ is a smooth form if $\mathrm{Bl}(X,L)=\varnothing$.

Remark 2.7. The Fubini–Study currents have the following geometric interpretation that justifies their name. Note that if X is compact and $\mathrm{Bl}(X,L)=\varnothing$, then $\gamma(L,h_L)$ is the pullback of the Fubini–Study form on the projective space by the Kodaira map defined by $H^0(X,L)$ (see, e.g., [56, (5.1.21)]). If X is noncompact and $H^0_{(2)}(X,L)$ is infinite dimensional, one can still define a Kodaira map to the infinite-dimensional projective space $\mathbb{CP}^\infty$, cf. [48]. This is a Hilbert manifold $\mathbb{CP}^\infty = \ell^2(\mathbb{C})\setminus\{0\}/\mathbb{C}^*$ modeled on the space $\ell^2(\mathbb{C})$. The Fubini–Study metric on $\mathbb{CP}^\infty$ is defined by $\omega_{\mathrm{FS},\infty} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|a\|^2$. Assume that $H^0_{(2)}(X,L)$ has no base locus, and consider an orthonormal basis $\{S_j\}_{j=1}^\infty$ of $H^0_{(2)}(X,L)$. With the representation (2.30), we define the Kodaira map $\Phi : X\to\mathbb{CP}^\infty$ by $\Phi(x) = [(f_j(x))_j]$ for $x\in U$. Then $\Phi^*\omega_{\mathrm{FS},\infty} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\sum_j|f_j|^2)$ coincides with formula (2.32).

Now, we are ready to prove Theorem 1.1. Let $\eta=\{\eta_j\}_{j=1}^d$ is a sequence of i.i.d. standard complex Gaussian random variables. Let ψ^S_η be the random holomorphic section defined in formula (2.8), and let $[\mathrm{Div}(\psi^S_\eta)]$ denote the (1,1)-current given by its zeros (cf. formula (2.27)).

Proof of Theorem 1.1. We fix a test form $\varphi\in\Omega_0^{n-1,n-1}(X)$ and evaluate $\mathbb{E}[\langle[\mathrm{Div}(\psi^S_\eta)],\varphi\rangle]$. By applying the Lelong–Poincaré formula (2.29) to ψ^S_η, we get

$$\begin{aligned}\langle[\mathrm{Div}(\psi^S_\eta)],\varphi\rangle &= \int_X\Big(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|\psi^S_\eta|^2_{h_L} + c_1(L,h_L)\Big)\wedge\varphi\\ &= \int_X c_1(L,h_L)\wedge\varphi + \frac{\sqrt{-1}}{2\pi}\int_X\log|\psi^S_\eta|^2_{h_L}\partial\bar{\partial}\varphi\\ &= \int_X c_1(L,h_L)\wedge\varphi + \frac{\sqrt{-1}}{2\pi}\int_{X\setminus\mathrm{Bl}(X,L)}\log|\psi^S_\eta|^2_{h_L}\partial\bar{\partial}\varphi.\end{aligned} \tag{2.34}$$

The last identity follows from that $\log|\psi^S_\eta|^2_{h_L}$ is almost surely locally integrable by Proposition 2.1 and Remark 2.4 (b), and that $\mathrm{Bl}(X,L)$ has Lebesgue measure zero in X. For $x\in X\setminus\mathrm{Bl}(X,L)$, we have $P(x,x)\neq 0$, and let $e_L(x)$ be a unit vector of L at x, define

$$b(x) = \big(P(x,x)^{-1/2}S_j(x)/e_L(x)\big)_{j\in\mathbb{N}}\in\ell^2(\mathbb{C}). \tag{2.35}$$

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We have $|b(x)|_{\ell^2} = 1$. Note that

$$P(x,x)^{-1/2}\psi_\eta^S = (\eta,\overline{b(x)})_{\ell^2}\, e_L(x). \tag{2.36}$$

Then

$$\mathbb{E}[\log|P(x,x)^{-1/2}\psi_\eta^S|^2_{h_L}] = \mathbb{E}[\log\big|(\eta,\overline{b(x)})_{\ell^2}\big|^2] = \mathbb{E}[\log|\eta_1|^2]. \tag{2.37}$$

Note that $\mathbb{E}[|\log|\eta_1|^2|] < \infty$. By Lemma 2.6, $\log P(x,x)$ is locally integrable on X, then we can apply the Fubini's theorem to the following integrals so that

$$\begin{aligned}\mathbb{E}\Big[\int_{X\setminus \mathrm{Bl}(X,L)} \log|P(x,x)^{-1/2}\psi_\eta^S|^2_{h_L}\partial\bar{\partial}\varphi\Big] &= \int_{X\setminus \mathrm{Bl}(X,L)} \mathbb{E}\big[\log|P(x,x)^{-1/2}\psi_\eta^S|^2_{h_L}\big]\partial\bar{\partial}\varphi \\ &= \mathbb{E}[\log|\eta_1|^2]\int_X \partial\bar{\partial}\varphi = 0.\end{aligned} \tag{2.38}$$

Then by formula (2.34), we get

$$\begin{aligned}\mathbb{E}\left[\langle[\mathrm{Div}(\psi_\eta^S)],\varphi\rangle\right] &= \int_X c_1(L,h_L)\wedge\varphi + \frac{\sqrt{-1}}{2\pi}\int_{X\setminus \mathrm{Bl}(X,L)} \log P(x,x)\cdot\partial\bar{\partial}\varphi \\ &= \big\langle c_1(L,h_L) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(P(x,x)),\varphi\big\rangle.\end{aligned} \tag{2.39}$$

This completes the proof. □

2.5. Geometric examples

We present in this subsection some interesting examples of Bergman spaces and Fubini–Study currents where our results apply. We start with some simple observations.

(i) If $P(x,x) > 0$ (equivalently, x is not in the base locus of $H^0_{(2)}(X,L)$), then the (1,1)-form $\sqrt{-1}\partial\bar{\partial}\log P(x,x)$ is smooth in a neighborhood of x, and hence $\gamma(L,h_L)$, too. In particular, if $\mathrm{Bl}(X,L) = \varnothing$, then $\gamma(L,h_L)$ is smooth.

(ii) If $P(x,x) > 0$, let $s_0 \in H^0_{(2)}(X,L)$ with $s_0(x) \neq 0$. Assume that there exist $s_1,\ldots,s_n \in H^0_{(0)}(X,L)$ such that $d(s_1/s_0)(x),\ldots,d(s_n/s_0)(x)$ are linearly independent (that is, sections of $H^0_{(2)}(X,L)$ give local coordinates at x). Then $\gamma(L,h_L)$ is strictly positive near x.

(iii) Thus, if $\mathrm{Bl}(X,L) = \varnothing$ and sections of $H^0_{(2)}(X,L)$ give local coordinates at any point in X, then $\gamma(L,h_L)$ defines a Kähler metric on X.

Example 2.8 (Bergman metric). We consider the case when L is the canonical bundle K_X of X (cf. [48, 78]). The space of holomorphic sections of K_X is the space $H^{n,0}(X)$ of holomorphic $(n,0)$-forms. Such a form can be written in local coordinates $(z_1,\ldots,z_n)$ as $f(z)dz_1\wedge\ldots\wedge dz_n$, with f a holomorphic function. We say that a measurable $(n,0)$-form β is an $\mathcal{L}^2$ section of K_X if

$$\|\beta\|^2 := 2^{-n}(\sqrt{-1})^{n^2}\int_X \beta\wedge\overline{\beta} < \infty. \tag{2.40}$$

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We denote by $H^{n,0}_{(2)}(X)$ the space of $\mathcal{L}^2$ holomorphic $(n,0)$-forms. We have $H^{n,0}_{(2)}(X) = H^0_{(2)}(X,K_X)$, where the right-hand side is defined with respect to an arbitrary metric Θ on X and the metric on K_X is induced by Θ.

We assume that $H^{n,0}_{(2)}(X) \neq \{0\}$ and let $\{\beta_j\}_{j=1}^d$ be an orthonormal basis of $H^{n,0}_{(2)}(X)$. In local coordinates $(U;z_1,\dots,z_n)$ write $\beta_j = f_j(z)dz_1\wedge\dots\wedge dz_n$. According to (2.32) the Fubini–Study current is given on U by $\gamma(K_X,h_{K_X})|_U = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\sum_j |f_j|^2)$. If the Fubini–Study current is actually a Kähler metric on X, then it is called the *Bergman metric* of X. We will denote it by ω_B. The metric ω_B is invariant by the group of biholomorphic transformations of X.

If X is an open set in $\mathbb{C}^n$, the canonical bundle is trivial, so we identify the space $H^{n,0}_{(2)}(X)$ of $\mathcal{L}^2$-holomorphic $(n,0)$-forms with the space $H^0_{(2)}(X)$ holomorphic functions which are $\mathcal{L}^2$ with respect to the Lebesgue measure. There is a vast literature on Bergman spaces and kernels on domains in $\mathbb{C}^n$; see, for example, [38, 43] and the references therein.

To give concrete examples let us recall the definition of Stein manifolds, which are interesting due to their rich function-theoretical structure [34]. For a complex manifold X, let $\mathcal{O}(X)$ denote the space of all holomorphic functions on X.

Definition 2.9. A complex manifold X is called Stein if the following two conditions are satisfied: (1) X is homomorphically convex, that is, for every compact subset $K \subset X$, its holomorphically convex hull $\widehat{K} = \left\{z \in X : |f(z)| \le \sup_{w\in K}|f(w)|, \forall f \in \mathcal{O}(X)\right\}$ is compact. (2) X is holomorphically separable, that is, if $x \neq y$ in X, then there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$.

Let $L \to X$ be a holomorphic line bundle. The cohomology vanishing theorem for coherent analytic sheaves on Stein manifolds (Cartan's theorem B, cf. [34]) yields the following:

(i) The holomorphic sections $H^0(X,L)$ give local coordinates at each point of X.

(ii) For any closed discrete set $A = \{p_k : k \in \mathbb{N}\} \subset X$ and any family $\{v_k \in L_{p_k} : k \in \mathbb{N}\}$, there exists $s \in H^0(X,L)$ with $s(p_k) = v_k$ for all $k \in \mathbb{N}$. In particular, for each $p \in X$ the evaluation map $H^0(X,L) \to L_p$ is surjective and we have $\dim H^0(X,L) = \infty$.

Example 2.10. Let X be a Stein manifold and $D \Subset X$ be a relatively compact domain. We consider a Hermitian metric on X whose associated $(1,1)$-form is denoted by Θ. Let $\mathrm{dV}_\Theta = \Theta^n/n!$ be the volume form induced by Θ, where $\dim X = n$. Let (L,h_L) be a Hermitian holomorphic line bundle. Consider the space $\mathcal{L}^2(D,L,h_L,\mathrm{dV}_\Theta)$ of measurable sections S of L over D satisfying $\int_D |S|^2_{h_L}\mathrm{dV}_\Theta < \infty$, and let $H^0_{(2)}(D,L,h_L,\mathrm{dV}_\Theta) = \mathcal{L}^2(D,L,h_L,\mathrm{dV}_\Theta)\cap H^0(X,L)$. The restriction map $H^0(X,L) \to H^0_{(2)}(D,L,h_L,\mathrm{dV}_\Theta)$ is well defined and injective. We deduce that the space $H^0_{(2)}(D,L,h_L,\mathrm{dV}_\Theta)$ is infinite dimensional, has empty base locus and sections of this space give local coordinates at any point of D. Therefore, $\gamma(L,h_L)$ is smooth on X and if (L,h_L) is semipositive (i.e., $c_1(L,h_L)$ is positive semidefinite), it is a Kähler form.

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We deduce from Theorem 1.1 and the discussion from Example 2.8 the following.

Corollary 2.11. *For any relatively compact domain $D \Subset X$ in a Stein manifold, the expectation of the zero divisors of the standard Gaussian random holomorphic $(n,0)$-forms defined from the $\mathcal{L}^2$-holomorphic $(n,0)$-forms on D is given by the Bergman metric on D. If $D \Subset \mathbb{C}^n$ this is true for standard Gaussian random holomorphic functions defined from the $\mathcal{L}^2$-holomorphic functions on D.*

One of the simplest examples is the unit disc $\mathbb{D} \subset \mathbb{C}$ endowed with the Lebesgue measure. Then $P(z,z) = \frac{1}{\pi(1-|z|^2)^2}$ and the Bergman metric

$$\omega_B = \frac{\sqrt{-1}}{\pi} \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} \tag{2.41}$$

is the hyperbolic metric (up to a constant factor) on the disc. We see on this example that the Bergman metric blows up as $|z| \to 1$, so the zeros accumulate towards the boundary of $\mathbb{D}$. This is a more general phenomenon. Assume that $D \Subset \mathbb{C}^n$ is a strictly pseudoconvex domain with smooth boundary. Then the Bergman metric blows up at the boundary in the nontangential directions. More precisely, let ϱ be a defining function of D so that $D = \{\varrho < 0\}$, $d\varrho \neq 0$ on ∂D and the Levi form of ϱ is positive definite on the complex tangent space of ∂D. For $v \in \mathbb{C}^n$ and $x \in D$, we denote by $|v|_{D,x}$ the norm of the vector $\sum_{j=1}^n v_j \frac{\partial}{\partial z_j}(x) \in T_x^{1,0}\mathbb{C}^n$ with respect to the Bergman metric. Then for any $x_0 \in \partial D$, there exists a neighborhood U of x_0 and $C > 0$ such that for any $x \in D \cap U$ and $v \in \mathbb{C}^n$ we have $|v|_{D,x} \geq C|\partial\varrho(x)\cdot v|/|\varrho(x)|$; see [23, 32]. There is a large amount of works about the boundary behavior of the Bergman metric. In addition to the accurate behavior of strictly pseudoconvex domains, there is another feature which indicates that zeros of random L^2 holomorphic sections tend to accumulate up to the boundary. This is the completeness of the Bergman metric. It is known for example that the Bergman metric is complete if D is a bounded pseudoconvex domain in $\mathbb{C}^n$ with a $\mathscr{C}^1$ boundary [66] or if D is a bounded hyperconvex domain in $\mathbb{C}^n$ (cf. [9, 39]).

Example 2.12 (Singular Hermitian metrics). As in the case of a smooth Hermitian metric, a singular Hermitian metric on L is defined in terms of an open cover $X = \bigcup U_\alpha$ for which there exist local holomorphic frames $e_\alpha : U_\alpha \to L$. Consider the transition functions $g_{\alpha\beta} = e_\beta/e_\alpha \in \mathcal{O}_X^\star(U_\alpha \cap U_\beta)$. If h_L is a singular Hermitian metric on L, then (see [21], also [56, p. 97]) $|e_\alpha|_{h_L}^2 = e^{-2\varphi_\alpha}$, where the functions $\varphi_\alpha \in L^1_{\mathrm{loc}}(U_\alpha)$ are called the local weights of the metric h. One has $\varphi_\alpha = \varphi_\beta + \log|g_{\alpha\beta}|$ on $U_\alpha \cap U_\beta$, and the curvature of h_L,

$$c_1(L,h_L)|_{U_\alpha} = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\varphi_\alpha,$$

is a well-defined closed (1,1) current on X. The Fubini–Study current $\gamma(L,h_L)$ is defined in this case by formula (2.32). We say that the metric h_L is (semi)positively curved if $c_1(L,h)$ is a positive current. Equivalently, the weights φ_α can be chosen to be plurisubharmonic (psh) functions. In this case, the Bergman kernel function can be defined by formula (2.7) (see [15]), $\log P(x,x)$ is still locally integrable [15, Lemma 3.2] and formula (1.4) holds. Adapting the proof of Theorem 1.1 to this context easily yields:

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Theorem 2.13. *Let (X,J,Θ) be a connected complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a singular Hermitian metric h_L so that $c_1(L,h_L)$ is a positive current. Assume that $H^0_{(2)}(X,L)\neq\{0\}$. Then as $(1,1)$-currents on X we have $\mathbb{E}[[\mathrm{Div}(\psi^S_\eta)]]=\gamma(L,h_L)$.*

Example 2.14 (Bargmann–Fock space: flat Gaussian holomorphic function). Let L be the trivial line bundle on $\mathbb{C}^n$, but we equip it with the Hermitian metric h_L such that $|1|^2_{h_L,z}=e^{-|z|^2}$, $z\in\mathbb{C}^n$. In this case, $R^L=\sum_{j=1}^n dz_j\wedge d\bar{z}_j$. We endow $\mathbb{C}^n$ with the flat metric $\Theta=\frac{\sqrt{-1}}{2\pi}\sum_{j=1}^n dz_j\wedge d\bar{z}_j$, so $\mathrm{dV}_\Theta=\pi^{-n}\Pi_{j=1}^n dx_j\wedge dy_j$. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, we write

$$S_\alpha(z)=\frac{z_1^{\alpha_1}\dots z_n^{\alpha_n}}{\sqrt{\alpha_1!\dots\alpha_n!}}. \tag{2.42}$$

A straightforward calculation then confirms that $\{S_\alpha\}_{\alpha\in\mathbb{N}^n}$ forms an orthonormal basis of $H^0_{(2)}(\mathbb{C}^n,L)$. In this case, we have

$$P(z,z)=\sum_{\alpha\in\mathbb{N}^n}\frac{|z_1|^{2\alpha_1}\dots|z_n|^{2\alpha_n}}{\alpha_1!\dots\alpha_n!}e^{-|z|^2}=1. \tag{2.43}$$

Denoting by $\eta=(\eta_\alpha)_{\alpha\in\mathbb{N}^n}$ a family of i.i.d. standard complex Gaussian random variables, we define the standard Gaussian random holomorphic function on $\mathbb{C}^n$ as

$$\psi^S_\eta=\sum_{\alpha\in\mathbb{N}^n}\eta_\alpha S_\alpha. \tag{2.44}$$

By Theorem 1.1, we have

$$\mathbb{E}[[\mathrm{Div}(\psi^S_\eta)]]=\gamma(L,h_L)=\frac{\sqrt{-1}}{2\pi}\sum_{j=1}^n dz_j\wedge d\bar{z}_j. \tag{2.45}$$

3. Equidistribution and large deviation for high tensor powers of line bundles

In the sequel, assume that $\eta=\{\eta_j\}_{j\in\mathbb{N}}$ is a sequence of i.i.d. standard complex Gaussian random variables, note that $\mathrm{Var}(\eta_1)=1$.

In this section, we consider the setting of Subsection 1.2: let (X,J,Θ) be a connected complex Hermitian manifold and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L. We assume Condition 1.2 holds. Let $\dot{R}^L\in\mathrm{End}(T^{(1,0)}X)$ such that $x\in X$, for $u,v\in T^{(1,0)}_xX$,

$$R^L_x(u,v)=g^{TX}_x(\dot{R}^Lu,v). \tag{3.1}$$

Set $a_0(x):=\det\dot{R}^L_x$ a smooth function on X, then by condition (1.6), $a_0(x)\geq\varepsilon^n$.

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3.1. Equidistribution of zeros of Gaussian random holomorphic sections

We consider the sequence of Hilbert spaces $H^0_{(2)}(X,L^p)$, $p\in\mathbb{N}$ large. Set

$$d_p = \dim H^0_{(2)}(X,L^p) \in \mathbb{N}\cup\{\infty\}. \tag{3.2}$$

We equip L^p with the induced Hermitian metric $h_p := h_L^{\otimes p}$. Let P_p denote the orthogonal projection from $\mathcal{L}^2(X,L^p)$ onto $H^0_{(2)}(X,L^p)$, and let $P_p(x,y)$ denote the corresponding Bergman kernel on X with respect to $\mathrm{dV}(x) = \frac{\Theta^n}{n!}$.

By Ma–Marinescu [56, Theorems 4.1.1 & 6.1.1], we have the following results on the asymptotics of on-diagonal Bergman kernels: There exist coefficients $\boldsymbol{b}_r \in \mathscr{C}^\infty(X,\mathbb{R})$, $r\in\mathbb{N}$, such that for any compact subset $K\subset X$, any $k,\ell\in\mathbb{N}$, there exists $C_{k,\ell,K}>0$ such that for $p\in\mathbb{N}^*$,

$$\left|\frac{1}{p^n}P_p(x,x) - \sum_{r=0}^{k}\boldsymbol{b}_r(x)p^{-r}\right|_{\mathscr{C}^\ell(K)} \le C_{k,\ell,K}\,p^{-k-1}, \tag{3.3}$$

where the $\mathscr{C}^\ell$-norm is induced by g^{TX}. In particular, we have

$$\boldsymbol{b}_0(x) = \det\left(\frac{\dot{R}^L_x}{2\pi}\right) = \frac{a_0(x)}{(2\pi)^n},$$

and $\boldsymbol{b}_1$ also has an explicit formula given as in [56, (4.1.9)],

$$\boldsymbol{b}_1(x) = \frac{1}{8\pi}\det\left(\frac{\dot{R}^L_x}{2\pi}\right)\Big(\boldsymbol{r}^X_\omega(x) - 2\Delta_\omega\big(\log(\det \dot{R}^L_x)\big)\Big), \tag{3.4}$$

where $\boldsymbol{r}^X_\omega$, Δ_ω denote the scalar curvature, and the Bochner Laplacian associated to the Riemannian metric $g^{TX}_\omega(\cdot,\cdot) := c_1(L,h_L)(\cdot,J\cdot)$ on X. An explicit formula for $\boldsymbol{b}_2$ can be found in [59, (0.14) & Remark 0.5].

