Can. J. Math., Vol. XLII, No. 2, 1990, pp. 239–258

THE n-DIMENSIONAL HILBERT TRANSFORM OF DISTRIBUTIONS, ITS INVERSION AND APPLICATIONS

O. P. SINGH AND J. N. PANDEY

1. Introduction. Pandey and Chaudhary [**13**] recently developed the theory of Hilbert transform of Schwartz distribution space $(D_{L^p})', p > 1$ in one dimension using Parseval's types of relations for one dimensional Hilbert transform [**17**] and noted that their theory coincides with the corresponding theory for the Hilbert transform developed by Schwartz [**16**] by using the technique of convolution in one dimension.

The corresponding theory for the Hilbert transform in n-dimension is considerably harder and will be successfully accomplished in this paper. We also develop the n-dimensional theory of the Hilbert transform to $D'(\mathbf{R}^n)$ by using a method analogous to that used by Ehrenpreis [**4**] to extend the theory of Fourier transform to D'. Further we exploit the result proved in Theorem 10.1 to give the intrinsic definition of the space $H(D(\mathbf{R}^n))$ and its topology. Some applications of our results to solve singular integral equations will be discussed. A related boundary value problem and its solutions will also be discussed.

2. The n-dimensional Hilbert transform. If $f \in L^p(\mathbf{R}^n), p > 1$ then it is well known that its Hilbert transform $(Hf)(x)$ defined by

$$(2.1)\qquad (Hf)(x) = \frac{1}{\pi^n} \lim \max_i \epsilon_i \longrightarrow 0^+ \int_{\substack{|t_i - x_i| > \epsilon_i \\ i=1,2,3,\ldots,n}} \frac{f(t)dt}{(x_1 - t_1)(x_2 - t_2)\cdots(x_n - t_n)}$$

exists a.e. and $(Hf)(x) \in L^p(\mathbf{R}^n)$.

It is also known that there exists a constant $C_p > 0$ independent of f satisfying

$$(2.2)\qquad \|(Hf)(x)\|_p \leqq C_p \|f\|_p.$$

The existence of the integral in (2.1) and its boundedness property as stated in (2.2) was proved by Riesz and Titchmarsh [**17**] for $n = 1$, and for $n > 1$ the results were proved by several authors such as Kokilashvile [**9**] and others. Riesz and Titchmarsh also obtained the following inversion formula

$$(2.3)\qquad (H^2 f)(x) = -f(x) \text{ a.e.}$$

for the one dimensional Hilbert transform.

In this paper we generalize the above inversion formula for $n > 1$ to the space $L^p(\mathbf{R}^n), p > 1$ and then to Schwartz distribution spaces $D'_{L^p}(\mathbf{R}^n)$ and $D'(\mathbf{R}^n)$.

Received September 21, 1988. This paper is dedicated to the memory of the late Professor Paul R. Beesack.

3. Schwartz testing functions space $D(\mathbf{R}^n)$. The space $D(\mathbf{R}^n), n \geqq 1$ is the Schwartz testing function space consisting of C^∞ functions defined on $\mathbf{R}^n$ having compact support and the C^∞ functions defined on $\mathbf{R}$ with compact support will be denoted by D or $D(\mathbf{R})$. The topology of $D(\mathbf{R}^n)$ is that defined by Schwartz [**16**]. Accordingly a sequence $\{\varphi_m\}_{m=1}^\infty$ in $D(\mathbf{R}^n)$ converges to zero in $D(\mathbf{R}^n)$ if and only if

(i) $\varphi_1, \varphi_2, \varphi_3, \dots$ have their support contained in a compact set K
(ii) $\varphi_m^{(k)}(x) \longrightarrow 0$ as $m \longrightarrow \infty$ uniformly for each $|k| = 0, 1, 2, \dots$ on arbitrary compact subset of $\mathbf{R}^n$.

The space $X(\mathbf{R}^n)$ is defined to be the collection of $\varphi \in D(\mathbf{R}^n)$ which are finite sums of the form

$$(3.1) \qquad \varphi(x) = \sum \varphi_{m_1}(x_1)\varphi_{m_2}(x_2)\cdots\varphi_{m_n}(x_n)$$

where $\varphi_{m_j} \in D, \forall j = 1, 2, \dots, n$. Then we have the following well-known result:

LEMMA 3.1. *The space $X(\mathbf{R}^n)$ is dense in the space $L^p(\mathbf{R}^n), p > 1$ with respect to the norm topology of $L^p(\mathbf{R}^n)$* [**18**, p. 71].