For $p\in\mathbb{N}_{>0}$, let $\psi^{S_p}_\eta$ be a standard Gaussian random holomorphic section constructed from $H^0_{(2)}(X,L^p)$, that is, for $\{S^p_j\}_{j=1}^{d_p}$ an orthonormal basis of $H^0_{(2)}(X,L^p)$ with respect to the $\mathcal{L}^2$-metric, and set

$$\psi^{S_p}_\eta = \sum_{j=1}^{d_p}\eta_j S^p_j. \tag{3.5}$$

Theorem 3.1. *Let (X,J,Θ) be a complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L, such that Condition 1.2 holds. Then we have:*

(i) *There exist functions $\boldsymbol{c}_r\in\mathscr{C}^\infty(X,\mathbb{R})$, $r\in\mathbb{N}$, such that for any compact subset $K\subset X$ and $k,\ell\in\mathbb{N}$, there exists a $p_{k,\ell,K}\in\mathbb{N}$, such that for all $p\ge p_{k,\ell,K}$, $\mathbb{E}[[\mathrm{Div}(\psi^{S_p}_\eta)]]$ is a smooth real $(1,1)$-form, and the following expansion holds in the $\mathscr{C}^\ell(K)$-norm induced from g^{TX} and associated Levi–Civita connection,*

$$\frac{1}{p}\mathbb{E}[[\mathrm{Div}(\psi^{S_p}_\eta)]] = c_1(L,h_L) + \frac{\sqrt{-1}}{2\pi}\sum_{r=0}^{k}\frac{\partial\overline{\partial}\boldsymbol{c}_r(x)}{p^{r+1}} + \mathcal{O}_{K,\ell,k}(p^{-k-2}). \tag{3.6}$$

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Moreover, each function $\boldsymbol{c}_r(x)$ *for* $r \geq 1$ *is a polynomial in terms of the coefficients* $\boldsymbol{b}_s/\boldsymbol{b}_0$, $s \leq r$, *given in (3.3). We have*

$$\begin{aligned} \boldsymbol{c}_0(x) &= \log \boldsymbol{b}_0(x),\, \boldsymbol{c}_1(x) = \frac{\boldsymbol{b}_1(x)}{\boldsymbol{b}_0(x)} = \frac{1}{8\pi}\Big(\boldsymbol{r}^X_\omega - 2\Delta_\omega\big(\log(\det \dot{R}^L)\big)\Big), \\ \boldsymbol{c}_2(x) &= \frac{\boldsymbol{b}_2(x)}{\boldsymbol{b}_0(x)} - \frac{1}{2}\left(\frac{\boldsymbol{b}_1(x)}{\boldsymbol{b}_0(x)}\right)^2. \end{aligned} \tag{3.7}$$

(ii) *As* $p \to +\infty$, *we have the weak convergence*

$$\frac{1}{p}\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]] \to c_1(L,h_L) \tag{3.8}$$

of (1,1)*-currents, that is, for any* $\varphi \in \Omega_0^{n-1,n-1}(X)$, *as* $p \to +\infty$,

$$\Big\langle \frac{1}{p}\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]],\varphi \Big\rangle \to \langle c_1(L,h_L),\varphi\rangle. \tag{3.9}$$

Proof. It is clear that the second part (ii) of this theorem follows directly from the expansion (3.6). We only need to prove the first part (i). By Theorem 1.1 for L^p,

$$\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]] = \gamma(L^p,h_p), \tag{3.10}$$

where $\gamma(L^p,h_p)$ is the corresponding Fubini–Study current defined via formula (1.4). To obtain the expansion (3.6), it is enough to study the expansion of $\frac{\sqrt{-1}}{2\pi p}\partial\overline{\partial}\log P_p(x,x)$ in p.

For this purpose, we apply the expansion (3.3) to $P_p(x,x)$. Note that $\boldsymbol{b}_0$ is uniformly bounded from below by a positive constant. Then for a given compact subset K of X, and $k,\ell \in \mathbb{N}$, we take the expansion (3.3) but with $\mathscr{C}^{\ell+2}(K)$-norm. We conclude that, for all sufficiently large p, we have $P_p(x,x) > 0$, and

$$\Big|\sum_{r=1}^{k} \boldsymbol{b}_r(x)p^{-r} + C_{k,\ell+2,K}\,p^{-k-1}\Big| \leq \frac{1}{2}\boldsymbol{b}_0(x). \tag{3.11}$$

Then $\frac{\sqrt{-1}}{2\pi p}\partial\overline{\partial}\log P_p(x,x)$ is a smooth (1,1)-form on K, and the expansion (3.6) is the consequence of the following identity and the Taylor expansion of $\log(1+x)$ with $|x|<1$,

$$\begin{aligned} \log P_p(x,x) &= n\log p + \log\Big(\sum_{r=0}^{k}\boldsymbol{b}_r(x)p^{-r} + R_k(p)\Big) \\ &= n\log p + \log \boldsymbol{b}_0(x) + \log\Big(1 + \sum_{r=1}^{k} p^{-r}\frac{\boldsymbol{b}_r(x)}{\boldsymbol{b}_0(x)} + R_k(p)\Big), \end{aligned} \tag{3.12}$$

where $R_k(p) = \mathcal{O}(p^{-k-1})$ denotes the remainder term. In particular, we get formulas (3.7). □

The convergence in the limit (3.8) can be improved by imposing further geometric assumptions, for instance the assumption of bounded geometry as in [60].

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Definition 3.2 (Bounded geometry). We say that (X,J,Θ), (L,h_L) have bounded geometry if J, g^{TX}, R^L and their derivatives of any order are uniformly bounded on X in the norm induced by g^{TX}, and the injective radius of (X,g^{TX}) is strictly positive.

One important class of examples are Galois coverings of compact projective manifolds M endowed with the pull-back of a positive holomorphic line bundle on M.

We recall the following results proved in [60, Theorem 3].

Theorem 3.3 [60]. *Let (X,J,Θ) be a complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L and Condition 1.2 holds. In addition, we assume that they have bounded geometry. Then for any $\ell \in \mathbb{N}$, we have the following expansion holding in the $\mathscr{C}^\ell$-norm induced from g^{TX} and the associated Levi–Civita connection on X,*

$$P_p(x,x) = \frac{a_0(x)}{(2\pi)^n} p^n + \mathcal{O}(p^{n-1}) \tag{3.13}$$

Moreover, there exists $p_0 \in \mathbb{N}$ such that for all $p > p_0$, X is holomorphically convex with respect to the bundle L^p and $H^0_{(2)}(X,L^p)$ separates points and gives local coordinates on X.

As a consequence, we get the following results.

Proposition 3.4. *Assume the same geometric hypothesis as in Theorem 3.3. Writing $\psi_\eta^{S_p}$ for the Gaussian random section constructed from $H^0_{(2)}(X,L^p)$, then for sufficiently large p, $\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]]$ is a smooth $(1,1)$-form on X. Then for any $\ell \in \mathbb{N}$, we have the following convergence in the $\mathscr{C}^\ell$-norm on X*

$$\frac{1}{p}\mathbb{E}\left[[\mathrm{Div}(\psi_\eta^{S_p})]\right] \to c_1(L,h_L), \text{ as } p \to +\infty. \tag{3.14}$$

Remark 3.5. Note that under the assumption of bounded geometry and for X noncompact, we have $d_p = \infty$, $p \gg 0$.

Example 3.6 (Scaled Bargmann–Fock spaces). We consider the line bundle (L,h_L) on $\mathbb{C}^n$ from Example 2.14, which satisfies the above assumptions. For $p \geq 1$, an orthonormal basis of $H^0_{(2)}(\mathbb{C}^n,L^p)$ is given by the family

$$S^p_\alpha(z) = p^{\frac{n}{2}} S_\alpha(\sqrt{p}z), \quad \alpha \in \mathbb{N}^n. \tag{3.15}$$

Then the Bergman kernel function is given

$$P_p(z,z) \equiv p^n. \tag{3.16}$$

Recall the flat Gaussian random holomorphic function ψ_η^S on $\mathbb{C}$ is defined by the equality (1.5). Then for $p \geq 1$, we have

$$\psi_\eta^{S_p}(z) = p^{n/2}\psi_\eta^S(\sqrt{p}z). \tag{3.17}$$

A direct computation then shows that

$$\frac{1}{p}\mathbb{E}[[\mathrm{Div}(\psi_\eta^{S_p})]] = \mathbb{E}[[\mathrm{Div}(\psi_\eta^{S})]] = \frac{\sqrt{-1}}{2\pi}\sum_{j=1}^n dz_j \wedge d\bar{z}_j. \tag{3.18}$$

Theorem 3.7. *Let (L,h_L) and (X,Θ) be as in Theorem 3.1. For any given test form $\varphi \in \Omega_0^{n-1,n-1}(X)$, we have*

$$\mathbb{P}\Big(\lim_{p\to+\infty}\frac{1}{p}\big\langle[\mathrm{Div}(\psi_\eta^{S_p})],\varphi\big\rangle = \langle c_1(L,h_L),\varphi\rangle\Big) = 1. \tag{3.19}$$

Proof. To prove this theorem, we mainly follow the arguments from proof of [56, Theorem 5.3.3], and the possibility of infinite dimension does not lead to complications in this setting. Fix a nontrivial test form $\varphi \in \Omega_0^{n-1,n-1}(X)$. Note that from the proof of Theorem 3.1, we have the convergence

$$\lim_{p\to\infty}\Big\langle\frac{1}{p}\gamma(L^p,h_p),\varphi\Big\rangle = \langle c_1(L,h_L),\varphi\rangle. \tag{3.20}$$

Defining the random variable

$$Y_p = \frac{1}{p}\Big\langle[\mathrm{Div}(\psi_\eta^{S_p})] - \gamma(L^p,h_p),\varphi\Big\rangle, \tag{3.21}$$

statement (3.19) is equivalent to proving the almost sure convergence

$$Y_p \to 0. \tag{3.22}$$

For any $x \in \operatorname{supp}\varphi$, let $e_L(x)$ denote a unit vector of $(L_x,h_{L,x})$. Set

$$b_p(x) = (P_p(x,x)^{-1/2}S_j^p(x)/e_L^{\otimes p}(x)) \in \ell^2(\mathbb{C}). \tag{3.23}$$

Then $\eta\cdot b_p(x)$ is a standard complex Gaussian variable. The covariance matrix of the Gaussian vector $(\eta\cdot b_p(x),\eta\cdot b_p(y))$ depends smoothly on $(x,y)\in\operatorname{supp}\varphi\times\operatorname{supp}\varphi$.

For $v=(v_1,v_2)\in\mathbb{C}^2$ with $\|v\|=1$, we consider the integral

$$\rho(v) := \frac{1}{4\pi^2}\int_{\mathbb{C}^2} e^{-\frac{1}{2}(|z_1|^2+|z_2|^2)}\big|\log|z_1|\cdot\log|v_1z_1+v_2z_2|\big|\mathrm{dV}(z). \tag{3.24}$$

The computations in [56, Eqs. (5.3.13) to (5.3.15)] then show that

$$C := \sup_{v\in\mathbb{C}^2,\,\|v\|=1}\rho(v) < \infty, \tag{3.25}$$

so for $x,y\in\operatorname{supp}\varphi$ we have

$$\mathbb{E}\Big[\Big|\log\Big|P_p(x,x)^{-1/2}\sum_j \eta_j S_j^p(x)\Big|_{h_p}\log\Big|P_p(y,y)^{-1/2}\sum_j \eta_j S_j^p(y)\Big|_{h_p}\Big|\Big] \le C. \tag{3.26}$$

Note that

$$\mathbb{E}[|Y_p|^2] = \frac{1}{p^2}\mathbb{E}\big[\big|\langle[\mathrm{Div}(\psi_\eta^{S_p})],\varphi\rangle\big|^2\big] - \frac{1}{p^2}\big|\langle\gamma(L^p,h_p),\varphi\rangle\big|^2. \tag{3.27}$$

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Then by relations (2.29) and (3.26) and the Fubini–Tonelli theorem we infer that

$$\begin{aligned}\mathbb{E}[|Y_p|^2] = \frac{1}{\pi^2 p^2}\int_{X\times X} &\left(\partial\overline{\partial}\varphi(x)\right)\left(\overline{\partial\overline{\partial}\varphi(y)}\right)\\ &\mathbb{E}\Big[\log\Big|P_p(x,x)^{-1/2}\sum_j \eta_j S_j^p(x)\Big|_{h_p}\log\Big|P_p(y,y)^{-1/2}\sum_j \eta_j S_j^p(y)\Big|_{h_p}\Big].\end{aligned}\tag{3.28}$$

By estimate (3.26), we conclude $\mathbb{E}[|Y_p|^2] = \mathcal{O}(\frac{1}{p^2})$. Hence, $\mathbb{E}[\sum_{p\geq 1}|Y_p|^2] = \sum_{p\geq 1}\mathbb{E}[|Y_p|^2] < \infty$, thus $Y_p \to 0$ almost surely. □

Corollary 3.8. *If Θ is a Kähler form and $\int_X c_1(L,h_L)\wedge\Theta^{n-1} < \infty$, or if $\int_X c_1(L,h_L)^n < \infty$, then*

$$\mathbb{P}\Big(\lim_{p\to+\infty}\frac{1}{p}[\mathrm{Div}(\psi_\eta^{S_p})] = c_1(L,h_L)\Big) = 1,\tag{3.29}$$

where the limit is taken with respect to the weak convergence of $(1,1)$-currents on X.

Proof. Let's assume at first that Θ is a Kähler form with $\int_X c_1(L,h_L)\wedge\Theta^{n-1} < \infty$. For all $\varphi\in\Omega_0^{n-1,n-1}(X)$, $s_p\in H^0(X,L^p)$, due to the positivity of the current $[\mathrm{Div}(s_p)]$, we have

$$|\langle[\mathrm{Div}(s_p)],\varphi\rangle| \leq |\varphi|_{\mathscr{C}^0(X)}\langle[\mathrm{Div}(s_p)],\Theta^{n-1}\rangle = p|\varphi|_{\mathscr{C}^0(X)}\int_X c_1(L,h_L)\wedge\Theta^{n-1}.\tag{3.30}$$

The last identity follows from the Poncaré–Lelong formula. We may set $C = \int_X c_1(L,h_L)\wedge\Theta^{n-1} < \infty$, then

$$\frac{1}{p}|\langle[\mathrm{Div}(s_p)],\varphi\rangle| \leq C|\varphi|_{\mathscr{C}^0(X)}.\tag{3.31}$$

By considering a countable $\mathscr{C}^0$-dense family of φ's in $\Omega_0^{n-1,n-1}(X)$, and applying Theorem 3.7, we get the equality (3.29).

If $\int_X c_1(L,h_L)^n < \infty$, we use $c_1(L,h_L)$ instead of Θ in the above arguments to get the equality (3.29). □

Remark 3.9. (1) The additional assumptions in the above corollary are necessary in our approach to the proof of the equality (3.29); however, it is an interesting question whether or not these extra assumptions could be removed.

(2) In the first assumption of Corollary 3.8, we can replace the Kähler condition on Θ by the condition that Θ is Gauduchon, which means $\partial\bar{\partial}\Theta^{n-1} = 0$.

(3) Due to the uniform estimate (3.30) for all holomorphic sections s_p with a given p, we can improve Corollary 3.8 as follows: Equation (3.29) holds true when the limit is taken as the weak convergence in the sense of current of order 0, that is, for any continuous $(n-1,n-1)$-form φ with compact support on X, we have

$$\lim_{p\to+\infty}\langle\frac{1}{p}[\mathrm{Div}(\psi_\eta^{S_p})],\varphi\rangle = \langle c_1(L,h_L),\varphi\rangle.\tag{3.32}$$

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Remark 3.10. For each $p \in \mathbb{N}_{>0}$, we can take a sequence of i.i.d. standard complex Gaussian random variables $\eta^p = \{\eta_j^p\}_{j=1}^{d_p}$ and assume that they are mutually independent for different p. We define the flat Gaussian random sections

$$\psi_{\eta^p}^{S_p} = \sum_{j=1}^{d_p} \eta_j^p S_j^p, \tag{3.33}$$

where $S_p = \{S_j^p\}_{j=1}^{d_p}$ is an orthonormal (Hilbert) basis of $H^0_{(2)}(X, L^p)$. Then the statements in Theorems 3.1, 3.7, Proposition 3.4 and Corollary 3.8 still hold true for the sequence of random sections $\psi_{\eta^p}^{S_p}, p \geq 1$.

3.2. Large deviation estimates and hole probability: proofs of Theorems 1.4 and 1.5

In this subsection, we study the large deviation estimates for random zeros in a given domain with respect to the high tensor powers as in [70], [25] and [27]. In particular, we prove Theorems 1.4 and 1.5. A key intermediate result in the approach to the above theorems is the proposition as follows, whose proof is deferred to the next subsection.

Proposition 3.11. *Let (X, J, Θ) be a connected complex Hermitian manifold, and let (L, h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L such that Condition 1.2 holds. Let U be a relatively compact open subset in X. For any $\delta > 0$, there exists $C_{U,\delta} > 0$ such that for all $p \gg 0$,*

$$\mathbb{P}\Big(\int_U \Big| \log |\psi_\eta^{S_p}(x)|_{h_p} \Big| \mathrm{dV}(x) \geq \delta p \Big) \leq e^{-C_{U,\delta} p^{n+1}}. \tag{3.34}$$

Proof of Theorem 1.4. The Poincaré–Lelong formula (2.29) shows that

$$\frac{\sqrt{-1}}{\pi} \partial\overline{\partial} \log |\psi_\eta^{S_p}|_{h_p} = [\mathrm{Div}(\psi_\eta^{S_p})] - p c_1(L, h_L) \tag{3.35}$$

as an identity of (1,1)-currents on X. Now, fix $\varphi \in \Omega_0^{n-1,n-1}(X)$, and fix a relatively compact open subset $U \subset X$ such that $\operatorname{supp} \varphi \subset U$. Then

$$\Big(\frac{1}{p} [\mathrm{Div}(s_p)], \varphi \Big) - \int_X c_1(L, h_L) \wedge \varphi = \frac{\sqrt{-1}}{p\pi} \int_X \log |\psi_\eta^{S_p}|_{h_p} \partial\overline{\partial}\varphi. \tag{3.36}$$

Since φ has a compact support in U, so has $\partial\overline{\partial}\varphi$. Set

$$S_\varphi = \max_{x \in U} \left| \frac{\sqrt{-1}\partial\overline{\partial}\varphi(x)}{\mathrm{dV}(x)} \right|. \tag{3.37}$$

We can and we may assume that $S_\varphi > 0$. Then

$$\left| \frac{\sqrt{-1}}{p\pi} \int_X \log |\psi_\eta^{S_p}|_{h_p} \partial\overline{\partial}\varphi \right| \leq \frac{S_\varphi}{p\pi} \int_U \left| \log |\psi_\eta^{S_p}(x)|_{h_p} \right| \mathrm{dV}(x). \tag{3.38}$$

Applying Proposition 3.11 to right-hand side of inequality (3.38) we get estimate (1.10). □

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Proof of Theorem 1.5. Estimate (1.14) is a direct consequence of estimate (1.13) by taking $\delta = n\,\mathrm{Vol}^L_{2n}(U)$. Hence, it is sufficient to prove estimate (1.13). For this purpose, let χ_U denote the characteristic function of U on X. Let $\delta > 0$ be arbitrary, and take ψ_1, $\psi_2 \in \mathcal{C}^\infty_0(X,\mathbb{R})$ such that $0 \le \psi_1 \le \chi_U \le \psi_2 \le 1$, and

$$\int_X \psi_1 \frac{c_1(L,h_L)^n}{n!} \ge \mathrm{Vol}^L_{2n}(U) - \delta, \quad \int_X \psi_2 \frac{c_1(L,h_L)^n}{n!} \le \mathrm{Vol}^L_{2n}(U) + \delta. \tag{3.39}$$

Note that the existence of such functions is guaranteed by the assumption that ∂U has measure 0 with respect to dV, hence also to $\frac{1}{n!}c_1(L,h_L)^n$. For $j \in \{1,2\}$, set $\varphi_j = \frac{1}{(n-1)!}\psi_j c_1(L,h_L)^{n-1}$. By applying Theorem 1.4 to φ_j separately, we get exactly estimate (1.13). □

3.3. Proof of Proposition 3.11

Let $U \subset X$ be a relatively compact open subset. For $s_p \in H^0(X,L^p)$, we set

$$\mathcal{M}^U_p(s_p) = \sup_{x\in U} |s_p(x)|_{h_p} < +\infty. \tag{3.40}$$

Before proving Proposition 3.11, we need to investigate the probabilities for both, $\mathcal{M}^U_p(\psi^{S_p}_\eta)$ taking atypically large and small values, respectively.

Proposition 3.12. *For any $\delta > 0$, there exists a constant $C_{U,\delta} > 0$ such that for $p \in \mathbb{N}_{>1}$,*

$$\mathbb{P}\big(\mathcal{M}^U_p(\psi^{S_p}_\eta) \ge e^{\delta p}\big) \le e^{-\delta p^{n+1} + C_{U,\delta}p^n \log p}. \tag{3.41}$$

Proof. The basic idea of the proof is that the local sup-norm of a holomorphic function is bounded by its local $\mathcal{L}^2$-norm as in estimate (2.9). We fix $\delta > 0$ and let $r > 0$ be sufficiently small so that we can choose a finite set of points $\{x_j\}^\ell_{j=1} \subset U$ such that the geodesic open balls $B^X(x_j,r)$, $j = 1,\ldots,\ell$ form an open covering of $\overline{U}$. Since r is sufficiently small, then we can assume that each larger ball $B^X(x_j,2r)$ lies in a complex chart (hence viewed as an open subset of $\mathbb{C}^n$), and that for each j, we can fix a local holomorphic frame $e_{L,j}$ of L on a neighborhood of $B^X(x_j,2r)$ with $\sup_{x\in B^X(x_j,2r)} |e_{L,j}(x)|_{h_L} = 1$. Set

$$\nu = \min\Big\{ \inf_{x\in B^X(x_j,2r)} |e_{L,j}(x)|_{h_L} \,:\, j = 1,\ldots,\ell \Big\}. \tag{3.42}$$

It is clear that $0 < \nu \le 1$. By fixing r small enough, we can and do assume that

$$-\log \nu \le \frac{\delta}{4}\cdot \tag{3.43}$$

As in estimate (2.9), since U is relatively compact, there exists a constant $C > 0$ such that for each $j = 1,\ldots,\ell$, if f is a holomorphic function on a neighborhood of $B^X(x_j,2r)$, then

$$\sup_{x\in B^X(x_j,r)} |f(x)| \le C\|f\|_{\mathcal{L}^2(B^X(x_j,2r))}, \tag{3.44}$$

where the volume form $\mathrm{dV}(x)$ on X is used in the norm $\|\cdot\|_{\mathcal{L}^2(B^X(x_j,2r))}$. Note that the choices of x_j, r, ℓ and the constants ν, C are independent of the tensor power p. Set

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$\widetilde{U} = \cup_j B^X(x_j,2r) \supset U$. For $p \in \mathbb{N}, s_p \in H^0(X,L^p)$, on each $B^X(x_j,2r)$, we write

$$s_p|_{B^X(x_j,2r)} = f_j e_{L,j}^{\otimes p}, \tag{3.45}$$

where f_j is a holomorphic function on the chart in $\mathbb{C}^n$ corresponding to $B^X(x_j,2r)$. Then we have

$$\begin{aligned} \mathcal{M}_p^U(s_p) = \sup_{x\in U} |s_p(x)|_{h_p} &\leq \max_j \sup_{x\in B^X(x_j,r)} |f_j(x)| \\ &\leq C \max_j \{\|f_j\|_{\mathcal{L}^2(B^X(x_j,2r))}\} \\ &\leq \frac{C}{\nu^p} \max_j \{\|s_p\|_{\mathcal{L}^2(B^X(x_j,2r),L^p)}\} \\ &\leq \frac{C}{\nu^p} \|s_p\|_{\mathcal{L}^2(\widetilde{U},L^p)}. \end{aligned} \tag{3.46}$$

The next step is to estimate the quantity $\mathbb{E}[\|\psi_\eta^{S_p}\|^{2p^n}_{\mathcal{L}^2(\widetilde{U},L^p)}]$ for $p \geq 2$. Applying Hölder's inequality with $\frac{1}{p^n} + \frac{p^n-1}{p^n} = 1$, we get

$$\mathbb{E}\big[\|\psi_\eta^{S_p}\|^{2p^n}_{\mathcal{L}^2(\widetilde{U},L^p)}\big] \leq \mathrm{Vol}(\widetilde{U})^{p^n-1}\mathbb{E}\Big[\int_{\widetilde{U}} |\psi_\eta^{S_p}(x)|^{2p^n}_{h_p}(x)\mathrm{dV}\Big]. \tag{3.47}$$

As in estimate (3.44), on a neighborhood of $B^X(x_j,2r)$, write

$$S_i^p = f_i^p e_{L,i}^{\otimes p}. \tag{3.48}$$

If $x \in B^X(x_j,2r)$, set

$$F_j(x) = \sum_{i=1}^{d_p} \eta_i f_i^p(x). \tag{3.49}$$

Then $F_j(x)$ is a complex Gaussian random variable with (total) variance $\sum_{i=1}^{d_p} |f_i^p(x)|^2$. By our assumption on the local frame $e_{L,j}$, we get

$$\sum_{i=1}^{d_p} |f_i^p(x)|^2 \leq \frac{1}{\nu^{2p}} P_p(x,x). \tag{3.50}$$

Then we have

$$\mathbb{E}\big[|F_j(x)|^{2p^n}\big] = p^n!\Big(\sum_{i=1}^{d_p} |f_i^p(x)|^2\Big)^{p^n}. \tag{3.51}$$

As a consequence, we get that for $x \in \widetilde{U}$,

$$\mathbb{E}\big[|\psi_\eta^{S_p}(x)|^{2p^n}_{h_p}\big] \leq \frac{p^n!}{\nu^{2p^{n+1}}} (P_p(x,x))^{p^n}. \tag{3.52}$$

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Since we are in the context of σ-finite measures and the integrands are nonnegative, Tonelli's theorem applies so that

$$\mathbb{E}\Big[\int_{\widetilde{U}} |\psi_\eta^{S_p}(x)|_{h_p}^{2p^n} \mathrm{dV}(x)\Big] \leq \frac{p^n!}{\nu^{2p^{n+1}}} \int_{\widetilde{U}} (P_p(x,x))^{p^n} \mathrm{dV}(x). \tag{3.53}$$

Moreover, by the on-diagonal estimate for the Bergman kernel on a given compact subset, there exists a constant $C_{\widetilde{U}} > 0$ (independent of p) such that for $p \in \mathbb{N}$, $x \in \widetilde{U}$,

$$P_p(x,x) \leq C_{\widetilde{U}} p^n. \tag{3.54}$$

Combining estimate (3.47) with the above inequalities, we infer that

$$\mathbb{E}\Big[\|\psi_\eta^{S_p}\|^{2p^n}_{\mathcal{L}^2(\widetilde{U},L^p)}\Big] \leq \big(C_{\widetilde{U}} \operatorname{Vol}(\widetilde{U})\big)^{p^n} \frac{p^n!}{\nu^{2p^{n+1}}} (p^n)^{p^n}. \tag{3.55}$$

By applying estimate (3.46) to $\psi_\eta^{S_p}$, we get

$$\mathbb{E}\big[\mathcal{M}_p^U(\psi_\eta^{S_p})^{2p^n}\big] \leq \Big(\frac{C}{\nu^p}\Big)^{2p^n} \mathbb{E}\big[\|\psi_\eta^{S_p}\|^{2p^n}_{\mathcal{L}^2(\widetilde{U},L^p)}\big] \leq \frac{(\widetilde{C}p^n)^{2p^n}}{\nu^{4p^{n+1}}}, \tag{3.56}$$

where $C > 0$, $\widetilde{C} > 0$ are constants independent of p. Then estimate (3.41) follows from Chebyshev's inequality and the inequality $\frac{1}{\nu} \leq e^{\frac{\delta}{4}}$ from (3.43). □

Remark 3.13. The choice to consider the p^n-th moment of $\|\psi_\eta^{S_p}\|^2$ leads to the exponent p^{n+1} in the exponential of the resulting probability estimate. One can consider arbitrary N-th moments to obtain a more general statement on this probability upper bound.

When X is compact, or if X is noncompact but d_p is bounded polynomially in p, then the upper bound $Ce^{-cp^{n+1}}$ can be obtained in a simpler way as in [27, 70] (and of course with a much sharper upper bound).

Now, we consider the probabilities of small values of $\mathcal{M}_p^U(\psi_\eta^{S_p})$, and we will adapt the ideas in [27, 70]. At first, we introduce a result on the near-diagonal estimate of Bergman kernel. We fix a point $x \in X$. Let $\{\boldsymbol{f}_j\}_{j=1}^n$ be an orthonormal basis of $(T_x^{1,0}X, g_x^{TX}(\cdot,\bar{\cdot}))$ such that

$$\dot{R}_x^L \boldsymbol{f}_j = \mu_j(x) \boldsymbol{f}_j, \tag{3.57}$$

where $\mu_j(x)$, $j = 1, \ldots, n$, are the eigenvalues of $\dot{R}_x^L$ (cf. formula (3.1)). Then by the first inequality in condition (1.6), we have

$$\mu_j(x) \geq \varepsilon. \tag{3.58}$$

Set $\boldsymbol{e}_{2j-1} = \frac{1}{\sqrt{2}}(\boldsymbol{f}_j + \overline{\boldsymbol{f}}_j)$, $\boldsymbol{e}_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(\boldsymbol{f}_j - \overline{\boldsymbol{f}}_j)$, $j = 1, \ldots, n$. Then they form an orthonormal basis of the (real) tangent vector space (T_xX, g_x^{TX}). If $v = \sum_{j=1}^{2n} v_j \boldsymbol{e}_j \in T_xX$, we can write

$$v = \sum_{j=1}^n (v_{2j-1} + \sqrt{-1} v_{2j}) \frac{1}{\sqrt{2}} \boldsymbol{f}_j + \sum_{j=1}^n (v_{2j-1} - \sqrt{-1} v_{2j}) \frac{1}{\sqrt{2}} \overline{\boldsymbol{f}}_j. \tag{3.59}$$

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Set $z=(z_1,\ldots,z_n)$ with $z_j=v_{2j-1}+\sqrt{-1}v_{2j}$, $j=1,\ldots,n$. We call z the complex coordinate of $v\in T_xX$. Then by formula (3.59),

$$\frac{\partial}{\partial z_j}=\frac{1}{\sqrt{2}}\boldsymbol{f}_j,\ \frac{\partial}{\partial \overline{z}_j}=\frac{1}{\sqrt{2}}\overline{\boldsymbol{f}}_j \tag{3.60}$$

so that

$$v=\sum_{j=1}^{m}\Big(z_j\frac{\partial}{\partial z_j}+\overline{z}_j\frac{\partial}{\partial \overline{z}_j}\Big). \tag{3.61}$$

Note that $|\frac{\partial}{\partial z_j}|^2_{g^{TX}}=|\frac{\partial}{\partial \overline{z}_j}|^2_{g^{TX}}=\frac{1}{2}$. For $v,v'\in T_xX$, let z,z' denote the corresponding complex coordinates. Define a weighted distance function $\Phi_x^{TX}(v,v')$ as follows,

$$\Phi_x^{TX}(v,v')^2=\sum_{j=1}^{n}\mu_j(x)|z_j-z'_j|^2. \tag{3.62}$$

For sufficiently small $\delta_0>0$, we identify the small open ball $B^X(x,2\delta_0)$ in X with the ball $B^{T_xX}(0,2\delta_0)$ in T_xX via the geodesic coordinate. Let $\mathrm{dist}(\cdot,\cdot)$ denote the Riemannian distance of (X,g^{TX}). There exists $C_1>0$ such that for $v,v'\in B^{T_xX}(0,2\delta_0)$, we have

$$C_1\mathrm{dist}(\exp_x(v),\exp_x(v'))\geq \Phi_x^{TX}(v,v')\geq \frac{1}{C_1}\mathrm{dist}(\exp_x(v),\exp_x(v')). \tag{3.63}$$

In particular,

$$\Phi_x^{TX}(0,v)\geq \varepsilon^{1/2}\mathrm{dist}(x,\exp_x(v)). \tag{3.64}$$

Moreover, if we consider a compact subset $K\subset X$, the constants δ_0 and C_1 can be chosen uniformly for all $x\in K$. For $p\in\mathbb{N}$, $x,y\in X$, the normalized Bergman kernel is defined as

$$N_p(x,y)=\frac{|P_p(x,y)|_{h_{p,x}\otimes h^*_{p,y}}}{\sqrt{P_p(x,x)}\sqrt{P_p(y,y)}}. \tag{3.65}$$

The following result was proved in [27, Theorem 5.1], where we use essentially the near-diagonal expansion of Bergman kernel from [56, Theorems 4.2.1 & 6.1.1].

Theorem 3.14. *Let (X,J,Θ) be a connected complex Hermitian manifold, and let (L,h_L) be a holomorphic line bundle on X with a smooth Hermitian metric h_L such that Condition 1.2 holds. Let U be a relatively compact open subset of X. Then the following uniform estimates on the normalized Bergman kernel hold for $x,y\in U$: For $k\geq 1$ and $b>\sqrt{16k/\varepsilon}$ fixed, we have for $p\gg 0$ (such that $b\sqrt{(\log p)/p}\leq 2\delta_0$) that*

$$N_p(x,y)=\begin{cases}\big(1+o(1)\big)\exp\Big(-\dfrac{p}{4}\Phi_x(0,v')^2\Big),\ \textit{uniformly for } \mathrm{dist}(x,y)\leq b\sqrt{(\log p)/p},\\ \qquad\qquad\qquad\qquad\qquad\qquad\textit{with } y=\exp_x(v'),v'\in T_xX;\\ \mathcal{O}(p^{-k}),\ \textit{uniformly for } \mathrm{dist}(x,y)\geq b\sqrt{(\log p)/p}.\end{cases} \tag{3.66}$$

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Proposition 3.15. *There exist constants $C_U > 0, C'_U > 0$ such that for all $\delta > 0$ and $p \in \mathbb{N}$,*

$$\mathbb{P}\big(\mathcal{M}_p^U(\psi_\eta^{S_p}) \le e^{-\delta p}\big) \le e^{-C_U\delta p^{n+1} + C'_U p^n \log p}. \tag{3.67}$$

Proof. For $x \in X$, we fix some $\lambda_x \in L_x$ with $|\lambda_x|_h = 1$ and set

$$\xi_x = \frac{\langle \lambda_x^{\otimes p}, \psi_\eta^{S_p}(x)\rangle_{h_p}}{\sqrt{P_p(x,x)}}. \tag{3.68}$$

Then ξ_x is a complex Gaussian random variable. Moreover, for any two points $x,y \in X$, we have

$$\big|\mathbb{E}[\xi_x \overline{\xi}_y]\big| = N_p(x,y). \tag{3.69}$$

Then by the asymptotics (3.66), using arguments similar to those used in [70, Subsection 3.2] or the proof of [27, Theorem 1.13], we can prove a more general version of estimate (3.67) as follows: for a sequence of positive numbers $\{\lambda_p\}_{p\in\mathbb{N}}$ with $\lambda_p \to 0$, we have

$$\mathbb{P}\big(\mathcal{M}_p^U(\psi_\eta^{S_p}) \le \lambda_p\big) \le e^{Cp^n \log \lambda_p + C'p^n \log p}, \quad p \gg 0. \tag{3.70}$$

Then, for any $\delta > 0$, choosing $\lambda_p = e^{-\delta p}$ in estimate (3.70), we recover (3.67). ☐

Combining Propositions 3.12 and 3.15, we arrive at the following.

Corollary 3.16. *For any relatively compact open subset $U \subset X$, and for $\delta > 0$, there exists a constant $C = C(U,\delta) > 0$ such that for $p \gg 1$,*

$$\mathbb{P}\left(\left|\log \mathcal{M}_p^U(\psi_\eta^{S_p})\right| \ge \delta p\right) \le e^{-Cp^{n+1}}. \tag{3.71}$$

Proof of Proposition 3.11. The proof of Proposition 3.11 follows by combining from the arguments in [70, Subsection 4.1] with Corollary 3.16. Here, we just sketch the proof.

For $t > 0$, set

$$\log^+ t = \max\{\log t, 0\},\ \log^- t := \log^+(1/t) = \max\{-\log t, 0\}. \tag{3.72}$$

Then

$$|\log t| = \log^+ t + \log^- t. \tag{3.73}$$

Let U be a relatively compact nonempty open subset in X. Then for any nonzero holomorphic section $s_p \in H^0(X, L^p)$, we have that $\left|\log |s_p|_{h_p}\right|$ is integrable on $\overline{U}$ with respect to dV. We now start with showing that

$$\mathbb{P}\left(\int_U \log^+ |\psi_\eta^{S_p}(x)|_{h_p} \mathrm{dV}(x) \ge \frac{\delta}{2} p\right) \le e^{-C_{U,\delta} p^{n+1}}. \tag{3.74}$$

For this purpose, observe that on U we have

$$\log^+ |\psi_\eta^{S_p}|_{h_p} \le \left|\log \mathcal{M}_p^U(\psi_\eta^{S_p})\right|, \tag{3.75}$$

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which then supplies us with

$$\mathbb{P}\left(\int_U \log^+ |\psi_\eta^{S_p}(x)|_{h_p} \mathrm{dV}(x) \geq \frac{\delta}{2}p\right) \leq \mathbb{P}\left(\left|\log \mathcal{M}_p^U(\psi_\eta^{S_p})\right| \geq \frac{\delta}{2\,\mathrm{Vol}(U)}p\right), \tag{3.76}$$

where $\mathrm{Vol}(U)$ denotes the volume of U with respect to dV. In combination with Corollary 3.16, this immediately implies estimate (3.74). The next step is to prove that

$$\mathbb{P}\left(\int_U \log^- |\psi_\eta^{S_p}(x)|_{h_p} \mathrm{dV}(x) \geq \frac{\delta}{2}p\right) \leq e^{-C_{U,\delta}\, p^{n+1}}. \tag{3.77}$$

Suppose that U contains an annulus $B(2{,}3) := \{z \in \mathbb{C}^n : 2 < |z| < 3\}$ (possibly after rescaling of coordinates) and the line bundle L on $B(4) := \{|z| < 4\}$ (still contained in U) has a holomorphic local frame e_L. We will mainly work on this annulus $B(2{,}3)$ instead of a coordinate disc. The reason is that, as in [70, p. 1991 in Subsection 4.1], we will need the two-sided positive bounds for the Poisson kernel of the disc and such bounds will hold on the annulus $B(2{,}3)$.

Set $\alpha(x) = \log |e_L(x)|^2_{h_L}$. We can then write

$$\psi_\eta^{S_p} = F_p e_L^{\otimes p}, \tag{3.78}$$

where F_p is a random holomorphic function on $B(4)$. Then

$$\log |\psi_\eta^{S_p}|_{h_p} = \log |F_p| + \frac{p}{2}\alpha. \tag{3.79}$$

In the following estimates, each K_i, $i \in \mathbb{N}$, denotes a sufficiently large positive constant. Then by relations (3.73) and (3.76), we have

$$\mathbb{P}\left(\int_{B(2,3)} \log^+ |F_p| \mathrm{dV} \geq K_1 p\right) \leq e^{-C_{U,K_1}\, p^{n+1}}. \tag{3.80}$$

Using the Poisson kernel and the submean inequality for $\log(|F_p|)$ (see [70, Subsection 4.1, pp. 1991]), we can improve estimate (3.80) to get

$$\mathbb{P}\left(\int_{B(2,3)} \left|\log |F_p|\right| \mathrm{dV} \geq K_2 p\right) \leq e^{-C_{U,K_2}\, p^{n+1}}. \tag{3.81}$$

From this point, we proceed as in [70, Section 4.1, p. 1992]. For $\delta \in (0, \frac{1}{2}]$, we fix a grid in the polar coordinate system of $B(2{,}3)$ so that, by enlarging a bit the grid cells, we obtain an open covering $\{U_j\}_{j=1}^q$ of $B(2{,}3)$ consisting of small boxes of diameters $\simeq \delta^{2n+2}$. Then for a finite set of points $\{z_j\}_{j=1}^q$ with $z_j \in U_j$ and for all s_p, $p \in \mathbb{N}$, we have

$$\begin{aligned} &-\int_{B(2,3)} \log |s_p|_{h_p} \mathrm{dV} \\ &\leq -\sum_{j=1}^q \mu_j \log |s_p|_{h_p}(z_j) + K_3\delta \int_{B(2,3)} \left|\log |f_p|\right| \mathrm{dV} + p\delta K_3 \sup_{z\in B(2,3)} |d\alpha(z)|_{g^{T^*X}}, \end{aligned} \tag{3.82}$$

where the constant K_3 does not depend on δ, but the quantities q and $\mu_j > 0$ only depend on δ, and we have $\sum_{j=1}^q \mu_j \simeq 1$. Applying the above inequality to $\psi_\eta^{S_p}$ and F_p, we can

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use Corollary 3.16 for each U_j: except an event of probability $\leq e^{-c_j p^{n+1}}$, we can always choose $z_j \in U_j$ such that $\log|\psi_\eta^{S_p}|_{h_p}(z_j) > -\delta p$. Then combining estimate (3.81) with estimate (3.82), we infer that

$$\mathbb{P}\left(-\int_{B(2,3)} \log|\psi_\eta^{S_p}|_{h_p} \mathrm{dV} \geq K_4 \delta p\right) \leq e^{-C_{U,\delta} p^{n+1}}, \forall\, p \gg 0. \tag{3.83}$$

Noting that $\log^- = -\log + \log^+$ and that a finite set of annuli of the form $B(2,3)$ covers U, we can infer estimate (3.77) from estimates (3.76) and (3.83). This completes our proof. □

Remark 3.17. Given the information we have for the law of large numbers and large deviations, it is natural to ask about a version of the central limit theorem. Sodin–Tsirelson [72, Main Theorem] introduced and proved the asymptotic normality of functionals of the zeros of certain random holomorphic functions on $\mathbb{C}$ or $\mathbb{D}$. This was extended by Shiffman–Zelditch [69, Theorem 1.2] for random holomorphic sections of line bundles on a compact Kähler manifold and for general random polynomials on $\mathbb{C}^n$ by Bayraktar [4].

One key ingredient in their approach is the normalized Bergman kernel defined in formula (3.65), viewed as the covariance function of a normalized Gaussian process on $\mathbb{C}$ or X, as constructed in the proof of Proposition 3.15. Using the estimates given in Theorem 3.14 and the seminal result by Sodin and Tsirelson [72, Theorem 2.2], one could obtain an extension of [69, Theorem 1.2], [72, Main Theorem], in the general setting considered here. This direction will be pursued elsewhere.

3.4. Remark on the lower bound for the hole probabilities

To obtain a lower bound of matching order $e^{-cp^{n+1}}$ for the hole probability in estimate (1.14) is generally more complicated. In the case of scaled Bargmann–Fock spaces (cf. Example 3.6), we can give a lower bound and and we sketch its proof below.