4. The inversion formula. Note that if $\varphi \in X(\mathbf{R}^n)$ and φ has the representation (3.1) then

$$(4.1) \qquad (H\varphi)(x) = \sum \prod_{i=1}^{n} (H_i\varphi_{m_i})(x_i)$$

where $H_i(\varphi_{m_i}) \stackrel{\Delta}{=} \hat{\varphi}_{m_i}$, the classical one dimensional Hilbert transform of φ_{m_i} defined by

$$(H_i\varphi_{m_i})(x_i) = \frac{1}{\pi} P \int_{\mathbf{R}} \frac{\varphi_{m_i}(t_i)dt_i}{(x_i - t_i)} = \hat{\varphi}_{m_i}(x_i).$$

We are now ready to prove our Inversion Theorem.

THEOREM 4.1. *Let H be the operator of the classical Hilbert transform as defined by* (2.1) *in n-dimensions. Then* $\forall f \in L^p(\mathbf{R}^n)$

$$(4.2) \qquad (H^2 f)(x) = (-1)^n f(x) \quad \text{a.e.}$$

Proof. Equations (4.1) and (2.3) imply that the inversion formula (4.2) is valid for the subspace $X(\mathbf{R}^n)$ of $L^p(\mathbf{R}^n)$. To prove it on $L^p(\mathbf{R}^n)$ let us assume that $f \in L^p(\mathbf{R}^n)$ and $\{\varphi_j\}_{j=1}^\infty$ is a sequence in $X(\mathbf{R}^n)$ tending to f in $L^p(\mathbf{R}^n)$ as $j \longrightarrow \infty$. Such a sequence exists by Lemma 3.1. Then

$$(4.3) \qquad \begin{aligned} \|H^2 f - (-1)^n f\|_p &= \|H^2 f - (-1)^n f - (H^2\varphi_j - (-1)^n\varphi_j)\|_p \\ &= \|H^2(f - \varphi_j) - (-1)^n(f - \varphi_j)\|_p. \end{aligned}$$

Now $H : L^p(\mathbf{R}^n) \longrightarrow L^p(\mathbf{R}^n)$ is a bounded linear operator [9], therefore H^2 is also a bounded linear operator from $L^p(\mathbf{R}^n)$ into itself. Therefore by (4.3)

$$\|H^2 f - (-1)^n f\|_p \leqq K_p \|f - \varphi_j\|_p \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

Hence

$$(4.4) \qquad H^2 f = (-1)^n f$$

in the $L^p(\mathbf{R}^n)$ sense and so a.e. as well.

5. The testing function space $D_{L^p}(\mathbf{R}^n)$. A complex valued function defined on $\mathbf{R}^n$ belongs to the space $D_{L^p}(\mathbf{R}^n), p > 1$ if and only if

(i) $\qquad \varphi \in C^\infty(\mathbf{R}^n)$,

(ii) $\qquad \varphi^{(k)}(t) \in L^p(\mathbf{R}^n), \quad \forall |k| \in \mathbf{N}$,

where

$$\begin{aligned} \varphi^{(k)}(t) &= D^k \varphi(t) \\ &= D_{t_1}^{k_1} d_{t_2}^{k_2} \cdots D_{t_n}^{k_n} \varphi(t) \\ D_{t_i}\varphi &= \frac{\partial \varphi}{\partial t_i}; \quad i = 1, 2, \dots, n. \\ k &= (k_1, k_2, \dots, k_n) \end{aligned}$$

and

$$|k| = \sum_{i=1}^n k_i, \quad k_i \in \mathbf{N}, \quad i = 1, 2, \dots, n.$$

The topology on the space $D_{L^p}(\mathbf{R}^n)$. The topology over $D_{L^p}(\mathbf{R}^n)$ is generated by the separating collection of seminorms $\{\gamma_{(k)}\}|k| \in \mathbf{N}$ where

$$(5.1) \qquad \gamma_{(k)}(\varphi) = \left(\int_{\mathbf{R}^n} |\varphi^{(k)}(t)|^p dt \right)^{1/p} \quad [\mathbf{20}].$$

Therefore, a sequence φ_j converges to φ in $D_{L^p}(\mathbf{R}^n)$ as $j \longrightarrow \infty$ if and only if

$$\gamma_{(k)}(\varphi_j - \varphi) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \forall |k| \in \mathbf{N}.$$

A sequence φ_j is said to be a Cauchy sequence in $D_{L^p}(\mathbf{R}^N)$ if and only if $\forall |k| \in \mathbf{N}$

$$\gamma_{(k)}(\varphi_m - \varphi_n) \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty$$

https://doi.org/10.4153/CJM-1990-014-4 Published online by Cambridge University Press

independently of each other.