Recall that for $p \in \mathbb{N}$, the family $\{S_\alpha^p\}_{\alpha \in \mathbb{N}^n}$ denotes an orthonormal basis of $H^0_{(2)}(\mathbb{C}^n, L^p)$. For $K > 0$, define the index set

$$I(K) = \big\{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \,:\, 0 \leq \alpha_j \leq K, j = 1, \ldots, n\big\}, \tag{3.84}$$

set $I^*(K) = I(K) \setminus \{(0, \ldots, 0)\}$ and put

$$q_p := \sharp I(Kp) = (1 + \lfloor Kp \rfloor)^n = \mathcal{O}(p^n). \tag{3.85}$$

For this canonical family of orthonormal basis, we can verify directly the following local concentration condition: For any relatively compact subset $U \subset \mathbb{C}^n$ and for any $c > 0$, there exist constants $K = K(U,c) > 0, C' = C'(U,c) > 0$ such that

$$\sup_{z \in \bar{U}} \sum_{\alpha \notin I(Kp)} |S_\alpha^p(z)|^2_{h_p} \leq C' e^{-cp}. \tag{3.86}$$

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In fact, to get the above inequality, one can start with the sum over $\alpha_1 \in \mathbb{N}$ for the monomial $z_1^{\alpha_1}$: after applying Stirling's formula for $\lceil Kp \rceil !$, $\sum_{\alpha_1 > Kp} \frac{|z_1|^{2\alpha_1}}{\alpha_1 !} = \mathcal{O}(e^{-c'p})$, when z_1 lies in a given compact subset.

Let $\psi_\eta^{S_p}$ be the random holomorphic section (actually, function) on $\mathbb{C}^n$ constructed in Example 3.6.

Lemma 3.18. *For any relatively compact open subset $U \subset \mathbb{C}^n$, there exists a constant $C'_U > 0$ such that for $p \gg 1$,*

$$\mathbb{P}\left(\mathrm{Div}(\psi_\eta^{S_p}) \cap U = \varnothing\right) \geq e^{-C'_U p^{n+1}}. \tag{3.87}$$

Proof. For $U = \varnothing$ the statement is trivial, so assume U nonempty is as in the assumptions. Fix a relatively compact open neighborhood U' of $\overline{U}$ and define the strictly positive quantity

$$\widetilde{M} := \min_{z \in \overline{U'}} e^{-\frac{|z|^2}{2}} \in (0,1). \tag{3.88}$$

Let the constants K and C' be the ones in estimate (3.86) for the constant $c = -2\log \widetilde{M} > 0$ and for U'. For $p \in \mathbb{N}$, write $S_0^p \equiv p^{n/2}$ for the unit constant section in $H^0_{(2)}(\mathbb{C}^n, L^p)$ corresponding to $\alpha = (0,\dots,0) \in \mathbb{N}^n$. Then

$$\min_{z \in \overline{U'}} |S_0^p(z)|_{h_p} = p^{n/2} \widetilde{M}^p. \tag{3.89}$$

Defining the random holomorphic sections

$$\psi_{\eta,\mathrm{I}}^{S_p}(z) := \sum_{\alpha \in I^*(Kp)} \eta_\alpha S_\alpha^p(z), \quad \psi_{\eta,\mathrm{II}}^{S_p}(z) := \sum_{\alpha \notin I(Kp)} \eta_\alpha S_\alpha^p(z), \tag{3.90}$$

we can decompose

$$\psi_\eta^{S_p} = \eta_0 S_0^p + \psi_{\eta,\mathrm{I}}^{S_p} + \psi_{\eta,\mathrm{II}}^{S_p}. \tag{3.91}$$

Note that the three random sections on the right-hand side of equality (3.91) are independent from each other. In the rest of the proof, we will regard the above sections as being holomorphic functions on $\mathbb{C}^n$, and let $|\cdot|$ denote the standard modulus on $\mathbb{C}$ (instead of considering the norm $|\cdot|_{h_p}$ on line bundle). Applying estimate (2.9) to the function $\psi_{\eta,\mathrm{II}}^{S_p}$ and using the estimate (3.86), we arrive at the upper bound

$$\begin{aligned}\mathbb{E}\Big[\sup_{z \in U} \left|\psi_{\eta,\mathrm{II}}^{S_p}(z)\right|^2\Big] &\leq C_{U'} \sigma^2 \int_{U'} \sum_{\alpha \notin I(Kp)} |S_\alpha^p(z)|^2 \mathrm{dV}(z) \\ &\leq \widetilde{C}_{U'} \operatorname{Vol}(U') \sigma^2 \widetilde{M}^{-2p} e^{-cp} \\ &= \widetilde{C}_{U'} \operatorname{Vol}(U') \sigma^2 =: \widetilde{C}',\end{aligned} \tag{3.92}$$

where the last equality follows from our choice $c = -2\log \widetilde{M}$. For any $\lambda > 0$, as a consequence of Chebyshev's inequality in combination with estimate (3.92), we have

$$\mathbb{P}\Big(\sup_{z \in U} \left|\psi_{\eta,\mathrm{II}}^{S_p}(z)\right| < \lambda\Big) \geq 1 - \frac{\widetilde{C}'}{\lambda^2}. \tag{3.93}$$

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We define the good event

$$\Omega_p = \left\{ |\eta_0| \geq 1\,;\, |\eta_\alpha| \leq \frac{1}{3\sqrt{q_p - 1}} \widetilde{M}^p, \alpha \in I^*(Kp)\,;\, \sup_{z\in U} \left| \psi^{S_p}_{\eta,\mathrm{II}}(z) \right| < \frac{1}{3} p^{n/2} \right\}. \tag{3.94}$$

For all sufficiently large $p \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{P}(\Omega_p) &= \mathbb{P}(|\eta_0| \geq 1) \cdot \mathbb{P}\left(\sup_{z\in U} \left| \psi^{S_p}_{\eta,\mathrm{II}}(z) \right| < \frac{1}{3} p^{n/2} \right) \\ &\quad \cdot \mathbb{P}\left(|\eta_\alpha| \leq \frac{1}{3\sqrt{q_p-1}} \widetilde{M}^p, \alpha \in I^*(Kp) \right) \\ &\geq e^{-1}\left(1 - \frac{9\widetilde{C}'}{p^n}\right) \cdot \Pi_{\alpha\in I^*(Kp)} \left(\frac{1}{18(q_p-1)} \widetilde{M}^{2p} \right). \end{aligned} \tag{3.95}$$

Then by formula (3.85), there exists $C'_U > 0$ such that for $p \gg 1$,

$$\mathbb{P}(\Omega_p) \geq e^{-C'_U p^{n+1}}. \tag{3.96}$$

Our lemma then follows once we show the inclusion

$$\Omega_p \subset \left\{ \mathrm{Div}(\psi^{S_p}_\eta) \cap U = \varnothing \right\}. \tag{3.97}$$

Indeed, if $|\eta_\alpha| \leq \frac{1}{3\sqrt{q_p-1}} \widetilde{M}^p$, $\alpha \in I^*(Kp)$, then for $z \in U$,

$$\begin{aligned} \left| \psi^{S_p}_{\eta,\mathrm{I}}(z) \right|^2 &\leq \Big(\sum_{\alpha \in I^*(pK)} |\eta_\alpha|^2 \Big) \Big(\sum_{\alpha\in I^*(pK)} |S^p_\alpha(z)|^2 \Big) \\ &\leq \frac{1}{\widetilde{M}^{2p}} \Big(\sum_{\alpha\in I^*(pK)} |\eta_\alpha|^2 \Big) P_p(z,z) \\ &\leq \frac{1}{9} p^n. \end{aligned} \tag{3.98}$$

As a consequence, on Ω_p and for $z \in U$, we get

$$\begin{aligned} \left| \psi^{S_p}_{\eta,\mathrm{I}}(z) + \psi^{S_p}_{\eta,\mathrm{II}}(z) \right| &\leq \left| \psi^{S_p}_{\eta,\mathrm{I}}(z) \right| + \left| \psi^{S_p}_{\eta,\mathrm{II}}(z) \right| \leq \frac{1}{3} p^{n/2} + \frac{1}{3} p^{n/2} \\ &< p^{n/2} \leq |\eta_0 S^p_0(z)|. \end{aligned} \tag{3.99}$$

The above strict inequality implies that the inclusion (3.97) holds. This finishes the proof of the lemma. □

We now shortly explain how applying our results to the special case of the Bargmann–Fock space recovers the results by Sodin–Tsirelson (for $\mathbb{C}$, [73, Theorem 1]) and Zrebiec (for $\mathbb{C}^n$, [82, Theorem 1.2]) about the hole probability. They proved that there exist constants $c_1 \geq c_2 > 0$ such that for $r > 0$ large,

$$\exp(-c_1 r^{2n+2}) \leq \mathbb{P}\big(\psi^S_\eta(z) \neq 0, \text{ for all } z \in \mathbb{B}(0,r)\big) \leq \exp(-c_2 r^{2n+2}), \tag{3.100}$$

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where $\mathbb{B}(0,r) = \{z \in \mathbb{C}^n : |z| < r\}$. Let us now fix $r_0 > 0$. Then by estimates (1.14) and (3.87), we get

$$\exp(-c\sqrt{p}^{2n+2}) \le \mathbb{P}\big(\psi_\eta^{S_p}(z) \neq 0, \text{ for all } z \in \mathbb{B}(0,r_0)\big) \le \exp(-c'\sqrt{p}^{2n+2}). \tag{3.101}$$

By using formula (3.17), the inequality (3.101) is equivalent to

$$\exp(-c\sqrt{p}^{2n+2}) \le \mathbb{P}\big(\psi_\eta^{S}(z) \neq 0, \text{ for all } z \in \mathbb{B}(0,\sqrt{p}r_0)\big) \le \exp(-c'\sqrt{p}^{2n+2}). \tag{3.102}$$

Therefore, we recover the estimates (3.100) by approximating a sufficiently large $r > 0$ by $\sqrt{p}r_0$.

Remark 3.19. In the context of a general complete Kähler manifold X, a question related to estimate (3.86) is to find for any relatively compact open subset $U \subset X$ a sequence of orthonormal basis $\{\widetilde{S}_j^p\}_{j=1}^{d_p}$ of $H^0_{(2)}(X,L^p)$, $p \in \mathbb{N}$, such that

$$\sup_{x\in\overline{U}} \sum_{j>K'p^n} |\widetilde{S}_j^p(x)|^2_{h_p} \le Ce^{-cp}, \tag{3.103}$$

where C, K', c are certain positive constants independent of p, and the sum in the left-hand side is taken to be 0 if $d_p = \dim H^0_{(2)}(X,L^p) \le K'p^n$. This question is trivial if $d_p = \mathcal{O}(p^n)$ for $p \gg 0$.

The existence of such a sequence of basis suggests that, on a relatively compact subset, the Bergman projections or Bergman kernels can be approximated by the orthogonal projections or their kernels of a sequence of finite-dimensional subspaces of $H^0_{(2)}(X,L^p)$. Moreover, one may expect a connection between the number (or dimension of the aforementioned subspace) $K'p^n$ and the integration of the dimension density on U,

$$\int_U P_p(x,x)\mathrm{dV}(x). \tag{3.104}$$

4. Random $\mathcal{L}^2$-holomorphic sections and Toeplitz operators

In this section, we always use the setting in Section 2: (X,J,Θ) is a connected complex Hermitian manifold (without boundary), and (L,h_L) is a Hermitian line bundle on X. We do not, however, assume any completeness for Θ or positivity for (L,h_L).

The goal of this section is to introduce a method of 'canonically randomizing' the $\mathcal{L}^2$-holomorphic sections of L on X, in particular when $d = \dim H^0_{(2)}(X,L) = \infty$.

As mentioned in the Introduction, this is achieved by the abstract Wiener space construction from probability theory with an approach via Toeplitz operators from the theory of geometric quantization. This induces a Gaussian probability measure on the space of $\mathcal{L}^2$-holomorphic sections.

4.1. Abstract Wiener spaces

To define a Gaussian probability measure on an infinite dimensional Hilbert space, we employ the construction of abstract Wiener spaces introduced by Gross [36]. We also refer to the article of Sheffield [67] for further motivation and developments on this topic. For

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a (complex) vector space $\mathcal{H}$, a Hermitian norm is a norm on $\mathcal{H}$ which is induced by a Hermitian inner product on it.

Definition 4.1. Let $(\mathcal{H},\|\cdot\|_{\mathcal{H}})$ be a separable Hilbert space of infinite dimension. A Hermitian norm $\|\cdot\|$ is called measurable if for all $\epsilon>0$, there exists a finite-dimensional subspace $F_\epsilon\subset\mathcal{H}$ such that for $F\subset\mathcal{H}$ a subspace of finite dimension with $F\perp F_\epsilon$, one has

$$\mu_{F,\,\|\cdot\|_{\mathcal{H}}}\big(\{x\in F\,:\,\|x\|\geq\epsilon\}\big)<\epsilon, \tag{4.1}$$

where $\mu_{F,\,\|\cdot\|_{\mathcal{H}}}$ denotes the standard Gaussian measure on F with respect to the Hermitian metric associated with $\|\cdot\|_{\mathcal{H}}$.

Proposition 4.2 (cf. [36],[49, Chapter I: Theorem 4.3]). *Let $\mathcal{H}$ be a separable Hilbert space with norm $\|\cdot\|_{\mathcal{H}}$, and $\|\cdot\|$ be a continuous (with respect to $\|\cdot\|_{\mathcal{H}}$) Hermitian norm on $\mathcal{H}$. Then the following two conditions are equivalent:*

1. *$\|\cdot\|$ is measurable.*
2. *There exists a one-to-one Hilbert–Schmidt operator T on $\mathcal{H}$ such that $\|x\|=\|Tx\|_{\mathcal{H}}$ for $x\in\mathcal{H}$.*

Given a measurable Hermitian norm $\|\cdot\|$ on $\mathcal{H}$, let $\mathcal{B}$ be the completion of $\mathcal{H}$ with respect to $\|\cdot\|$. Then $(\mathcal{B},\|\cdot\|)$ is a separable Hilbert space containing $\mathcal{H}$ as a dense subspace. Let $\mathcal{B}^*$ be the topological dual space of $\mathcal{B}$. If $\alpha\in\mathcal{B}^*$, then $\alpha|_{\mathcal{H}}$ is a continuous linear functional on $(\mathcal{H},\|\cdot\|_{\mathcal{H}})$. If α vanishes identically on $\mathcal{H}$, then it vanishes on $\mathcal{B}$. This way, we can regard $\mathcal{B}^*$ as a (dense) subspace of $\mathcal{H}^*$, where $\mathcal{H}^*$ can be identified with $\mathcal{H}$ via the Hilbert metric associated with $\|\cdot\|_{\mathcal{H}}$.

By a slight abuse of notation, we denote by $\mathcal{S}$ the Borel σ-algebra of $\mathcal{B}$. Then each $\alpha\in\mathcal{B}^*$ is a Borel-measurable function from $\mathcal{B}$ to $\mathbb{C}$. For $F\subset\mathcal{B}^*\subset\mathcal{H}$ an arbitrary finite-dimensional subspace we introduce the notation

$$\phi_F:\mathcal{B}\to F,\,\phi_F(b)=\sum_{j=1}^{\dim_{\mathbb{C}}F}(b,v_j)v_j, \tag{4.2}$$

where $\{v_j\}$ is an orthonormal basis of $(F,\|\cdot\|_{\mathcal{H}})$. Gross [36] proved the following result.

Theorem 4.3. *Fix a measurable norm $\|\cdot\|$ on $\mathcal{H}$ as above. There exists a unique probability measure $\mathcal{P}$ on $(\mathcal{B},\mathcal{S})$ such that for $F\subset\mathcal{B}^*$ any finite-dimensional subspace,*

$$\mathcal{P}(\phi_F^{-1}(U))=\mu_{F,\,\|\cdot\|_{\mathcal{H}}}(U), \tag{4.3}$$

for all Borel subset U of F. The triple $(\mathcal{B},\mathcal{S},\mathcal{P})$ is called an abstract Wiener space.

If $\alpha\in\mathcal{B}^*$, then as a function on $\mathcal{B}$, it is an element of $\mathcal{L}^2(\mathcal{B},\mathcal{S},\mathcal{P})$. We denote this map by

$$\Phi_0:\mathcal{B}^*\to\mathcal{L}^2(\mathcal{B},\mathcal{S},\mathcal{P}). \tag{4.4}$$

Moreover, for $\alpha \in \mathcal{B}^*$, $\Phi_0(\alpha)$ is a Gaussian random variable with zero mean and variance $\|\alpha\|^2_{\mathcal{H}}$. The map Φ_0 extends to a continuous linear map

$$\Phi : \mathcal{H}^* \simeq \mathcal{H} \to \mathcal{L}^2(\mathcal{B},\mathcal{S},\mathcal{P}), \tag{4.5}$$

where for $y \in \mathcal{H}$, $\Phi(y)$ is a Gaussian random variable with zero mean and variance $\|y\|^2_{\mathcal{H}}$.

Remark that the above construction is trivial if $\mathcal{H}$ is finite dimensional; indeed, in this case the Hilbert space $\mathcal{B}$ is reduced to $\mathcal{H}$ itself. The probability measure constructed in Theorem 4.3 is the standard Gaussian probability measure on $\mathcal{H}$ with respect to the norm $\|\cdot\|_{\mathcal{H}}$.

4.2. Toeplitz operators on $H^0_{(2)}(X,L)$

Recall that P denotes the orthogonal projection from $\mathcal{L}^2(X,L)$ onto $H^0_{(2)}(X,L)$, and $P(x,y)$, $x,y \in X$, denotes the corresponding Bergman kernel. W.l.o.g. we may and do always assume that $d = \dim H^0_{(2)}(X,L) \geq 1$ in the following.

Definition 4.4. For a bounded function $f \in \mathscr{C}^\infty(X,\mathbb{C})$, set

$$T_f : H^0_{(2)}(X,L) \to H^0_{(2)}(X,L), \quad T_f := Pf, \tag{4.6}$$

where the action of f is the pointwise multiplication by f. The operator T_f is called Toeplitz operator associated with f. Equivalently, we consider $T_f : \mathcal{L}^2(X,L) \to \mathcal{L}^2(X,L)$, $T_f := PfP$.

The integral kernel of T_f is given by

$$T_f(x,x') = \int_X P(x,x'')f(x'')P(x'',x')\mathrm{dV}(x''). \tag{4.7}$$

Note also that the Hilbert adjoint of T_f is $T_{\overline{f}}$. We introduce a class of bounded smooth functions on X whose associated Toeplitz operators are Hilbert–Schmidt.

Definition 4.5. Let $\mathcal{Q}(X,L;\mathbb{C})$ be the vector space of bounded smooth complex functions f on X such that

$$\int_X |f(x)|P(x,x)\mathrm{dV}(x) < \infty, \tag{4.8}$$

where P is the Bergman kernel of L.

Example 4.6. (1) It is clear that $\mathscr{C}^\infty_c(X,\mathbb{C})$ is a subspace of $\mathcal{Q}(X,L;\mathbb{C})$. In particular, if X is compact, then

$$\mathcal{Q}(X,L;\mathbb{C}) = \mathscr{C}^\infty(X,\mathbb{C}). \tag{4.9}$$

(2) Let $\mathscr{C}^\infty_b(\mathbb{C}^n,\mathbb{C})$ denote the set of bounded smooth functions on $\mathbb{C}^n$. In the case of the Bargmann–Fock space (see Example 2.14), we have

$$\mathcal{Q}(\mathbb{C}^n,L;\mathbb{C}) = \mathscr{C}^\infty_b(\mathbb{C}^n,\mathbb{C}) \cap \mathcal{L}^1(\mathbb{C}^n,\mathrm{dV}). \tag{4.10}$$

(3) In general, with the assumptions as in Section 3, if we assume further that (X,J,Θ), (L,h_L) have bounded geometry, by [60, Theorem 6], there exist $c > 0$, $C > 0$ and $p_0 \in \mathbb{N}^*$

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such that for $p \geq p_0$,

$$cp^n \leq \inf_{x\in X} P_p(x,x) \leq \sup_{x\in X} P_p(x,x) \leq Cp^n, \tag{4.11}$$

that is, the Bergman kernel function $P_p(x,x)$ is bounded from above and away from zero on X. As a consequence, we get that for $p \geq p_0$,

$$\mathcal{Q}(X,L^p;\mathbb{C}) = \mathscr{C}_b^\infty(X,\mathbb{C}) \cap \mathcal{L}^1(X,\mathrm{dV}). \tag{4.12}$$

Proposition 4.7. *For $f \in \mathcal{Q}(X,L;\mathbb{C})$, the operator T_f on $H^0_{(2)}(X,L)$ has smooth Schwartz kernel and is Hilbert–Schmidt.*

Proof. If $d = \dim H^0_{(2)}(X,L) < \infty$, then the statement is trivial. Hence, we assume $d = \infty$ w.l.o.g. in the sequel. Let $\{S_j\}_{j=1}^\infty$ be a complete Hilbert basis of $H^0_{(2)}(X,L)$. Note that by [2, Proposition (2.4)], for any compact set $K \subset X$, the series

$$\sum_{j=1}^{\infty} |S_j(x)|^2_{h_L} \tag{4.13}$$

converges uniformly for $x \in K$. As a consequence, for $K_1,K_2 \subset X$ compact, the series

$$\sum_{j=1}^{\infty} S_j(x) \otimes (S_j(y))^* \tag{4.14}$$

converges absolutely and uniformly for $x \in K_1$ and $y \in K_2$.

As follows from the properties of holomorphic functions, if we replace $S_j(x)$, $(S_j(y))^*$ by their respective covariant derivatives, then the series (4.13) and (4.14) are still absolutely convergent on any given compact subsets.

By Definition 4.5, for $j \in \mathbb{N}^*$, the function $X \ni x \mapsto f(x)|S_j(x)|^2_{h_L}$ is integrable on X with respect to dV. Furthermore, for $x' \in X$, $i,j \in \mathbb{N}^*$, we have

$$f(x')(S_i(x'))^* S_j(x') = f(x') h_{L,x'}(S_j(x'),S_i(x')), \tag{4.15}$$

and

$$\int_X \left| f(x')\,(S_i(x'))^*\, S_j(x')\right| \mathrm{dV}(x') \leq \left\| \sqrt{|f|} S_i \right\|_{\mathcal{L}^2(X,L)} \cdot \left\| \sqrt{|f|} S_j \right\|_{\mathcal{L}^2(X,L)}. \tag{4.16}$$

Now, we fix two compact subsets K_1, $K_2 \subset X$. For $x \in K_1$, $y \in K_2$ and $i,j \in \mathbb{N}^*$ we have

$$\left| S_i(x) \otimes (S_i(x'))^* f(x') S_j(x') \otimes (S_j(y))^* \right| \leq |S_i(x)|_{h_L} \cdot \left| f(x')(S_i(x'))^* S_j(x') \right| \cdot |S_j(y)|_{h_L}, \tag{4.17}$$

where the norm in the left-hand side is given by $h_{L,x} \otimes h^*_{L,y}$. By inequality (4.16) this entails

$$\begin{aligned} &\int_X \left| S_i(x) \otimes (S_i(x'))^* f(x') S_j(x') \otimes (S_j(y))^* \right| \mathrm{dV}(x') \\ &\quad \leq |S_i(x)|_{h_L} \left\| \sqrt{|f|} S_i \right\|_{\mathcal{L}^2(X,L)} \cdot \left\| \sqrt{|f|} S_j \right\|_{\mathcal{L}^2(X,L)} |S_j(y)|_{h_L}. \end{aligned} \tag{4.18}$$

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Putting things together, we arrive at

$$\begin{aligned}\sum_{i=1}^{\infty}|S_i(x)|_{h_L}\|\sqrt{|f|}S_i\|_{\mathcal{L}^2(X,L)} &\leq \Big(\sum_{i=1}^{\infty}|S_i(x)|^2_{h_L}\Big)^{1/2}\Big(\sum_{i=1}^{\infty}\|\sqrt{|f|}S_i\|^2_{\mathcal{L}^2(X,L)}\Big)^{1/2}\\ &= \Big(\sum_{i=1}^{\infty}|S_i(x)|^2_{h_L}\Big)^{1/2}\Big(\int_X |f(x')|P(x',x')\mathrm{dV}(x')\Big)^{1/2}\\ &< \infty,\end{aligned} \tag{4.19}$$

and the above estimates still hold if we replace $S_i(x)$ by its covariant derivatives at x.

Recalling the Schwartz kernel of T_f from (4.7), the above calculations show that $T_f(x,y)$ is a smooth section on $X\times X$. For proving that T_f is Hilbert–Schmidt, it only remains to show that

$$\sum_{i,j}\big|\langle T_f S_i,S_j\rangle_{\mathcal{L}^2(X,L)}\big|^2<\infty. \tag{4.20}$$

Indeed, by inequality (4.16), we have

$$\big|\langle T_f S_i,S_j\rangle_{\mathcal{L}^2(X,L)}\big|^2 \leq \Big\|\sqrt{|f|}S_i\Big\|^2_{\mathcal{L}^2(X,L)}\cdot\Big\|\sqrt{|f|}S_j\Big\|^2_{\mathcal{L}^2(X,L)}. \tag{4.21}$$

Then

$$\begin{aligned}\sum_{i,j}\big|\langle T_f S_i,S_j\rangle_{\mathcal{L}^2(X,L)}\big|^2 &\leq \sum_{i,j=1}^{\infty}\Big\|\sqrt{|f|}S_i\Big\|^2_{\mathcal{L}^2(X,L)}\cdot\Big\|\sqrt{|f|}S_j\Big\|^2_{\mathcal{L}^2(X,L)}\\ &= \Big(\int_X |f(x)|P(x,x)\mathrm{dV}(x)\Big)^2<\infty.\end{aligned} \tag{4.22}$$

This completes our proof. □

Corollary 4.8. *If $f\in\mathscr{C}^\infty(X,\mathbb{C})$ is with compact support, then T_f is a Hilbert–Schmidt operator on $H^0_{(2)}(X,L)$. Moreover, T_f is trace class, and*

$$\mathrm{Tr}[T_f]=\int_X f(x)P(x,x)\mathrm{dV}(x). \tag{4.23}$$

4.3. Random $\mathcal{L}^2$-holomorphic sections of L

Let $\mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$ denote the subspace of $\mathcal{Q}(X,L;\mathbb{C})$ consisting of the functions valued in $\mathbb{R}_{\geq 0}$. For $f\in\mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$, T_f is a nonnegative self-adjoint Hilbert–Schmidt (hence compact) operator on $H^0_{(2)}(X,L)$.

Lemma 4.9. *For $0\neq f\in\mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$, the operator $T_f: H^0_{(2)}(X,L)\to H^0_{(2)}(X,L)$ is injective.*

Proof. Since $f\neq 0$, there exists an open subset U of X on which f is strictly positive. If $s\in H^0_{(2)}(X,L)$ is such that $T_f s=0$, then $0=\langle T_f s,s\rangle=\int_X f(x)|s(x)|^2_{h_L}\mathrm{dV}(x)$, and hence

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$s|_U = 0$. Since U is open and s is holomorphic on X, we get $s = 0$. This proves the lemma. $\square$

Fix $f \in \mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$, $f \neq 0$. If $d < \infty$, then the above T_f is actually an isomorphism on the vector space $H^0_{(2)}(X,L)$. Now, we focus on the case of $d = \infty$. Since T_f is compact and injective, it cannot be surjective. Hence, it does not admit a bounded inverse. Moreover, for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$, the operator $T_f - \lambda$ is Fredholm with closed range and Fredholm index 0. Set $D(T_f^{-1}) = \mathrm{Range}(T_f : H^0_{(2)}(X,L) \to H^0_{(2)}(X,L)) \subset H^0_{(2)}(X,L)$, which is a dense subspace. The inverse of T_f is defined as

$$T_f^{-1} : D(T_f^{-1}) \subset H^0_{(2)}(X,L) \to H^0_{(2)}(X,L). \tag{4.24}$$

Let $\sigma(T_f) \subset \mathbb{R}_{\geq 0}$ denote the spectrum of T_f, which is a countable set consisting of two parts: the point spectrum $\sigma_{\mathrm{p}}(T_f) \subset \mathbb{R}_{>0}$ (eigenvalues) and the residual spectrum $\sigma_{\mathrm{res}}(T_f) = \{0\}$. In this case, the point spectrum of T_f is a decreasing sequence of strictly positive eigenvalues of finite multiplicity,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq \ldots \to 0. \tag{4.25}$$

Since any separable (complex) Hilbert space is isometric to the Hilbert space $\ell^2(\mathbb{C})$ by choosing an orthonormal basis, we can choose an orthonormal basis $\{S_j\}_{j=1}^{\infty}$ of $H^0_{(2)}(X,L)$ with respect to the $\mathcal{L}^2$-metric such that

$$T_f S_j = \lambda_j S_j. \tag{4.26}$$

If $S \in H^0_{(2)}(X,L)$, we can write uniquely

$$S = \sum_{j\geq 1} a_j S_j, \quad a_j \in \mathbb{C}. \tag{4.27}$$

Then $(a_j)_j \in \ell^2(\mathbb{C})$, yielding the identification between $H^0_{(2)}(X,L)$ and $\ell^2(\mathbb{C})$.

Since T_f is one-to-one and Hilbert–Schmidt, by Proposition 4.2, $\|\cdot\|_f := \|T_f \cdot\|$ defines a Hermitian measurable norm on $H^0_{(2)}(X,L)$. We denote by $\mathcal{B}_f(X,L)$ the completion of $H^0_{(2)}(X,L)$ with respect to $\|\cdot\|_f$ and set

$$\ell^2_f(\mathbb{C}) = \Big\{ (a_j \in \mathbb{C})_{j\geq 1} \, : \, \sum_{j\geq 1} \lambda_i^2 |a_j|^2 < \infty \Big\}. \tag{4.28}$$

It is clearly a separable Hilbert space, and using the basis as in (4.26), we have

$$\mathcal{B}_f(X,L) \simeq \ell^2_f(\mathbb{C}). \tag{4.29}$$

Proposition 4.10. *Assume $d = \infty$, $0 \neq f \in \mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$. Then the operator T_f extends uniquely to an isomorphism of Hilbert spaces*

$$\widehat{T}_f : \big(\mathcal{B}_f(X,L), \|\cdot\|_f\big) \to \big(H^0_{(2)}(X,L), \|\cdot\|_{\mathcal{L}^2(X,L)}\big). \tag{4.30}$$

Given $0 \neq f \in \mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$, if $d < \infty$, we set

$$\big(\mathcal{B}_f(X,L), \|\cdot\|_f\big) = \big(H^0_{(2)}(X,L), \|\cdot\|_f\big), \quad \text{and} \quad \widehat{T}_f := T_f. \tag{4.31}$$

Then we unify our notation for both cases $d < \infty$ and $d = \infty$.

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Definition 4.11. Denote by $\mathcal{P}_f$ the probability measure from Theorem 4.3 with the choice $\mathcal{B} = \mathcal{B}_f(X,L)$. Let $\mathbb{P}_f$ be the Gaussian probability measure on $H^0_{(2)}(X,L)$ given by the pushforward of $\mathcal{P}_f$ through the isomorphism (4.30). We call $\mathbb{P}_f$ the Gaussian probability measure induced by the spectral decomposition of the Toeplitz operator T_f. This way, we randomize the sections in $H^0_{(2)}(X,L)$.

Remark 4.12. When $0 < d < \infty$, then $\mathcal{B}_f(X,L) = H^0_{(2)}(X,L)$, and $\mathcal{P}_f = \mathbb{P}_{\mathrm{st}}$ is exactly the standard Gaussian probability measure on $H^0_{(2)}(X,L)$ with respect to the $\mathcal{L}^2$ inner product. So that $\mathbb{P}_f$ is the pushforward of $\mathbb{P}_{\mathrm{st}}$ via the isomorphism T_f.

Lemma 4.13. *Assume $d \geq 1$, $0 \neq f \in \mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$. For any nonzero $S \in H^0_{(2)}(X,L)$, the random variable on $(H^0_{(2)}(X,L),\mathbb{P}_f)$ defined as $H^0_{(2)}(X,L) \ni s \mapsto \langle s,S\rangle_{\mathcal{L}^2(X,L)} \in \mathbb{C}$ is a centered complex Gaussian variable with variance $\|T_f S\|^2_{\mathcal{L}^2(X,L)}$.*

Proof. Note that $T_f S$ is nonzero in $H^0_{(2)}(X,L)$, the linear form

$$H^0_{(2)}(X,L) \ni s' \mapsto \langle s',T_f S\rangle_{\mathcal{L}^2(X,L)} \in \mathbb{C} \tag{4.32}$$

extends to a bounded linear form on $(\mathcal{B}_f(X,L),\|\cdot\|_f)$, hence defines an element in $\mathcal{B}_f(X,L)^*$, denoted by Ψ_S. Then by property (4.3), the random variable $\Psi_S(s')$ with s' having the law $\mathcal{P}_f$, is a centered complex Gaussian variable with variance $\|T_f S\|^2_{\mathcal{L}^2(X,L)}$. Put differently, by construction, for $s' \in \mathcal{B}_f(X,L)$,

$$\Psi_S(s') = \langle \widehat{T}_f s',S\rangle_{\mathcal{L}^2(X,L)}. \tag{4.33}$$

Thus, as a random variable, it is exactly the same as $\langle s,S\rangle_{\mathcal{L}^2(X,L)}$ with s having distribution $\mathbb{P}_f$. This completes the proof. $\square$

4.4. Zeros of random $\mathcal{L}^2$-holomorphic sections: proof of Theorem 1.6

We assume $d \geq 1$, and we fix $0 \neq f \in \mathcal{Q}(X,L;\mathbb{R}_{\geq 0})$. The operator $T_f^2 := T_f \circ T_f$ on $H^0_{(2)}(X,L)$ is a positive self-adjoint operator of trace class. Let $T_f^2(x,y)$ denote the Schwartz kernel of T_f^2.

Lemma 4.14. *The function $X \ni x \mapsto \log T_f^2(x,x)$ is locally integrable on X so that the $(1,1)$-current $\partial\bar{\partial}\log T_f^2(x,x)$ is well defined on X.*

Proof. Let $\{S_j\}_{j=1}^d$ be the orthonormal basis of $H^0_{(2)}(X,L)$ satisfying the equalities (4.26). Then

$$T_f^2(x,x) = \sum_{j=1}^{d} \lambda_j^2 |S_j(x)|^2_{h_L}, \quad \text{for } x \in X. \tag{4.34}$$

If $d = \infty$, the above sum is uniformly convergent on any compact subset of X. Similar to the proof of Lemma 2.6, we get that the function $\log T_f^2(x,x)$ is a quasi-plurisubharmonic function on X, hence locally integrable. This completes our proof. $\square$

We can now prove Theorem 1.6 for the zeros of the random $\mathcal{L}^2$-holomorphic sections constructed in last subsection.

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Proof of Theorem 1.6. By (4.34), $T_f^2(x,x)$ vanishes exactly on $\mathrm{Bl}(X,L)$. Let $\{S_j\}_{j=1}^d$ be the orthonormal basis of $H^0_{(2)}(X,L)$ satisfying the equalities (4.26). By Lemma 4.13, the complex random variables

$$\eta_j := \frac{1}{\lambda_j}\langle \boldsymbol{s}, S_j\rangle_{\mathcal{L}^2(X,L)},\ j=1,2,\dots \tag{4.35}$$

form an i.i.d. sequence of standard centered complex Gaussian variable. As a consequence, we get that for $x\in X$,

$$\boldsymbol{s}(x) = \sum_j \eta_j\lambda_j S_j(x). \tag{4.36}$$

Recalling the definition of Fubini–Study currents from formula (1.17), we can proceed as in the proof of Theorem 1.1, replacing $P(x,x)$ by $T_f^2(x,x)$ given in formula (4.34), and we conclude formula (1.18). □

Remark 4.15. In the above proof, we observe that the random $\mathcal{L}^2$-holomorphic section $\boldsymbol{s}$ on $(H^0_{(2)}(X,L),\mathbb{P}_f)$ is equivalent to the construction given in formula (4.36), as we explained in the Introduction (cf. Equation (1.2)). More precisely, denote by $S=\{S_j\}_{j=1}^d$ the orthonormal basis of $H^0_{(2)}(X,L)$ as given in formula (4.26), and write ψ_η^S for the Gaussian random holomorphic section defined by formula (2.8), which can be regarded as a random variable valued in $\mathcal{B}_f(X,L)$. Then the distribution of the identity on $(H^0_{(2)}(X,L),\mathbb{P}_f)$ coincides with the distribution of the random section $\widehat{T}_f\psi_\eta^S$.

Remark 4.16. The Fubini–Study currents (1.17) associated to f have the following geometric interpretation, analogous to that of the usual Fubini–Study currents (1.4); see Remark 2.7. Assume that that $H^0_{(2)}(X,L)$ has no base locus, and consider the Toeplitz–Kodaira map $\Phi_f : X\to\mathbb{CP}^{d-1}$, given by $\Phi_f(x) = [(T_fS_j(x)/e_L(x))_j]$, where e_L is a local holomorphic frame around x ($d=\infty$ is allowed). As in Remark 2.7, formula (4.34) yields $\Phi_f^*\omega_{\mathrm{FS}} = \gamma_f(L,h_L)$. Thus, the Fubini–Study current $\gamma_f(L,h_L)$ is the pull-back of the Fubini–Study metric by the Toeplitz–Kodaira map Φ_f.

Remark 4.17. Note that in the above constructions we consider a nonnegative real function f in order to guarantee the injectivity of T_f on $H^0_{(2)}(X,L)$. We can also consider another setting. Here, we do not need the injectivity of T_f. Let $f\in\mathcal{Q}(X,L;\mathbb{R})$, which can be negative on a subset of X. Set

$$H^0_{(2)}(X,L,f) := (\ker T_f)^\perp = \overline{T_fH^0_{(2)}(X,L)} \subset H^0_{(2)}(X,L). \tag{4.37}$$

It is a Hilbert space, and the sections in $H^0_{(2)}(X,L,f)$ are the $\mathcal{L}^2$-holomorphic sections of L *detected* by f. We consider the (self-adjoint) Hilbert–Schimdt operator

$$T_f^\sharp := T_f|_{H^0_{(2)}(X,L,f)} : H^0_{(2)}(X,L,f)\to H^0_{(2)}(X,L,f). \tag{4.38}$$

Then we can proceed as in Subsection 4.3 to construct a Gaussian probability measure $\mathbb{P}^{\sharp}_f$ on $H^0_{(2)}(X,L,f)$. Let $\boldsymbol{s}^{\sharp}$ denotes the corresponding random section in $H^0_{(2)}(X,L,f)$ given by identity map, then

$$\mathbb{E}^{\mathbb{P}^{\sharp}_f}\left[[\mathrm{Div}(\boldsymbol{s}^{\sharp})]\right] = \gamma_f(L,h_L), \tag{4.39}$$

where $\gamma_f(L,h_L)$ is given by formula (1.17).

One step further, set $m(f) := \dim\ker T_f \in \mathbb{N}\cup\{\infty\}$, and let $\{S_j\}_{j=1}^{m(f)}$ be an orthonormal Hilbert basis of $\ker T_f$, then the Schwartz kernel of the orthogonal projection $P_{\ker T_f}$ is given as

$$P_{\ker T_f}(x,y) = \sum_{j=1}^{m(f)} S_j(x)\otimes (S_j(y))^*. \tag{4.40}$$

Let $\psi_{\ker T_f}$ be a standard Gaussian holomorphic section constructed from the Hilbert space $\ker T_f$. We define a Gaussian random holomorphic section as follows,

$$\hat{\boldsymbol{s}} = \psi_{\ker T_f} + \boldsymbol{s}^{\sharp}. \tag{4.41}$$

If $m(f) < \infty$, then $\hat{\boldsymbol{s}}$ is valued in $H^0_{(2)}(X,L)$; otherwise, it is almost never $\mathcal{L}^2$-integrable on X. Then we have

$$\mathbb{E}\left[[\mathrm{Div}(\hat{\boldsymbol{s}})]\right] = c_1(L,h_L) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(T_f^2(x,x) + P_{\ker T_f}(x,x)\right). \tag{4.42}$$

Note that since f is bounded on X, then we always have

$$T_f^2(x,x) \leq T_f^2(x,x) + P_{\ker T_f}(x,x) \leq \max\{\|f\|_\infty^2, 1\}P(x,x), \tag{4.43}$$

where $\|f\|_\infty$ is the $\mathcal{L}^\infty$-norm of f on X.

We will consider the above different settings in Subsection 5.4 for high tensor powers of a prequantum line bundle on a complete Kähler manifold.

5. Random $\mathcal{L}^2$-holomorphic sections for high tensor powers

By analogy with Section 3, we want to study the asymptotic behavior of the zeros of the random $\mathcal{L}^2$-holomorphic sections for high tensor powers of a given positive line bundle on X. We make the same assumptions for (X,Θ) and (L,h_L) as in the beginning of Section 3 (or in Subsection 1.2), in particular, we assume the condition (1.6) and the completeness of g^{TX}.

To construct in a natural way the sequence of random $\mathcal{L}^2$-holomorphic sections of L^p, $p\in\mathbb{N}_{>0}$, we use the Toeplitz operators $\{T_{f,p}\}_{p\in\mathbb{N}_{>0}}$ associated with a suitable positive function f on X. Such operators $\{T_{f,p}\}_{p\in\mathbb{N}_{>0}}$ are already well studied in the context of Berezin–Toeplitz quantization.

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5.1. Asymptotics of Toeplitz operators

Recall that P_p denotes the orthogonal projection from $\mathcal{L}^2(X,L^p)$ onto $H^0_{(2)}(X,L^p)$. For a smooth bounded function f on X and $p\in\mathbb{N}_{>0}$, we set

$$T_{f,p}=P_pfP_p. \tag{5.1}$$

This defines a bounded linear operator acting on $H^0_{(2)}(X,L^p)$.

To obtain the asymptotic expansion of the Schwartz kernels of $\{T_{f,p}\}$, we need further assumptions either on the function f or on the geometry of X and L. We are mainly concerned with the following two cases (note that we always assume g^{TX} to be complete).

(I) The geometric data (X,J,Θ) and (L,h_L) are as in Condition 1.2 and the function f be smooth and constant outside a compact subset of X.

(II) The geometric data (X,J,Θ) and (L,h_L) are as in Condition 1.2 and have bounded geometry (cf. Definition 3.2), and f is a bounded smooth function on X with (globally) bounded derivatives (with respect to ∇^{TX} and g^{TX}) of any order.

Theorem 5.1 (cf. [56, Chapter 7],[60], [33, Lemmas 3.11, 3.14 & 4.6]). *Assume that either (I) or (II) hods. Then the Toeplitz operator $\{T_{f,p}\}_{p\in\mathbb{N}}$ has the following properties:*

(i) *For every compact subset $K\subset X$, every $\epsilon>0$, and every $\ell,m\in\mathbb{N}$, there exists $C_{\ell,m,\varepsilon}>0$ such that for $p\geq 1$, $x,x'\in X$ with $d(x,x')>\varepsilon$, we have*

$$|T_{f,p}(x,x')|_{\mathscr{C}^m(K\times K)}\leq C_{\ell,m,\epsilon}p^{-\ell}, \tag{5.2}$$

where the $\mathscr{C}^m$-norm is induced by ∇^{TX}, and h_L, g^{TX}.