The space $D_{L^p}(\mathbf{R}^n)(1 < p < \infty)$ is sequentially complete, locally convex Hausdorff topological vector space [**20**].

Note (1). If $\varphi \in D_{L^p}(\mathbf{R}^n)$ then $\varphi^{(k)}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ for each $|k| \in \mathbf{N}$ [**16**].

(2) If ϕ_j is a sequence tending to zero in $D_{L^p}(\mathbf{R}^n)$ as $j \longrightarrow \infty$ then for each $|k| \in \mathbf{N}$

$$\varphi_j^{(k)}(x) \longrightarrow 0 \quad \text{uniformly on } \mathbf{R}^n \text{ as } j \longrightarrow \infty.$$

This result is well known [**5, 16**].

THEOREM 5.1. *The operator H of n-dimensional Hilbert transform as defined by* (2.1) *is a homemorphism from $D_{L^p}(\mathbf{R}^n)$ onto itself.*

Proof. The result is well known for $n = 1$, see [**13, 16**] and we use this fact to prove the result for $n > 1$. For $\varphi(t)$ in $D_{L^p}(\mathbf{R}^n), p > 1$, let us define

$$\begin{aligned} (5.2) \qquad & (H_i\varphi)(t_1, t_2, \dots, t_{i-1}, x_i, t_{i+1}, \dots, t_n) \\ &= \frac{1}{\pi} P \int_{\mathbf{R}} \frac{\phi(t_1, t_2, \dots, t_{i-1}, y_i, t_{i+1}, \dots, t_n)}{x_i - y_i} \, dy_i \\ &= \bar{\varphi}(t_1, t_2, \dots, t_{i-1}, x_i, t_{i+1}, \dots, t_n). \end{aligned}$$

It is easy to see that if $f \in L^p(\mathbf{R}^n)$ then

$$\begin{aligned} (Hf)(x) &= (H_1 H_2 \cdots H_{i-1} H_i H_{i+1} \cdots H_n f)(x) \\ &= (H_i(H_1 H_2 \cdots H_{i-1} H_{i+1} \cdots H_n) f)(x) \end{aligned}$$

(operators $H_1, H_2, H_3, \dots$ are commutative).

Therefore, for $\varphi \in D_{L^p}(\mathbf{R}^n), p > 1$, we have

$$(H(\varphi)(x) = H_i(\bar{\varphi}(x_1, x_2, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n)),$$

where

$$\begin{aligned} & \bar{\varphi}(x_1, x_2, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n) \\ &= \frac{1}{(\pi)^{n-1}} \left[P \int_{\mathbf{R}^{n-1}} \frac{\varphi(y_1, y_2, \dots, y_{i-1}, t_i, y_{i+1}, \dots, y_n)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (x_j - y_j)} \right. \\ & \left. \times \, dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_n \right]. \end{aligned}$$

By successive application of Theorem 5.1 for $n = 1$, it follows that

$$\bar{\varphi}(x_1, x_2, \dots, x_{i-1}, t_i, x_{i+1}, \dots, x_n) \in D_{L^p}(\mathbf{R}^n).$$

When $x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ are kept fixed then it follows that

$$(5.3)\qquad \begin{aligned} \frac{\partial}{\partial x_i}(H\varphi)(x) &= H_i \frac{\partial}{\partial t_i}\bar{\varphi}(x_1, x_2, \ldots, x_{i-1}, t_i, x_{i+1}, \ldots, x_n) \\ &= H_i H_1 H_2, \ldots, H_{i-1} H_{i+1} \cdots H_n \frac{\partial}{\partial t_i}\varphi(t_1, \ldots, t_n) \\ &= H\left(\frac{\partial \varphi}{\partial t_i}\right), \quad i = 1, 2, \ldots, n. \end{aligned}$$

By successive application of this result it can be shown that

$$(5.4)\qquad D^k(H\varphi)(x) = H(D^k\varphi)(x).$$

Therefore, using (2.2) we have

$$\|D^k(H\varphi)(x)\|_p = \|H(D^k\varphi)(x)\|_p \leqq C_p\|D^k\varphi\|_p.$$

Hence,

$$(5.5)\qquad \varphi \in D_{L^p}(\mathbf{R}^n) \Rightarrow H\varphi \in D_{L^p}(\mathbf{R}^n).$$

In view of the inversion formula (4.2), we have

$$(5.6)\qquad H\varphi = 0 \Rightarrow \varphi = 0$$

i.e., H is one to one.