(ii) *We have the uniform asymptotic expansion as $p\to\infty$ on any compact subset of X,*

$$T_{f,p}(x,x)=\sum_{\ell=0}^{\infty}\boldsymbol{b}_\ell(f)(x)p^{n-l}+\mathcal{O}(p^{-\infty}), \tag{5.3}$$

where $\boldsymbol{b}_\ell(f)\in\mathscr{C}^\infty(X,\mathbb{C})$, in particular,

$$\boldsymbol{b}_0(f)(x)=\boldsymbol{b}_0(x)f(x), x\in X. \tag{5.4}$$

(iii) *The operator norms of $T_{f,p}$, $p\in\mathbb{N}$, satisfy*

$$\lim_{p\to\infty}||T_{f,p}||=||f||_\infty. \tag{5.5}$$

(iv) *If g is a bounded smooth function on X in the same class as f (Case (I) or (II)), then on any compact subset $K\subset X$, we have the uniform expansion*

$$(T_{f,p}T_{g,p})(x,x)=p^n\boldsymbol{b}_0(x)f(x)g(x)+\mathcal{O}(p^{n-1}), \tag{5.6}$$

the expansion still holds if we take the derivatives with respect to x of any given order on both sides. In particular, in Case (II), we can refine estimate (5.2) to an exponential decay with respect to $\sqrt{p}$, and the results (5.2), (5.3) and (5.6) hold uniformly on the whole manifold X.

The above theorem for Case (I) was mainly proved by Ma and Marinescu in [56, Chapter 7], [57]. For Case (II), it can be proved by a variation of the arguments in [56, Chapter 7] by using the exponential estimate for the Bergman kernel obtained in [60]. This is explained by Finski in [33, Sections 3 & 4].

Remark 5.2. The condition that f has bounded derivatives of any order in case (II) is used to guarantee the full expansion (5.3) and can be relaxed for our purposes. If (X,J,Θ) and (L,h_L) are as in Condition 1.2 and have bounded geometry, and if we assume that two smooth functions f, g have bounded derivatives up to order 2, then the expansion (5.6) still holds uniformly on any given compact subset in the $\mathscr{C}^2$-norm (cf. the proof of [56, Lemma 7.2.4] and [3, Theorem 3.5 and Remark 3.6]).

Remark 5.3. Assume $0 \neq f \in \mathscr{C}^\infty_0(X,\mathbb{R}_{\geq 0})$ to have compact support, hence $T_{f,p}$ is compact for $p \geq 1$. We use the spectrum of the Toeplitz operators $T_{f,p}$ to define probability measures $\mathbb{P}_{f,p}$ on $H^0_{(2)}(X,L^p)$ (Definitions 4.11 and 5.6). It is thus interesting to know what is the asymptotic distribution of the spectra $\sigma(T_{f,p})$ as $p \to \infty$. They play a role in several areas of geometric quantization [7, 41, 51]. On $(0,|f|_{\mathscr{C}^0}]$, the spectral density measure $\mu_{f,p}$ of $T_{f,p}$ is defined as the sum of the Dirac masses at all the eigenvalues (counted with multiplicities) in $\sigma_{\mathrm{p}}(T_{f,p})$, which is locally finite. A result of [41] shows that as $p \to +\infty$, we have the weak convergence of measures on $(0,|f|_{\mathscr{C}^0}]$,

$$p^{-n}\mu_{f,p} \to f_*\left(\frac{1}{n!}c_1(L,h_L)^n\right). \tag{5.7}$$

This extends the results for compact Kähler manifolds or domains in $\mathbb{C}^n$, such as [7, 51].

Our results in the sequel will mainly employ the expansion (5.6) with $g = f$. Note that with further geometric conditions on (X,Θ) and (L,h_L), we have a refined version of formula (5.6). Let Ric denote the Ricci curvature tensor, and set $\mathrm{Ric}_\Theta =: \mathrm{Ric}(J\cdot,\cdot)$. Let $\boldsymbol{r}^X$ denote the scalar curvature of (X,g^{TX}), and let Δ be the (positive) Bochner Laplacian associated with g^{TX} acting on the functions. We will use $\langle\cdot,\cdot\rangle$ to denote the $\mathbb{C}$-linear extension of the inner product $g^{\Lambda^\bullet T^*X}$. Consider the connection ∇^{T^*X} : $\mathscr{C}^\infty(X,T^*X\otimes\mathbb{C}) \to \mathscr{C}^\infty(X,T^*X\otimes T^*X\otimes\mathbb{C})$, let $D^{0,1}$, $D^{1,0}$ denote the its respective $(1,0)$, $(0,1)$ components.

The following theorem was proved in [59] for a compact Kähler manifold equipped with a prequantum line bundle. It was showed in [56, 57, 58, 59] that the problem can be localized and since the computations are local the result extends to the case of complete (noncompact) Kähler manifolds. In particular, as a consequence of [56, Sections 7.4 & 7.5] (for the Case (I)) and [60] [33, Sections 3 & 4] (for Case (II)), these results hold for both our cases (I), (II).

Theorem 5.4. *Assume that (X,J,Θ) is connected, complete Kähler manifold and that (L,h_L) is a prequantum holomorphic line bundle on X with smooth Hermitian metric h_L, that is, $\Theta = c_1(L,h_L)$. Let f,g be bounded smooth functions where are constants outside a compact subset (Case (I)), or if in addition $(X,\Theta),(L,h_L)$ have the bounded geometry, let f,g be two bounded smooth functions on X such that their derivatives of any order are also bounded on X (Case (II)). Then for $\ell \in \mathbb{N}$, there exists a smooth function on X, denoted by $\boldsymbol{b}_\ell(f,g)$, which is a polynomial in the derivatives of f,g with coefficients depending only*

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on Θ and h_L, such that on any compact subset $K \subset X$, we have the uniform expansion as follows ($N \geq 0$),

$$(T_{f,p}T_{g,p})(x,x) = \sum_{\ell=0}^{N} p^{n-\ell} \boldsymbol{b}_\ell(f,g)(x) + \mathcal{O}(p^{n-N-1}). \tag{5.8}$$

Furthermore, we have

$$\begin{aligned} \boldsymbol{b}_0(f,g) &= fg, \\ \boldsymbol{b}_1(f,g) &= \frac{\boldsymbol{r}^X}{8\pi} fg - \frac{1}{4\pi}\big((\Delta f)g + f(\Delta g)\big) + \frac{1}{2\pi}\langle \overline{\partial} f, \partial g\rangle, \\ \boldsymbol{b}_2(f,g) &= \frac{1}{32\pi^2}\Big(f(\Delta^2 g) + (\Delta^2 f) g - \boldsymbol{r}^X \big(f(\Delta g) + (\Delta f) g\big)\Big) \\ &\quad - \frac{\sqrt{-1}}{8\pi^2}\langle \mathrm{Ric}_\Theta, f\partial\bar{\partial} g + g\partial\bar{\partial} f\rangle \\ &\quad + \frac{1}{8\pi^2}\Big\{ \frac{1}{2}\Delta f \cdot \Delta g + \frac{\boldsymbol{r}^X}{2}\langle \overline{\partial} f, \partial g\rangle + \langle D^{0,1}\overline{\partial} f, D^{1,0}\partial g\rangle_{g^{T^*X\otimes T^*X}} \\ &\quad - \langle \overline{\partial}\Delta f, \partial g\rangle - \langle \overline{\partial} f, \partial \Delta g\rangle \Big\}. \end{aligned} \tag{5.9}$$

Remark 5.5. Actually, by [59, Remark 0.5], a version of Theorem 5.4 holds also, without the prequantum condition $\Theta = c_1(L,h_L)$. In this case, we have

$$(T_{f,p}T_{g,p})(x,x) = \sum_{\ell=0}^{N} p^{n-\ell} \boldsymbol{b}^*_\ell(f,g)(x) + \mathcal{O}(p^{n-N-1}). \tag{5.10}$$

with explicit formulas for $\boldsymbol{b}^*_0(f,g)$, $\boldsymbol{b}^*_1(f,g)$, $\boldsymbol{b}^*_2(f,g)$ (cf. [59, (0.30)]), which generalize the expressions (5.9). We will come back to this point in Remark 5.15.

5.2. Random zeros on the support: proofs of Theorems 1.8 and 1.9

Fix a $p_0 \in \mathbb{N}_{>0}$, set

$$\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0}) := \cap_{p\geq p_0} \mathcal{Q}(X,L^p;\mathbb{R}_{\geq 0}). \tag{5.11}$$

Now, we consider a nontrivial $f \in \mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0})$. Note that such function always exists, take for instance a nonnegative smooth function with compact support, or in the case of Bargmann–Fock space, a nonnegative Schwartz function on $\mathbb{C}^n$. In the rest of this section, we always consider an integer $p \geq p_0$.

Definition 5.6. Following the construction in Definition 4.11, let $\mathbb{P}_{f,p}$ be the probability measure on $H^0_{(2)}(X,L^p)$ induced by the spectral decomposition of the Toeplitz operator $T_{f,p}$. We will denote by $\boldsymbol{S}_{f,p}$ the random section in $H^0_{(2)}(X,L^p)$ given by the probability distribution $(H^0_{(2)}(X,L^p),\mathbb{P}_{f,p})$.

Let U be an open subset of X, and let $\Omega_0^{n-1,n-1}(U)$ denote the smooth $(n-1,n-1)$-forms on $\overline{U}$ with compact support in U. For any $(1,1)$-current α on X, let $\alpha|_U$ denote its restriction on U by acting on sections in $\Omega_0^{n-1,n-1}(U)$.

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Theorem 5.7. *Assume that* (X,J,Θ), (L,h_L), *and* f *are as in one of the cases (I) or (II), with* f *a nontrivial function in* $\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0})$. *Let* U *be an open subset of* X *such that* $f>0$ *on* U. *Then for every compact subset of* $K\subset U$, *there exists* $p_K\in\mathbb{N}$ *such that for all* $p\geq p_K$, $\gamma_f(L^p,h_p)$ *is smooth on* K, *and we have the expansion of smooth forms on* K *in any* $\mathscr{C}^\ell$*-norm,*

$$\gamma_f(L^p,h_p)=pc_1(L,h_L)+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\boldsymbol{b}_0f^2)+\mathcal{O}(p^{-1}),\quad p\to+\infty. \tag{5.12}$$

Proof. By formula (1.17), we get

$$\gamma_f(L^p,h_p)=pc_1(L,h_L)+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log T^2_{f,p}(x,x). \tag{5.13}$$

For any compact subset $K\subset U$, set $m_K:=\max_{x\in K}\boldsymbol{b}_0(x)f^2(x)$, $c_K:=\min_{x\in K}\boldsymbol{b}_0(x)f^2(x)>0$. By formula (5.6), for $\ell\in\mathbb{N}$, the following identity holds uniformly in any $\mathscr{C}^\ell(K)$-norm,

$$T^2_{f,p}(x,x)=\boldsymbol{b}_0(x)f^2(x)p^n+\mathcal{O}(p^{n-1}),\ x\in K. \tag{5.14}$$

If we are in Case (II), the asymptotics (5.14) hold uniformly over the whole manifold X.

As a consequence, there exists $p_K\in\mathbb{N}$ such that for $p\gg p_K$, $x\in K$, we have

$$2m_Kp^n\geq T^2_{f,p}(x,x)\geq\frac{1}{2}c_Kp^n. \tag{5.15}$$

Then for $p\geq p_K$, $\gamma_f(L^p,h_p)$ is a smooth form on an open neighborhood of K. Then by the uniform expansion (5.14) and estimate (5.15), we get, in any $\mathscr{C}^\ell(K)$-norm,

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log T^2_{f,p}(x,x)=\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\boldsymbol{b}_0f^2)+\mathcal{O}(p^{-1}),\quad p\to+\infty. \tag{5.16}$$

This way, we conclude the expansion (5.12). □

Remark 5.8. (1) The expansion (5.12) is a generalization of Tian's approximation theorem for a given integral Kähler form ([76, Theorem A], [56, Theorem 5.1.4]) for the case of the Toeplitz–Kodaira map (cf. Remark 4.16).

(2) By the arguments in the proof of [56, Lemma 5.1.6] and the conclusions drawn in Theorem 5.7, we infer that for any relatively compact open subset $V\subset X$ such that $f>0$ on $\overline{V}$, the Toeplitz–Kodaira map restricted to V,

$$\Phi_{f,p}|_V:V\subset X\to\mathbb{CP}^{d_p-1},$$

is an immersion for p large enough. This remains true even when d_p is infinite.

(3) On an open subset where $\boldsymbol{b}_0f^2$ is a nonzero positive constant, we have a sharper remainder for the approximations

$$\frac{1}{p}\gamma_f(L^p,h_p)=c_1(L,h_L)+\mathcal{O}(p^{-2}),\quad p\to+\infty. \tag{5.17}$$

(4) For a compact Kähler manifold X with a prequantum line bundle (L,h_L), Ancona–Le Floch [1] proved the approximation theorem in sense of currents under the condition

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that f vanishes transversally on X:

$$\gamma_f(L^p,h_p) - pc_1(L,h_L) \to \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\ \log f^2,\ p\to+\infty.$$

Combining Theorem 1.6 with Theorem 5.7, we get the following results.

Theorem 5.9. *Assume that (X,J,Θ), (L,h_L) and f are as in one of the cases (I) or (II), with f a nontrivial function in $\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0})$. Let U be an open subset of X such that $f>0$ on U. Let V be an open subset of U which is relatively compact in X, then there exists $p_V\in\mathbb{N}$ such that for all $p\geq p_V$ $\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_V]$ is the smooth form $\gamma_f(L^p,h_p)|_V$ on V, and we have the convergence of smooth forms on V in any $\mathscr{C}^\ell$-norms as $p\to+\infty$,*

$$\frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_V] \to c_1(L,h_L)|_V. \tag{5.18}$$

Moreover, we have the expansion in any $\mathscr{C}^\ell$-norms on V for $p\gg 0$,

$$\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_V] - pc_1(L,h_L)|_V = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\boldsymbol{b}_0 f^2) + \mathcal{O}(p^{-1}). \tag{5.19}$$

As a consequence, we have the weak convergence of $(1,1)$-currents on U as $p\to+\infty$,

$$\begin{aligned}\frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U] &\to c_1(L,h_L)|_U,\\ \mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U] - pc_1(L,h_L)|_U &\to \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\boldsymbol{b}_0 f^2).\end{aligned} \tag{5.20}$$

Remark 5.10. In line with Remark 5.2, for the case of bounded geometry, to show the weak convergence in terms of currents in the limit (5.20), it is sufficient to require that the function f has bounded derivatives up to order 2, since we only need the asymptotics (5.14) to hold in local $\mathscr{C}^2$-norms.

By considering sequences of random sections in the product probability space,

$$(\boldsymbol{S}_{f,p})_p \in \prod_p \left(H^0_{(2)}(X,L^p),\mathbb{P}_{f,p}\right), \tag{5.21}$$

we also have the following convergence with probability one.

Theorem 5.11. *Let f be as in Theorem 5.9. Let U be an open subset of X such that $f>0$ on U, then for any $\varphi\in\Omega_0^{n-1,n-1}(U)$, we have*

$$\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}\left\langle[\mathrm{Div}(\boldsymbol{S}_{f,p})],\varphi\right\rangle = \langle c_1(L,h_L),\varphi\rangle\right) = 1. \tag{5.22}$$

If there exists a smooth closed positive $(1,1)$-form α on X such that $\int_X c_1(L,h_L)\wedge\alpha^{n-1}<\infty$ and that α is strictly positive on $\overline{U}$, then we have

$$\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U = c_1(L,h_L)|_U\right) = 1, \tag{5.23}$$

where the limit is taken with respect to the weak convergence of $(1,1)$-currents on U.

Proof. Fix a nonzero $\varphi\in\Omega_0^{n-1,n-1}(U)$. Note that from the proof of Theorem 5.7, we have the convergence

$$\lim_{p\to\infty}\Big\langle\frac{1}{p}\gamma_f(L^p,h_p),\varphi\Big\rangle=\langle c_1(L,h_L),\varphi\rangle. \tag{5.24}$$

Defining the random variable

$$Y_{f,p}=\frac{1}{p}\Big\langle[\mathrm{Div}(\boldsymbol{S}_{f,p})]-\gamma_f(L^p,h_p),\varphi\Big\rangle, \tag{5.25}$$

the statement (5.22) is equivalent to proving that almost surely one has

$$Y_{f,p}\to 0. \tag{5.26}$$

Note that if we use the construction from the proof of Theorem 1.6, we can write

$$\boldsymbol{S}_{f,p}=\sum_{j=1}^{d_p}\eta_j^p\lambda_j^pS_j^p, \tag{5.27}$$

where $\{\eta_j^p\}_j$ is a sequence of i.i.d. standard complex Gaussian random variables, $\{\lambda_j^p\}_j$ is the point spectrum of $T_{f,p}$, and $\{S_j^p\}_j$ is the orthonormal basis of $H^0_{(2)}(X,L^p)$ given by the eigensections of $T_{f,p}$. Then, as explained in Remark 3.10, we can proceed as in the proof of Theorem 3.7, and we get $\mathbb{E}[|Y_{f,p}|^2]=\mathcal{O}(p^{-2})$, which entails the convergence (5.26), and hence equality (5.22).

Now, we prove (5.23). Note that by our assumptions on α, the analogous arguments in the proof of Corollary 3.8 show that there exists $C_{U,\alpha}>0$ such that for all $\varphi\in\Omega_0^{n-1,n-1}(U)$,

$$\frac{1}{p}|\langle[\mathrm{Div}(s_p)],\varphi\rangle|\leq C_{U,\alpha}|\varphi|_{\mathscr{C}^0(U,\alpha)}, \tag{5.28}$$

where $\mathscr{C}^0(U,\alpha)$ denotes the $\mathscr{C}^0$-norm for differential forms on U with respect to the Hermitian metric induced by α. Then the equality (5.23) follows as a consequence of the equality (5.22) by considering a countable $\mathscr{C}^0(U,\alpha)$-dense family of φ's in $\Omega_0^{n-1,n-1}(U)$. □

It is natural to investigate a relaxation of the assumptions of Theorem 5.11 as follows. For f as above, consider U an open subset of $\operatorname{supp} f$. In general, f might vanish at some points in U, and it is a natural and interesting question to understand for which kind of conditions on the zeros of f in U we still can have the equidistribution results for the random zeros on U as above. Since f is nonnegative, if $f(x_0)=0$, the least possible vanishing order of f at x_0 is 2. In the sequel, we will explain that, if f has only zeros of order 2 at which Δf does not vanish, then the above results still hold. For this purpose, we will employ the results in Theorem 5.4 so that we need the prequantum condition.

Proposition 5.12. *Assume that (X,J,Θ), (L,h_L), f are as in either cases (I) or (II) with f a nontrivial function in $\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0})$ and the prequantum condition $\Theta=c_1(L,h_L)$ holds. Let U be an open subset of X. If f only vanishes up to order 2 in U and Δf is* *nonzero at all zeros of f, then for any compact subset K of U, there exists a constant* $c_K > 0$ *and* $p_K \geq p_0$ *such that for* $x \in K$, $p \geq p_K$,

$$T_{f,p}^2(x,x) \geq c_K p^{n-2}. \tag{5.29}$$

Moreover, $\log f^2$ *is locally integrable on U, and we have the weak convergence of currents on U as* $p \to +\infty$,

$$\partial\overline{\partial}\log T_{f,p}^2(x,x) \to \partial\overline{\partial}\log f^2. \tag{5.30}$$

Around a point x where $f(x) > 0$*, the convergence in (5.30) holds in any local* $\mathscr{C}^\ell$*-norms.*

Proof. With our assumptions, we can apply Theorem 5.4 to $T_{p,f}^2$. Let $x_0 \in U$ with $f(x_0) = 0$. By our assumption on f, we have

$$-\Delta f(x_0) \neq 0. \tag{5.31}$$

By taking a geodesic normal coordinate system ($Y = (y_j)_{j=1}^{2n} \in \mathbb{R}^{2n}$) centered at x_0, the Taylor expansion of f near x_0 reads

$$f(Y) = \sum_j c_j(x_0) y_j^2 + \mathcal{O}(|Y|^3), \tag{5.32}$$

where $c_j(x_0) \geq 0$ since $f \geq 0$. Then

$$-\Delta f(x_0) = \sum_j c_j(x_0) > 0. \tag{5.33}$$

Now, we compute the terms $\boldsymbol{b}_\ell(f,f)$, $\ell = 1,2$, from (5.9) near x_0,

$$\begin{aligned} \boldsymbol{b}_1(f,f) &= \frac{1}{8\pi}(\boldsymbol{r}^X f - 4\Delta f)f + \frac{1}{2\pi}|\partial f|^2, \\ \boldsymbol{b}_2(f,f) &= \frac{1}{4\pi^2}\Big(\sum_j c_j(x_0)\Big)^2 + \frac{1}{8\pi^2}|D^{0,1}\overline{\partial} f(x_0)|^2_{g^{T^*X\otimes T^*X}} + \mathcal{O}(|Y|). \end{aligned} \tag{5.34}$$

Setting

$$\mu(f,x_0) = \frac{1}{4\pi^2}\Big(\sum_j c_j(x_0)\Big)^2 + \frac{1}{8\pi^2}|D^{0,1}\overline{\partial} f(x_0)|^2_{g^{T^*X\otimes T^*X}} > 0, \tag{5.35}$$

we can choose a small open neighborhood V_{x_0} of x_0 such that for $x \in V_{x_0}$,

$$\boldsymbol{r}_x^X f(x) - 4\Delta f(x) \geq 0, \quad \text{and} \quad \boldsymbol{b}_2(f,f)(x) \geq \frac{1}{2}\mu(f,x_0), \tag{5.36}$$

and so

$$\boldsymbol{b}_1(f,f)(x) \geq 0. \tag{5.37}$$

Since $\boldsymbol{b}_0(f,f) = f^2$, then from the above computations and estimate (5.15), we get estimate (5.29).

By formula (5.32), on a sufficiently small open neighborhood of x_0, we have

$$f(Y) \geq \frac{1}{2}\sum_j c_j(x_0)y_j^2. \tag{5.38}$$

Then it is clear that $\log f^2$ is integrable near x_0. Then the current $\partial\overline{\partial}\log f^2$ is well defined on U. Near a point where f does not vanish, we get the strong convergence in the limit (5.30) by means of asymptotics (5.6) and (5.16).

Next, we focus on a point x_0 with $f(x_0) = 0$. Note that

$$p^{-n}T^2_{f,p}(x,x) = f^2 + \boldsymbol{b}_1(f,f)p^{-1} + \boldsymbol{b}_2(f,f)p^{-2} + \mathcal{O}(p^{-3}). \tag{5.39}$$

By formulas (5.34), we can take a small open neighborhood V'_{x_0} of x_0 such that for $x \in V'_{x_0}$, $p \gg 0$,

$$\boldsymbol{b}_1(f,f)(x)p^{-1} + \boldsymbol{b}_2(f,f)(x)p^{-2} + \mathcal{O}(p^{-3}) \geq 0, \quad \text{and} \quad f^2(x) \leq p^{-n}T^2_{f,p}(x,x) \leq 1. \tag{5.40}$$

Then on V'_{x_0}, we have

$$\left|\log(p^{-n}T^2_{f,p}(x,x))\right| \leq |\log f^2(x)|. \tag{5.41}$$

At the same time we have the pointwise convergence of functions as $p \to \infty$,

$$\log(p^{-n}T^2_{f,p}(x,x)) \to \log f^2(x). \tag{5.42}$$

Since $\log f^2$ is integrable near x_0, by the dominated convergence theorem, we get the convergence of (1,1)-currents in (5.30) on V'_{x_0}, hence on U. This completes the proof. □

Remark 5.13. In the proof of Proposition 5.12, we see that if f has at least one zero in $K \subset U$, then the power $(n-2)$ in estimate (5.29) cannot be improved; otherwise, a lower bound of $T^2_{f,p}(x,x)$ on K is given in estimate (5.15). When X is compact, this observation indicates that if $f \geq 0$ has only proper zeros of order 2 and at least one of such vanishing point, then the lowest eigenvalue of $T_{f,p}$ should behave like $\mathcal{O}(\frac{1}{p})$ as p grows. For this kind of results, we refer to the papers [19, 20] of Deleporte. In particular, when X is compact, the lower bound in estimate (5.29) can be deduced from [19].

As a direct consequence of Proposition 5.12, we obtain:

Theorem 5.14. *Assume that (X,J,Θ), (L,h_L), f are as in either cases (I) or (II) with f a nontrivial function in $\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0})$ and the prequantum condition $\Theta = c_1(L,h_L)$ holds. Let U be an open subset of* $\operatorname{supp} f$ *such that f only vanishes up to second order in U with nonzero Δf at the zeros. Then as $p \to \infty$,*

(i) *We have the weak convergence of* (1,1)*-currents on U as $p \to +\infty$,*

$$\begin{aligned} \frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U] &\to c_1(L,h_L)|_U, \\ \mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U] - pc_1(L,h_L)|_U &\to \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log f^2. \end{aligned} \tag{5.43}$$

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For any $\varphi \in \Omega_0^{n-1,n-1}(U)$, *we have*

$$\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}\langle[\mathrm{Div}(\boldsymbol{S}_{f,p})],\varphi\rangle = \langle c_1(L,h_L),\varphi\rangle\right) = 1. \tag{5.44}$$

(ii) *If there is a smooth closed positive* (1,1)*-form* α *on* X *with* $\int_X c_1(L,h_L)\wedge\alpha^{n-1} < \infty$ *such that* α *is strictly positive on* $\overline{U}$, *then we have*

$$\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U = c_1(L,h_L)|_U\right) = 1, \tag{5.45}$$

where the limit is taken with respect to the weak convergence of (1,1)*-currents on* U.

Remark 5.15. Following Remark 5.5, we have more general versions of Proposition 5.12 and Theorem 5.14 without the prequantum condition. In fact, let $\boldsymbol{b}_j^*(f,g)$, $j = 0,1,2,\ldots$, denote the coefficients in the general expansion (5.10), valid without assuming the prequantum condition. We still consider the same function f in Proposition 5.12, and a point $x_0 \in U$ such that $f(x_0) = 0$, $-\Delta f(x_0) > 0$. Then by [59, (0.17), (0.30) and Remark 0.5], we get that, on a very small neighbourhood of x_0, we have

$$\begin{aligned}
&\boldsymbol{b}_0^*(f,f)(x) = \boldsymbol{b}_0(x)f(x)^2 \geq 0,\\
&\boldsymbol{b}_1^*(f,f)(x) \geq 0,\\
&\boldsymbol{b}_2^*(f,f)(x) = \boldsymbol{b}_0(x)\mu(f,x_0) + \mathcal{O}(|Y|),
\end{aligned} \tag{5.46}$$

where $\boldsymbol{b}_0(x)$ is the positive function given in (3.3), and Y denote the normal coordinates centered at x_0. Note that for $\boldsymbol{b}_1^*(f,f)(x) \geq 0$, we use that $-\Delta f(x_0)$ dominates the terms f, $\partial f/\partial y_j$, $j = 1,\ldots,2n$, near the point x_0. Then we can proceed as in the proof of Proposition 5.12 to get the same conclusions. All the results in Theorem 5.14 still hold except that we need to change the second line in (5.43) to the following limit as $p \to +\infty$,

$$\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U] - pc_1(L,h_L)|_U \to \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log f^2 + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\boldsymbol{b}_0. \tag{5.47}$$

Example 5.16. We give here simple examples for the situation of Theorem 5.14.

(1) Let $X = \mathbb{D}$ be the unit disc in $\mathbb{C}$ endowed with the hyperbolic metric $\omega := \omega_B$ given by Equation (2.41). This metric induces a metric $h_{K_\mathbb{D}}$ on the canonical bundle $K_\mathbb{D}$ with $c_1(K_\mathbb{D},h_{K_\mathbb{D}}) = \omega$, hence $(K_\mathbb{D},h_{K_\mathbb{D}})$ is a prequantum line bundle on $(\mathbb{D},\omega)$. For $r \in (0,1)$ consider a cutoff function $\chi \in \mathscr{C}_0^\infty(\mathbb{D})$ such that $\chi = 1$ on $B_r(0)$. We define $f \in \mathscr{C}_0^\infty(\mathbb{D})$ by $f(z) = \chi(z)|z|^2$. Then we have on $U = B_r(0)$, with δ_0 the delta distribution at 0,

$$\frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]|_U] \to \omega|_U, \quad \mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})|_U]] - p\omega|_U \to 2\delta_0,\ p \to \infty, \tag{5.48}$$

since by Lelong–Poincaré formula we have $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log|z|^2 = \delta_0$.

(2) Let $X = \mathbb{CP}^1$ endowed with the Fubini–Study metric ω_{FS} and $(L,h_L) = (\mathscr{O}(1),h_{FS})$ be the hyperplane line bundle endowed with the Fubini–Study metric, so $c_1(L,h_L) = \omega_{FS}$. We define $f \in \mathscr{C}^\infty(\mathbb{CP}^1)$ in homogeneous coordinates $[z_0,z_1]$ on $\mathbb{CP}^1$ by $f([z_0,z_1]) = |z_0|^2/(|z_0|^2 + |z_1|^2)$ (note that f is a rescaling of the height function on the sphere $S^2 \subset \mathbb{R}^3$).

Then we have on $\mathbb{CP}^1$,

$$\frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]] \to \omega_{FS}, \quad \mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]] - (p+2)\omega_{FS} \to 2\delta_{[0,1]}, \; p \to \infty, \tag{5.49}$$

since $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log f = \delta_{[0,1]} - \omega_{FS}$, with $\delta_{[0,1]}$ the delta distribution at the point $[0,1] \in \mathbb{CP}^1$.

(3) The previous example can be generalized as follows. Let X be a compact Kähler manifold, (L,h_L) be a prequantum holomorphic line bundle on X. Let $D \subset X$ be a smooth complex hypersurface of X, and $F = \mathscr{O}(D)$ be the associated holomorphic line bundle. Let $s_D \in H^0(X,F)$ be a canonical holomorphic section of F, that vanishes at first order on D. It is uniquely determined up to a multiplicative constant. We endow F with a Hermitian metric h_F and consider $f \in \mathscr{C}^\infty(X)$, $f = |s_D|^2_{h_F}$. By the Lelong–Poincaré formula (2.29), we have on X as $p \to \infty$,

$$\begin{aligned} \frac{1}{p}\mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]] &\to c_1(L,h_L), \\ \mathbb{E}^{\mathbb{P}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}_{f,p})]] - pc_1(L,h_L) &\to 2([D] - c_1(F,h_F)), \end{aligned} \tag{5.50}$$

where $[D]$ is the current of integration on D.

These examples illustrate how one can recover the geometric input (the curvature form $c_1(L,h_L)$) and the classical observable f in the semiclassical limit $p \to \infty$. By recovering the zero divisor D of s_D in the limit (5.50), we have determined s_D up to a constant factor.

5.3. Higher fluctuation of random zeros near points of vanishing order two

In this subsection, we work under the assumption that (X,J,Θ), (L,h_L), f are given in either cases (I) or (II) with f a nontrivial function in $\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0})$ and with prequantum condition $\Theta = c_1(L,h_L)$. We aim to examine the random zeros of $\boldsymbol{S}_{f,p}$ near a proper zero of f with vanishing order 2, up to Planck scale $1/\sqrt{p}$. Note that in [1], for a compact Kähler manifold X and under a different assumption on f, Ancona and Le Floch proved that the random zeros fluctuate more near the zeros of f. We will prove a similar result for our setting. For this purpose, we need to refine the computations in formulas (5.34) in a complex coordinate system centered at x_0 where f vanishes with order 2.

Suppose $f \geq 0$ and that x_0 is a zero of f with $\Delta f(x_0) < 0$. Then we can choose a holomorphic coordinate system centered at x_0, denoted by $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$ such that

$$g^{TX}_z = g^{\mathbb{C}^n}_{\mathrm{st}} + \mathcal{O}(|z|^2), \tag{5.51}$$

where $g^{\mathbb{C}^n}_{\mathrm{st}}$ denotes the standard Euclidean metric on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Note that we view z as a column vector, and let $(\cdot)^T$ denote the transpose of a matrix. In this coordinate system, we can write

$$f(z) = z^T A\bar{z} + z^T Bz + \bar{z}^T \bar{B}\bar{z} + \mathcal{O}(|z|^3), \tag{5.52}$$

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where the matrix A is Hermitian and semipositive definite, B is symmetric complex matrix and they are determined uniquely by the Hessian of f at x_0. Set

$$\hat{f}_{x_0}(z) = z^T A\bar{z} + z^T Bz + \bar{z}^T \bar{B}\bar{z}. \tag{5.53}$$

Since $f \geq 0$, then for any $z \in \mathbb{C}^n$ with $\|z\| = 1$,

$$z^T A\bar{z} \geq 2|\Re(z^T Bz)|, \tag{5.54}$$

where $\Re(\cdot)$ denotes the real part. In particular, $\hat{f}_{x_0}(z) \geq 0$. Using this complex coordinate system, we compute

$$\begin{aligned} \Delta f(z) &= -4\,\mathrm{Tr}[A] + \mathcal{O}(|z|), \\ |\partial f(z)|^2 &= 2|A\bar{z} + 2Bz|^2 + \mathcal{O}(|z|^3), \\ |D^{0,1}\bar{\partial} f(z)|^2_{g^{T^*X\otimes T^*X}} &= 16\,\mathrm{Tr}[B\bar{B}^T] + \mathcal{O}(|z|). \end{aligned} \tag{5.55}$$

Note that $\mu(f,x_0)$ is defined in (5.35), then we have

$$\mu(f,x_0) = \frac{1}{\pi^2}(\mathrm{Tr}[A])^2 + \frac{2}{\pi^2}\mathrm{Tr}[B\bar{B}^T] > 0. \tag{5.56}$$

Then we rewrite the computations in formulas (5.34) as follows:

$$\begin{aligned} \boldsymbol{b}_0(f,f)(z) &= \hat{f}^2_{x_0}(z) + \mathcal{O}(|z|^5), \\ \boldsymbol{b}_1(f,f)(z) &= \frac{2}{\pi}\mathrm{Tr}[A]\hat{f}_{x_0}(z) + \frac{1}{\pi}|A\bar{z} + 2Bz|^2 + \mathcal{O}(|z|^3), \\ \boldsymbol{b}_2(f,f)(z) &= \mu(f,x_0) + \mathcal{O}(|z|). \end{aligned} \tag{5.57}$$

Definition 5.17. Associated with the Kähler form Θ and f near x_0, we define a (strictly) positive function on $\mathbb{C}^n$ as follows:

$$F_{f,x_0}(z) = \hat{f}^2_{x_0}(z) - \frac{1}{2\pi}(\Delta f)(x_0)\hat{f}_{x_0}(z) + \frac{1}{\pi}|A\bar{z} + 2Bz|^2 + \mu(f,x_0). \tag{5.58}$$

Note that this function does not depend on the choice of the holomorphic coordinate systems centered at x_0 satisfying the equality (5.51). Equivalently, we have for $z \in \mathbb{C}^n \simeq (T_{x_0}X, J_{x_0})$,

$$F_{f,x_0}(z) = \lim_{p\to\infty}\left\{p^2\boldsymbol{b}_0(f,f)(z/\sqrt{p}) + p\boldsymbol{b}_1(f,f)(z/\sqrt{p}) + \boldsymbol{b}_2(f,f)(z/\sqrt{p})\right\}. \tag{5.59}$$

We also define the following positive quadratic function in $z \in \mathbb{C}^n$,

$$\widehat{\boldsymbol{b}}_1(z) = \lim_{p\to\infty} p\boldsymbol{b}_1(f,f)(z/\sqrt{p}) = -\frac{1}{2\pi}(\Delta f)(x_0)\hat{f}_{x_0}(z) + \frac{1}{\pi}|A\bar{z} + 2Bz|^2. \tag{5.60}$$

Proposition 5.18. *With the above notation,*

$$\beta_{f,x_0} := \partial\bar{\partial}\widehat{\boldsymbol{b}}_1 = \partial\bar{\partial}F_{f,x_0}(0) \in \Lambda^{(1,1)}T^*_{x_0}X \tag{5.61}$$

is a positive $(1,1)$*-form on* $\mathbb{C}^n$*, more precisely,*

$$\beta_{f,x_0} = (dz)^T \wedge K_{f,x_0} d\bar{z}, \tag{5.62}$$

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where K_{f,x_0} *is the semipositive definite Hermitian matrix given by*

$$K_{f,x_0} = \frac{2}{\pi}\mathrm{Tr}[A]A + \frac{1}{\pi}(A^2 + 4B\bar{B}). \tag{5.63}$$

We have the convergence of $(1,1)$*-forms at* x_0 *as* $p \to \infty$*,*

$$\frac{1}{p}\partial\overline{\partial}\log T^2_{f,p}(x,x)|_{x=x_0} \to \frac{1}{\mu(f,x_0)}\beta_{f,x_0} = \partial\overline{\partial}\log F_{f,x_0}(0). \tag{5.64}$$

Proof. The first part follows directly from the formulae (5.58) and (5.60). We now prove the convergence (5.64). In complex coordinates z centered at x_0, we have for $|z| < 1$,

$$p^{-n}T^2_{f,p}(z,z) = p^{-2}F_{f,x_0}(\sqrt{p}z) + \mathcal{O}(p^{-3}) + \mathcal{O}(|z|^5) + p^{-1}\mathcal{O}(|z|^3) + p^{-2}\mathcal{O}(|z|), \tag{5.65}$$

hence

$$\begin{aligned} &\frac{1}{p}\partial\overline{\partial}\log(p^{-n}T^2_{f,p}(z,z)) \\ &= \frac{\big(\partial\overline{\partial}F_{f,x_0}\big)(\sqrt{p}z) + \mathcal{O}(p^{-1}) + p\mathcal{O}(|z|^3) + \mathcal{O}(|z|)}{F_{f,x_0}(\sqrt{p}z) + \mathcal{O}(p^{-1}) + p^2\mathcal{O}(|z|^5) + p\mathcal{O}(|z|^3) + \mathcal{O}(|z|)} \\ &\quad - \frac{\big(\partial F_{f,x_0} \wedge \overline{\partial}F_{f,x_0}\big)(\sqrt{p}z) + p^3\mathcal{O}(|z|^7) + p^2\mathcal{O}(|z|^5) + p\mathcal{O}(|z|^3) + \mathcal{O}(|z|) + \mathcal{O}(p^{-1})}{\big\{F_{f,x_0}(\sqrt{p}z) + \mathcal{O}(p^{-1}) + p^2\mathcal{O}(|z|^5) + p\mathcal{O}(|z|^3) + \mathcal{O}(|z|)\big\}^2}. \end{aligned} \tag{5.66}$$

Taking $z = 0$ in the equality (5.66) and letting $p \to \infty$ yields (5.64). □

Definition 5.19. For a zero x_0 of f as above, we define for $R > 0$ the linear function

$$\Phi^R_{f,x_0} : \Lambda^{(n-1,n-1)}_{x_0}T^*X \to \mathbb{C}, \quad \Phi^R_{f,x_0}(\alpha) := \frac{\sqrt{-1}}{2\pi}\int_{B^{\mathbb{C}^n}(0,R)} \partial\overline{\partial}\log F_{f,x_0}(z) \wedge \alpha, \tag{5.67}$$

where $\alpha \in \Lambda^{(n-1,n-1)}_{x_0}$ is viewed as a constant $(n-1,n-1)$-form on $\mathbb{C}^n \simeq (T_{x_0}X, J_{x_0})$.

Remark 5.20. The expression of $\Phi^R_{f,x_0}(\alpha)$ can be made more concrete by using the formula (5.58), especially if f has a nice shape near x_0 (for example, if $B = 0$). We will demonstrate this in Example 5.21, but our expectation is that the calculations in general will be much more complicated, so we will not attempt to do so in this paper.

Example 5.21. Now, we assume f near x_0 is given by expansion (5.52), where $B = 0$ and

$$A = \mathrm{Id}_n. \tag{5.68}$$

Then

$$F_{f,x_0}(z) = |z|^4 + \frac{(2n+1)}{\pi}|z|^2 + \frac{n^2}{\pi^2}. \tag{5.69}$$

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Set $\omega_0 = \sqrt{-1}\sum_j dz_j \wedge d\bar{z}_j$. Then we have

$$\begin{aligned}
&\sqrt{-1}\partial\overline{\partial}\log F_{f,x_0}(z) \wedge \frac{\omega_0^{n-1}}{(n-1)!} \\
&= \pi\Big[\frac{(2n-2)\pi^3|z|^6 + (6n^2-n-2)\pi^2|z|^4 + (6n^3+2n^2-3n-1)\pi|z|^2 + 2n^4+n^3}{\pi^4|z|^8 + (4n+2)\pi^3|z|^6 + (6n^2+4n+1)\pi^2|z|^4 + (4n^3+2n^2)\pi|z|^2 + n^4}\Big]\frac{\omega_0^n}{n!}.
\end{aligned} \tag{5.70}$$

In the case $n=1$, we have

$$\sqrt{-1}\partial\overline{\partial}\log F_{f,x_0}(z) = \pi\Big[\frac{3\pi^2|z|^4 + 4\pi|z|^2 + 3}{\pi^4|z|^8 + 6\pi^3|z|^6 + 11\pi^2|z|^4 + 6\pi|z|^2 + 1}\Big]\omega_0. \tag{5.71}$$

Theorem 5.22. *Assume that (X,J,Θ), (L,h_L), f are as in either Case (I) or Case (II) with f a nontrivial function in $\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}_{\geq 0})$ and the prequantum condition $\Theta = c_1(L,h_L)$ holds. Let x_0 be a zero of f with $\Delta f(x_0) < 0$. Then for any fixed $R > 0$, $\varphi \in \Omega_0^{n-1,n-1}(X)$, and for all $p > p_0$,*

$$\frac{\sqrt{-1}}{2\pi}\int_{B(x_0,R/\sqrt{p})} \partial\overline{\partial}\log(T^2_{f,p}(x,x)) \wedge \varphi = p^{-n+1}\Phi^R_{f,x_0}\big(\varphi(x_0)\big) + \mathcal{O}(p^{-n+1/2}). \tag{5.72}$$

Proof. For $p \gg 0$, we identify $B(x_0,R/\sqrt{p}) \simeq B^{\mathbb{C}^n}(0,R/\sqrt{p})$. Then for $z \in B^{\mathbb{C}^n}(0,R/\sqrt{p})$, $l \in \mathbb{N}$,

$$p^l\mathcal{O}(|z|^{2l+1}) = \mathcal{O}(p^{-1/2}). \tag{5.73}$$

Also, note for $z \in B^{\mathbb{C}^n}(0,R)$,

$$\varphi(z/\sqrt{p}) = \varphi(x_0) + \mathcal{O}(p^{-1/2}). \tag{5.74}$$

Then the asymptotics (5.72) follows from the asymptotics (5.66). This completes the proof. □

As explained in Subsection 1.4, formula (5.72) gives different powers of p in the asymptotics (1.25), which exhibit two different behaviors of the fluctuations of the random zeros depending on whether $f(x) = 0$ or $f(x) \neq 0$.