The fact that H is onto follows by the same inversion formula. For if $\varphi \in D_{L^p}(\mathbf{R}^n)$, we have

$$(5.7)\qquad H[(H\varphi)(-1)^n] = \varphi,$$

and note that $(-1)^n H\varphi \in D_{L^p}(\mathbf{R}^n)$. Therefore H^{-1} exists, and using (4.2) we have

$$(5.8)\qquad H^{-1} = (-1)^n H.$$

Since H is linear and continuous, in view of (5.8) H^{-1} is also linear and continuous; thus proving the theorem.

6. The n-dimensional distributional Hilbert transform. For $p > 1$, assume that $f \in L^p(\mathbf{R}^n)$ and $g \in L^q(\mathbf{R}^n)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Then it is easy to show that

$$(6.1)\qquad \int_{\mathbf{R}^n} (Hf)(x)g(x)dx = \int_{\mathbf{R}^n} f(x)(-1)^n(Hg)(x)dx.$$

https://doi.org/10.4153/CJM-1990-014-4 Published online by Cambridge University Press

In the adjoint notation (6.1) can be written as

$$(6.2) \qquad \langle Hf, g\rangle = \langle f, (-1)^n Hg\rangle.$$

We are motivated by the equation (6.2) to define the Hilbert transform of distributions in n-dimension.

In conformity with the notation used by Laurent Schwartz we will denote $D'_{L^p}(\mathbf{R}^n), p > 1$ or some time abbreviated as D'_{L^p} as the dual space of $D_{L^q}(\mathbf{R}^n)$ where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Definition. For $f \in D'_{L^p}(\mathbf{R}^n)$, we define the n-dimensional Hilbert transform Hf of f as an element of $D'_{L^p}(\mathbf{R}^n)$ satisfying

$$(6.3) \qquad \langle Hf, \varphi\rangle = \langle f, (-1)^n H\varphi\rangle, \quad \forall\varphi \in D_{L^q}(\mathbf{R}^n).$$

$H\varphi$ in (6.3) stands for the classical n-dimensional Hilbert transform of φ.

It can be easily shown that the functional Hf defined by (6.3) is linear and continuous on $D_{L^q}(\mathbf{R}^n)$.

Example 1. Find $H\delta$ where $\delta \in D'_{L^p}(\mathbf{R}^n)$.

From the definition (6.3), we have

$$\begin{aligned}
\langle H\delta, \varphi\rangle &= \langle \delta, (-1)^n H\varphi\rangle \\
&= \left\langle \delta, (-1)^n \frac{1}{\pi^n} P\int_{\mathbf{R}^n} \frac{\varphi(t)dt}{(x_1 - t_1)\cdots(x_n - t_n)}\right\rangle \\
&= \frac{(-1)^n}{(-\pi)^n} P\int_{\mathbf{R}^n} \frac{\varphi(t)dt}{t_1 t_2 \cdots, t_n} \\
&= \left\langle \frac{1}{\pi^n} p.v.\left(\frac{1}{t_1 t_2\cdots t_n}\right), \varphi\right\rangle, \quad \forall\varphi \in D_{L^q}(\mathbf{R}^n).
\end{aligned}$$

Therefore

$$(6.4) \qquad H\delta = \frac{1}{\pi^n} p.v.\left(\frac{1}{t_1 t_2 \cdots t_n}\right) \stackrel{\Delta}{=} \frac{1}{\pi^n} p.v.\left[\frac{1}{t}\right].$$

Example 2. Find

$$H\left(p.v.\left[\frac{1}{t}\right]\right).$$

Operating both sides of (6.4) by H we get

$$H^2\delta = \frac{1}{\pi^n} H\left(p.v.\left[\frac{1}{t}\right]\right).$$

https://doi.org/10.4153/CJM-1990-014-4 Published online by Cambridge University Press

Hence

$$H\left(p.v.\left[\frac{1}{t}\right]\right) = (-\pi)^n\delta.$$

Since the operators $H_1, H_2, \dots H_n$ as defined in Section 5 are commutative, we can see that

$$p.v.\left(\frac{1}{t_1 t_2 \cdots t_n}\right) = p.v.\left(\frac{1}{t_{i_1} t_{i_2} \cdots t_{i_n}}\right)$$

where $i_1, i_2, i_3, \dots, i_n$ is a permutation of $1, 2, \dots, n$.

7. Calculus on $D'_{L^p}(\mathbf{R}^n)$. Let $f \in D'_{L^p}(\mathbf{R}^n)$. Then the distributional differentiation on $D'_{L^p}(\mathbf{R}^n)$ is defined as follows

(7.1) $$\langle D^k f, \varphi\rangle = \langle f, (-1)^{|k|} D^k \varphi\rangle, \forall \varphi \in D_{L^q}(\mathbf{R}^n), q = \frac{p}{p-1}, p > 1.$$

Now we prove the following

THEOREM 7.1. *Let $f \in D'_{L^p}(\mathbf{R}^n)$ then*

$$D^k Hf = HD^k f.$$

Proof.

$$\begin{aligned}\langle D^k Hf, \varphi\rangle &= \langle Hf, (-1)^{|k|} D^k \varphi\rangle, \forall \varphi \in D_{L^q}(\mathbf{R}^n) \\ &= \langle f, (-1)^n H(-1)^{|k|} D^k \varphi\rangle \\ &= \langle D^k f, (-1)^n H\varphi\rangle \\ &= \langle HD^k f, \varphi\rangle.\end{aligned}$$

Hence the Theorem 7.1 is established.

Example 3. Solve in $D'_{L^p}(\mathbf{R}^n)$ the operator equation

(7.2) $$y = Hy + f,$$

where $f \in D'_{L^p}(\mathbf{R}^n), n > 1$.

Solution. Operating both sides of (7.2) by H and applying the inversion Theorem 4.1 and using (7.3) we get

(7.3) $$y[1 - (-1)^n] = f + Hf.$$

Case (i): n is odd

$$(7.4) \qquad (7.3) \Rightarrow y = \frac{Hf + f}{2}.$$

Case (ii): n is even

$$(7.5) \qquad (7.3) \Rightarrow Hf = -f.$$

Therefore solution to (7.2) does not exist if

$$(7.6) \qquad Hf \neq -f.$$

If $Hf = -f$ is satisfied then there exists infinitely many solutions and in this case

$$y = \frac{f}{2} \text{ is a solution to (7.2).}$$

If g_i's are such that they satisfy

$$(7.7) \qquad Hy = y$$

then

$$(7.8) \qquad y = \frac{f}{2} + \sum_{i=1}^{m} C_i g_i,$$

where C_i's are arbitrary constants, satisfies (7.2).

The fact that there exists non-zero solutions to $Hy = y$ (n even) follows easily; for

$$y = \varphi_1(y_1)\varphi_2(y_2)\cdots\varphi_n(y_n) + (H_1\varphi_1)(y_1)(H_2\varphi_2)(y_2)\cdots(H_n\varphi_n)(y_n),$$

where $\varphi_i \in D$, satisfies $Hy = y$ when n is even, and

$$y = (H_1\varphi_1)(y_1)\cdots(H_n\varphi_n)(y_n) - \varphi_1(y_1)\cdots\varphi_n(y_n)$$

satisfies

$$Hy = -y.$$

There do exist non-zero y's not satisfying

$$Hy = -y,$$

when n is even.

As an example if we choose

$$y = \prod_{i=1}^{n}(H_i\varphi_i)(y_i) + \prod_{i=1}^{n}\varphi_i(y_i)$$

where $\varphi_i \in D$ such that $y \not\equiv 0$, then it does not satisfy $Hy = -y$, when n is even. It is still an open problem to determine the whole class of solutions to

$$y = Hy + f$$

when $Hf = -f$ is satisfied for n even.