5.4. Case of real functions with negative values

This subsection continues the discussion from Remark 4.17. We examine the equidistribution of random zeros of $\mathcal{L}^2$-holomorphic sections *detected* by a given real function f, which is not necessarily nonnegative. Now, we consider the case of complete Kähler manifold (X,Θ) equipped with a prequantum holomorphic line bundle (L,h_L). Recall that $\mathcal{Q}(X,L^p;\mathbb{R})$ is the subspace of $\mathcal{Q}(X,L^p;\mathbb{C})$ consisting of real-valued functions and that

$$\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R}) := \cap_{p\geq p_0}\mathcal{Q}(X,L^p;\mathbb{R}). \tag{5.75}$$

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Definition 5.23. Let f be a real smooth function on X, for $x \in X$, we say f is vanishing properly at x up to order 2 if one of the following cases holds:

- $f(x) \neq 0$, or
- $f(x) = 0$, $df(x) \neq 0$, or
- $f(x) = 0$, $df(x) = 0$, $\Delta f(x) \neq 0$ with $f\Delta f \leq 0$ on an open neighborhood of x.

For any subset $U \subset X$, we say f is vanishing properly on U up to order 2 if it does so for every point in U. Given such a function, we also set

$$\kappa(K) := \max_{x\in K} \operatorname{ord}_x(f) \in \{0,1,2\}. \tag{5.76}$$

The following proposition is an extension of Proposition 5.12.

Proposition 5.24. *Assume that* (X,J,Θ), (L,h_L), f *are given in either cases (I) or (II) with* f *a nontrivial function in* $\mathcal{Q}_{\geq p_0}(X,L;\mathbb{R})$ *and with prequantum condition* $\Theta = c_1(L,h_L)$. *Let* U *be an open subset of* $\operatorname{supp} f$ *such that* f *vanishes properly on* U *up to second order. Then for any compact subset* K *of* U, *there exists a constant* $c_K > 0$ *and* $p_K \geq p_0$ *such that for* $x \in K$, $p \geq p_K$,

$$T^2_{f,p}(x,x) \geq c_K p^{n-\kappa(K)}. \tag{5.77}$$

Moreover, $\log f^2$ *is locally integrable on* U, *and we have the weak convergence of currents on* U,

$$\partial\overline{\partial}\log T^2_{f,p}(x,x) \to \partial\overline{\partial}\log f^2, \quad \text{as } p\to\infty. \tag{5.78}$$

Around a point x *with* $f(x)\neq 0$, *the convergence in (5.78) holds in any local* $\mathscr{C}^\ell$*-norms.*

Proof. We start with proving estimate (5.77). For $x_0 \in U$, if $f(x_0) \neq 0$, then $f^2(x_0) > 0$, so estimate (5.77) holds near x_0. If $f(x_0) = 0$, $df(x_0) \neq 0$, then in a sufficiently small neighborhood of x_0, there is a constant $c_{x_0} > 0$ such that have

$$\boldsymbol{b}_1(f,f) = \frac{1}{8\pi}(\boldsymbol{r}^X f - 4\Delta f)f + \frac{1}{2\pi}|\partial f|^2 \geq c_{x_0}|df(x_0)|^2_{g^{T^*X}_{x_0}} > 0 \tag{5.79}$$

so that near x_0,

$$T^2_{f,p}(x,x) \geq \frac{1}{2}c_{x_0}p^{n-1}. \tag{5.80}$$

If $\operatorname{ord}_{x_0}(f) = 2$, we can adapt the proof of Proposition 5.12. The condition that $\Delta f(x_0)$ is nonzero with $f\Delta f \leq 0$ near x_0 implies that on a small neighborhood of x_0,

$$(\boldsymbol{r}^X f - 4\Delta f)f \geq 0,\ \mu(f,x_0) > 0. \tag{5.81}$$

Then estimate (5.77) still holds near x_0. The second part of the proposition follows from arguments analogous to those used to prove Proposition 5.12. □

For $f \in \mathcal{Q}_{\geq p_0}(X,L;\mathbb{R})$, the operator $T_{f,p}$ might not be injective so that, in Remark 4.17, we introduce a closed subspace $H^0_{(2)}(X,L^p,f) = (\ker T_{f,p})^\perp$ of $H^0_{(2)}(X,L^p)$ and the

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Gaussian probability measure $\mathbb{P}^{\sharp}_{f,p}$ on it. Consider the sequence of random $\mathcal{L}^2$-holomorphic sections *detected* by f

$$(\boldsymbol{S}^{\sharp}_{f,p})_{p\geq p_0} \in \Pi_{p\geq p_0}\big(H^0_{(2)}(X,L^p,f),\, \mathbb{P}^{\sharp}_{f,p}\big), \tag{5.82}$$

and a sequence of random holomorphic sections defined as in the decomposition (4.41),

$$\widehat{\boldsymbol{S}}_{f,p} = \psi_{\ker T_{f,p}} + \boldsymbol{S}^{\sharp}_{f,p},\ \ p\geq p_0. \tag{5.83}$$

Recall that the $\mathcal{L}^2$-integrability on X of $\widehat{\boldsymbol{S}}_{f,p}$ depends on the finiteness of $\dim\ker T_{f,p}$.

From the inequality (4.43) and by Proposition 5.24, we get:

Theorem 5.25. *Under the assumptions of Proposition 5.24, let U be an open subset of* $\operatorname{supp} f$ *such that f vanishes properly on U up to order* 2*. Then the following assertions hold.*

(i) *We have the convergence of* (1,1)*-currents on U as $p\to\infty$,*

$$\begin{aligned} &\frac{1}{p}\mathbb{E}^{\mathbb{P}^{\sharp}_{f,p}}[[\mathrm{Div}(\boldsymbol{S}^{\sharp}_{f,p})]|_U] \to c_1(L,h_L)|_U,\\ &\frac{1}{p}\mathbb{E}[[\mathrm{Div}(\widehat{\boldsymbol{S}}_{f,p})]|_U] \to c_1(L,h_L)|_U. \end{aligned} \tag{5.84}$$

(ii) *For any $\varphi\in\Omega_0^{n-1,n-1}(U)$, we have*

$$\begin{aligned} &\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}\langle[\mathrm{Div}(\boldsymbol{S}^{\sharp}_{f,p})],\varphi\rangle = \langle c_1(L,h_L),\varphi\rangle\right)\\ &=\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}\langle[\mathrm{Div}(\widehat{\boldsymbol{S}}_{f,p})],\varphi\rangle = \langle c_1(L,h_L),\varphi\rangle\right)=1. \end{aligned} \tag{5.85}$$

(iii) *If there is a smooth closed positive* (1,1)*-form α on X with $\int_X c_1(L,h_L)\wedge\alpha^{n-1}<\infty$ such that α is strictly positive on $\overline{U}$, then we have*

$$\mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(\boldsymbol{S}^{\sharp}_{f,p})]|_U = c_1(L,h_L)|_U\right) = \mathbb{P}\left(\lim_{p\to\infty}\frac{1}{p}[\mathrm{Div}(\widehat{\boldsymbol{S}}_{f,p})]|_U = c_1(L,h_L)|_U\right)=1, \tag{5.86}$$

where the limit is taken with respect to the weak convergence of (1,1)*-currents on U.*

Remark 5.26. If X is compact, then $H^0_{(2)}(X,L^p)=H^0(X,L^p)$, $p\in\mathbb{N}$, are finite dimensional, and we can take f to be any real smooth function vanishing properly up to order 2 in the above theorem. If $\kappa(X)\leq 1$, then the first limit (5.84) is already proved by Ancona and Le Floch [1]. As mentioned in Subsection 5.3, Ancona and Le Floch studied the fluctuations of the random zeros near a vanishing point of f with order 1. Their computations, as well as ours in the proof of Proposition 5.18 for points with vanishing order 2, were done by rescaling the expansion (5.8) of $T^2_{f,p}(x,x)$. Since the problem has local character, their results [1, Theorem 1.5 and Corollary 1.6] also hold in the noncompact setting for a vanishing point of f of order 1.

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References

[1] M. Ancona and Y. Le Floch, Berezin–Toeplitz operators, Kodaira maps, and random sections. arXiv:2206.15112 (2022).

[2] M. F. Atiyah, Elliptic operators, discrete groups and von Neumann algebras. in *Colloque* 'Analyse et Topologie' *en l'Honneur de Henri Cartan (Orsay, 1974)*, Astérisque, No. 32-33. (Paris: Soc. Math. France, Paris, 1976), 43–72.

[3] T. Barron, X. Ma, G. Marinescu and M. Pinsonnault, *Semi-classical properties of Berezin-Toeplitz operators with $\mathscr{C}^k$-symbol*, *J. Math. Phys.* **55** (2014), 042108, 25.

[4] T. Bayraktar, Asymptotic normality of linear statistics of zeros of random polynomials. *Proc. Amer. Math. Soc.* **145** (2017), 2917–2929.

[5] T. Bayraktar, D. Coman, H. Herrman and G. Marinescu, A survey on zeros of random holomorphic sections. *Dolomites Res. Notes Approx.* **11** (2018), 1–19.

[6] T. Bayraktar, D. Coman and G. Marinescu, Universality results for zeros of random holomorphic sections. *Trans. Amer. Math. Soc.* **373** (2020), 3765–3791.

[7] B. Berndtsson, Bergman kernels related to Hermitian line bundles over compact complex manifolds. In *Explorations in Complex and Riemannian Geometry, Contemp. Math.* vol. 332. (Amer. Math. Soc., Providence, RI, 2003), 1–17.

[8] P. Bleher, B. Shiffman and S. Zelditch, Poincaré–Lelong approach to universality and scaling of correlations between zeros. *Comm. Math. Phys.* **208** (2000), 771–785.

[9] Z. Błocki and P. Pflug, Hyperconvexity and Bergman completeness. *Nagoya Math. J.* **151** (1998), 221–225.

[10] T. Bloom and N. Levenberg, Random polynomials and pluripotential-theoretic extremal functions. *Potential Anal.* **42** (2015), 311–334.

[11] E. Bogomolny, O. Bohigas and P. Leboeuf, Quantum chaotic dynamics and random polynomials. *Journal of Statistical Physics* **85** (1996), 639–679.

[12] M. Bordemann, E. Meinrenken and M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and $N \to \infty$ limit*s. Comm. Math. Phys.* **165** (1994), 281–296.

[13] D. Coman, X. Ma and G. Marinescu, Equidistribution for sequences of line bundles on normal Kähler spaces. *Geom. Topol.* **21** (2017), 923–962.

[14] D. Coman and G. Marinescu, Convergence of Fubini-Study currents for orbifold line bundles. *Internat. J. Math.* **24** (2013), 1350051, 27.

[15] D. Coman and G. Marinescu, Equidistribution results for singular metrics on line bundles. *Ann. Sci. Éc. Norm. Supér. (4)* **48** (2015), 497–536.

[16] D. Coman, G. Marinescu and V. A. Nguyên, Hölder singular metrics on big line bundles and equidistribution. *Int. Math. Res. Not. IMRN*, (2016), 5048–5075.

[17] D. Coman, G. Marinescu and V. A. Nguyên. *Approximation and equidistribution results for pseudo-effective line bundles. J. Math. Pures Appl. (9)* **115** (2018), 218–236.

[18] G. de Rham, *Variétés différentiables. Formes, courants, formes harmoniques*, Publications de l'Institut de Mathématique de l'Université de Nancago, III. (Hermann, Paris, 1973).

Troisième édition revue et augmentée, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1222b.

[19] A. Deleporte, Low-energy spectrum of Toeplitz operators: The case of wells. *J. Spectr. Theory* **9** (2019), 79–125.

[20] A. Deleporte, Low-energy spectrum of Toeplitz operators with a miniwell. *Comm. Math. Phys.* **378** (2020), 1587–1647.

[21] J. P. Demailly, Singular Hermitian metrics on positive line bundles. In *Complex Algebraic Varieties (Bayreuth, 1990)*, Lecture Notes in Math., vol. 1507. (Springer, Berlin, 1992) 87–104.

[22] J. P. Demailly, *Complex Analytic and Differential Geometry* (2012). Available at https://www-fourier.ujf-grenoble.fr/~demailly/documents.html.

[23] K. Diederich, Das Randverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudo-konvexen Gebieten. *Math. Ann.* **187** (1970), 9–36.

[24] T. C. Dinh, X. Ma and G. Marinescu, Equidistribution and convergence speed for zeros of holomorphic sections of singular Hermitian line bundles. *J. Funct. Anal.* **271** (2016), 3082–3110.

[25] T. C. Dinh, G. Marinescu and V. Schmidt, Equidistribution of zeros of holomorphic sections in the non-compact setting. *J. Stat. Phys.* **148** (2012), 113–136.

[26] T. C. Dinh and N. Sibony, Distribution des valeurs de transformations méromorphes et applications. *Comment. Math. Helv* **81** (2006), 221–258.

[27] A. Drewitz, B. Liu and G. Marinescu, *Large deviations for zeros of holomorphic sections on punctured Riemann surfaces. Michigan Mathematical Journal* (2023), 1–41. Advance publication.

[28] A. Drewitz, B. Liu and G. Marinescu, Toeplitz operators and zeros of square-integrable random holomorphic sections. arXiv:2404.15983 (2024).

[29] A. Edelman and E. Kostlan. How many zeros of a random polynomial are real? *Bull. Amer. Math. Soc. (N.S.)* **32** (1995), 1–37.

[30] A. Edelman and E. Kostlan. Erratum: 'How many zeros of a random polynomial are real?' [Bull. Amer. Math. Soc. (N.S.) 32(1) (1996), 1–37, 1995; MR1290398 (95m:60082)]. *Bull. Amer. Math. Soc. (N.S.)* **33**, 325.

[31] H. Federer, *Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften*, Band 153. (Springer-Verlag New York, Inc., New York, 1969).

[32] C. Fefferman, The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Invent. Math.* **26** (1974), 1–65.

[33] S. Finski, Complex embeddings, Toeplitz operators and transitivity of optimal holomorphic extensions. arXiv:2201.04102 (2022).

[34] H. Grauert and R. Remmert, *Theory of Stein Spaces, Classics in Mathematics.* (Springer-Verlag, Berlin, 2004).

[35] P. Griffiths and J. Harris, *Principles of Algebraic Geometry, Wiley Classics Library*, (John Wiley & Sons, Inc., New York, 1994).

[36] L. Gross, Abstract Wiener spaces. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. II: Contributions to Probability Theory, Part 1.* (Univ. California Press, Berkeley, CA, 1967), 31–42.

[37] J. H. Hannay, Chaotic analytic zero points: exact statistics for those of a random spin state. *J. Phys. A* **29** (1996), L101–L105.

[38] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman Spaces, Graduate Texts in Mathematics*, vol. 199. (Springer-Verlag, New York, 2000).

[39] G. Herbort, The Bergman metric on hyperconvex domains. *Math. Z.* **232** (1999), 183–196.

[40] J. B. Hough, M. Krishnapur, Y. Peres and B. Virág, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes, University Lecture Series*, vol. 51. (American Mathematical Society, Providence, RI, 2009).
[41] L. Ioos, W. Lu, X. Ma and G. Marinescu, Berezin–Toeplitz quantization of non-compact manifolds. In preparation.
[42] S. Janson, *Gaussian Hilbert Spaces, Cambridge Tracts in Mathematics*, vol. 29. (Cambridge University Press, Cambridge, 1997).
[43] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis, De Gruyter Expositions in Mathematics*, vol. 9. (Walter de Gruyter GmbH & Co. KG, Berlin, 2013).
[44] Z. Kabluchko and D. Zaporozhets. Asymptotic distribution of complex zeros of random analytic functions. *Ann. Probab.* **42** (2014), 1374–1395.
[45] M. Kac, On the average number of real roots of a random algebraic equation. *Bull. Amer. Math. Soc.* **49** (1943), 314–320.
[46] J. P. Kahane, *Some Random Series of Functions*, 2nd edn., *Cambridge Studies in Advanced Mathematics*, vol. 5. (Cambridge University Press, Cambridge, 1985).
[47] A. Klenke, *Probability Theory*, (Springer, Universitext., London, 2014), German edition.
[48] S. Kobayashi, Geometry of bounded domains. *Trans. Amer. Math. Soc.* **92** (1959), 267–290.
[49] H. H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Mathematics, Vol. 463. (Springer-Verlag, Berlin-New York, 1975).
[50] M. Ledoux, *The Concentration of Measure Phenomenon, Mathematical Surveys and Monographs*, vol. 89. (American Mathematical Society, Providence, RI, 2001).
[51] N. Lindholm, Sampling in weighted L^p spaces of entire functions in $\mathbb{C}^n$ and estimates of the Bergman kernel. *J. Funct. Anal.* **182** (2001), 390–426.
[52] J. E. Littlewood and A. C. Offord, On the distribution of the zeros and a-values of a random integral function. *I. J. London Math. Soc.* **20** (1945), 130–136.
[53] J. E. Littlewood and A. C. Offord, On the distribution of zeros and a-values of a random integral function. II. *Ann. of Math. (2)* **49** (1948), 885–952; errata 50, 990–991 (1949).
[54] B. Liu and D. Zielinski, Semipositive line bundles on punctured Riemann surfaces: Bergman kernels and random zeros. arXiv:2407.15106 (2024).
[55] X. Ma, Geometric quantization on Kähler and symplectic manifolds. In *Proceedings of the International Congress of Mathematicians*, vol. II. (Hindustan Book Agency, New Delhi, 2010), 785–810.
[56] X. Ma and G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels, Progress in Mathematics*, vol. 254. (Birkhäuser Verlag, Basel, 2007).
[57] X. Ma and G. Marinescu, *Toeplitz operators on symplectic manifolds. J. Geom. Anal.* **18** (2008), 565–611.
[58] X. Ma and G. Marinescu, Berezin–Toeplitz quantization and its kernel expansion. In *Geometry and Quantization*, Trav. Math., vol. 19. (Univ. Luxemb., Luxembourg, 2011) 125–166.
[59] X. Ma and G. Marinescu, Berezin–Toeplitz quantization on Kähler manifolds. *J. Reine Angew. Math.* **662** (2012), 1–56.
[60] X. Ma and G. Marinescu, Exponential estimate for the asymptotics of Bergman kernels. *Math. Ann.* **362** (2015), 1327–1347.
[61] G. Marinescu and D. V. Vu, Bergman kernel functions associated to measures supported on totally real submanifolds. *J. Reine Angew. Math.* **810** (2024), 217–251.
[62] S. Nonnenmacher and A. Voros. Chaotic eigenfunctions in phase space. *J. Statist. Phys.* **92** (1998), 431–518.

[63] A. C. Offord, The distribution of the values of an entire function whose coefficients are independent random variables. *Proc. London Math. Soc. (3)* **14a** (1965), 199–238.
[64] A. C. Offord, The distribution of zeros of power series whose coefficients are independent random variables. *Indian J. Math.* **9** (1967), 175–196.
[65] A. C. Offord, The distribution of the values of an entire function whose coefficients are independent random variables. *II. Math. Proc. Cambridge Philos. Soc.* **118** (1995), 527–542.
[66] T. Ohsawa, A remark on the completeness of the Bergman metric. *Proc. Japan Acad. Ser. A Math. Sci.* **57** (1981), 238–240.
[67] S. Sheffield, Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** (2007), 521–541.
[68] B. Shiffman and S. Zelditch, Distribution of zeros of random and quantum chaotic sections of positive line bundles. *Comm. Math. Phys.* **200** (1999), 661–683.
[69] B. Shiffman and S. Zelditch, Number variance of random zeros on complex manifolds, II: smooth statistics. *Pure Appl. Math. Q.* **6** (2010), 1145–1167.
[70] B. Shiffman, S. Zelditch and S. Zrebiec, Overcrowding and hole probabilities for random zeros on complex manifolds. *Indiana Univ. Math. J.* **57** (2008), 1977–1997.
[71] M. Sodin, Zeros of Gaussian analytic functions. *Math. Res. Lett.* **7** (2000), 371–381.
[72] M. Sodin and B. Tsirelson, Random complex zeroes. *I. Asymptotic normality. Israel J. Math.* **144** (2004), 125–149.
[73] M. Sodin and B. Tsirelson, Random complex zeroes. III. Decay of the hole probability. *Israel J. Math.* **147** (2005) 371–379.
[74] M. Sodin and B. Tsirelson, Random complex zeroes. *II. Perturbed lattice. Israel J. Math.* **152** (2006), 105–124.
[75] T. Tao and V. Vu, *Local universality of zeroes of random polynomials. Int. Math. Res. Not. IMRN* (2015), 5053–5139.
[76] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds. *J. Differential Geom.* **32** (1990), 99–130.
[77] S. R. S. Varadhan, *Probability Theory*, Courant Lecture Notes in Mathematics, vol. 7. New York: New York University, Courant Institute of Mathematical Sciences; (American Mathematical Society, Providence, RI, 2001).
[78] A. Weil, *Introduction à l'étude des variétés kählériennes*, Publications de l'Institut de Mathématique de l'Université de Nancago, VI. Actualités Sci. Ind. no. 1267. (Hermann, Paris, Paris, 1958).
[79] S. Zelditch, Quantum ergodicity on the sphere. *Comm. Math. Phys.* **146** (1992), 61–71.
[80] S. Zelditch, Szegö kernels and a theorem of Tian. *Internat. Math. Res. Notices* (1998) 317–331.
[81] S. Zelditch, From random polynomials to symplectic geometry. In *XIIIth International Congress on Mathematical Physics (London, 2000).* (Int. Press, Boston, MA, 2001), 367–376.
[82] S. Zrebiec, The zeros of flat Gaussian random holomorphic functions on $\mathbb{C}^n$, and hole probability. *Michigan Math. J.* **55** (2007), 269–284.