8. The testing function space $H(D(\mathbf{R}^n))$. A complex valued C^∞ function φ defined on $\mathbf{R}^n$ belongs to the space $H(D(\mathbf{R}^n))$ if and only if $\varphi(x)$ is the n-dimensional Hilbert transform of some $\psi(t)$ in $D(\mathbf{R}^n)$. Hence $\varphi \in H(D(\mathbf{R}^n)) \Leftrightarrow$ there exists $\psi(t)$ in $D(\mathbf{R}^n)$ such that

$$(8.1) \qquad \varphi(x) = \frac{1}{\pi^n} P \int_{\mathbf{R}^n} \frac{\psi(t)}{x-t}\, dt = H\psi,$$

where the integral is being taken in the Cauchy principal value sense and $(x-t)$ in (8.1) is interpreted as

$$\prod_{i=1}^{n}(x_i - t_i).$$

The topology of $H(D(\mathbf{R}^n))$ is the same as that transported from the space $D(\mathbf{R}^n)$ to $H(D(\mathbf{R}^n))$ by means of the Hilbert transform H. Therefore a sequence φ_n in $H(D(\mathbf{R}^n))$ converges to zero in $H(D(\mathbf{R}^n))$ if and only if its associated sequence ψ_n converges to zero in $D(\mathbf{R}^n)$, where $H\psi_n = \varphi_n, \forall n \in \mathbf{N}$.

THEOREM 8.1. *Let $H(D(\mathbf{R}^n))$ and $D_{L^p}(\mathbf{R}^n)$ be the spaces defined as before. Then*

(i) *$H(D(\mathbf{R}^n)) \subset D_{L^p}(\mathbf{R}^n)$ and $H(D(\mathbf{R}^n))$ is dense in $D_{L^p}(\mathbf{R}^n)$.*

(ii) *Convergence of a sequence in $H(D(\mathbf{R}^n))$ implies its convergence in $D_{L^p}(\mathbf{R}^n)$.*

Hence the restriction of any $f \in D'_{L^p}(\mathbf{R}^n)$ to $H(D(\mathbf{R}^n))$ is in $H'(D(\mathbf{R}^n))$. Therefore

$$H'(D(\mathbf{R}^n)) \supset D'_{L^p}(\mathbf{R}^n).$$

Proof. (i) Since $D(\mathbf{R}^n)$ is dense in $D_{L^p}(\mathbf{R}^n)$ and

$$H : D_{L^p}(\mathbf{R}^n) \xrightarrow{\text{onto}} D_{L^p}(\mathbf{R}^n)$$

is homemorphism, we conclude that $H(D(\mathbf{R}^n))$ is dense in $D_{L^p}(\mathbf{R}^n)$. [See also **14**.]

(ii) Let $\varphi_j \longrightarrow 0$ in $H(D(\mathbf{R}^n))$. Then there exists a sequence $\psi_j \longrightarrow 0$ in $D(\mathbf{R}^n)$ as $j \longrightarrow \infty$ such that $H\psi_j = \varphi_j$. Now using equation (2.2) and (5.4), we have

$$\|\varphi_j^{(k)}\|_p \leqq C_p\|\psi_j^{(k)}\|_p \longrightarrow 0 \quad \text{as } j \longrightarrow \infty.$$

Remark. In view of the Inversion Theorem 4.1,

$$H : H(D(\mathbf{R}^n)) \longrightarrow D(\mathbf{R}^n)$$

is linear and continuous.

9. The *n*-dimensional generalized Hilbert transform. The generalized Hilbert transform Hf of $f \in D'(\mathbf{R}^n)$ is defined to be an ultradistribution $Hf \in H'(D(\mathbf{R}^n))$ such that

$$(9.1) \qquad \langle Hf, \varphi\rangle = \langle f, (-1)^n H\varphi\rangle, \quad \forall \varphi \in H(D(\mathbf{R}^n))$$

where $H\varphi$ is the classical Hilbert transform defined by (8.1). If $g \in H'(D(\mathbf{R}^n))$, its Hilbert transform Hg is defined to be a Schwartz distribution by the relation

$$(9.2) \qquad \langle Hg, \varphi\rangle = \langle g, (-1)^n H\varphi\rangle, \quad \forall \varphi \in D(\mathbf{R}^n).$$

Let $g = Hf$, for some $f \in D'(\mathbf{R}^n)$. Then

$$(9.3) \qquad \begin{aligned} \langle H^2 f, \varphi\rangle &= \langle Hf, (-1)^n H\varphi\rangle \\ &= \langle f, H^2\varphi\rangle \\ &= \langle f, (-1)^n\varphi\rangle \\ \Rightarrow H^2 &= (-1)^n I \text{ on } D'(\mathbf{R}^n). \end{aligned}$$

Definition 9.1. The *derivative* $D^k g$ of an ultra distribution $g \in H'(D(\mathbf{R}^n))$ is defined as follows:

$$(9.4) \qquad \langle D^k g, \varphi\rangle = \langle g, (-1)^{|k|} D^k\varphi\rangle,$$

for every $\varphi \in H(D(\mathbf{R}^n))$.

THEOREM 9.2. *Let* $f \in D'(\mathbf{R}^n)$, *then*

$$(9.5) \qquad (Hf)^{(k)} = H(f^{(k)}).$$

Proof.

$$\begin{aligned}\langle D^k Hf, \varphi\rangle &= \langle Hf, (-1)^{|k|} D^k\varphi\rangle, \quad \forall\varphi \in H(D(\mathbf{R}^n))\\ &= \langle f, (-1)^{|k|+n} HD^k\varphi\rangle\\ &= \langle f, (-1)^{|k|+n} D^k H\varphi\rangle \quad \text{(from (5.4))}\\ &= \langle D^k f, (-1)^n H\varphi\rangle\\ &= \langle HD^k f, \varphi\rangle.\end{aligned}$$

Example. Solve in $D'(\mathbf{R}^n)$

$$\frac{\partial y}{\partial x_1} + H\frac{\partial f}{\partial x_1} = \delta(x).$$

We rewrite the equation in the form

$$\frac{\partial}{\partial x_1}[y + Hf] = \delta(x) = \delta(x_1) * \delta(x_2) * \cdots * \delta(x_n).$$

Then

$$\begin{aligned}y + Hf &= h(x_1) * \delta(x_2) * \cdots * \delta(x_n)\\ &+ C(x_2, x_3, \ldots, x_n).\end{aligned}$$

10. An intrinsic definition of the space $H(D(\mathbf{R}^n))$ and its topology. In this section we will give an intrinsic definition of the space $H(D(\mathbf{R}^n))$ and its topology. We also now give some lemmas to be used in the sequel.

LEMMA 10.1. *Let $\{\varphi_\nu\}_{\nu=1}^\infty$ be a sequence of functions tending to zero in $D_{L^p}(\mathbf{R}^n)$ as $\nu \longrightarrow \infty$ i.e.,*

$$\gamma_{(k)}(\varphi_\nu) \longrightarrow 0 \quad \text{as } \nu \longrightarrow \infty\ \forall|k| \in \mathbf{N},$$

then for each $|k| = 0, 1, 2, \ldots$

$$\varphi_\nu^{(k)} \longrightarrow 0 \quad \text{as } \nu \longrightarrow \infty \text{ uniformly } \forall x \in \mathbf{R}^n.$$

Proof. The lemma is well known [**5, 16**], but a very simple proof can be given as follows:

$$(10.1) \qquad \varphi^{(k)}(x) = \langle\delta(t), \varphi^{(k)}(x-t)\rangle, \quad \forall\varphi \in D_{L^p}(\mathbf{R}^n).$$

In view of the boundedness property of generalized functions [**20**] there exists a constant $c > 0$ and an $r = (r_1, r_2, \ldots, r_n)$ and $|r| = r_1 + r_2 + \cdots + r_n$ such that

$$\begin{aligned}|\varphi^{(k)}(x)| &\leqq C\gamma'_{|r|}(\varphi^{(k)}(x-t)) \quad [\mathbf{20}, \text{p. 8–19}]\\ &\leqq C\gamma'_{|r|}(\varphi^{(k)}(t))\end{aligned}$$

where

$$\gamma'_{|0|} = \gamma_{|0|} \quad \text{and} \quad \gamma'_{|r|} = \max_{|j| \leqq |r|} \gamma_{(j)}.$$

Therefore

$$|\varphi_\nu^{(k)}(x)| \leqq C\gamma'_{|r|}(\varphi_\nu^{(k)}(t)) \longrightarrow 0 \quad \text{as } \nu \longrightarrow \infty$$

independently of x. This completes the proof of the lemma.

LEMMA 10.2. *Let* $\varphi(t) \in D(\mathbf{R}^n)$ *then as* $y \longrightarrow 0^+$ *i.e.,* $y_i \longrightarrow 0^+$ $\forall i = 1, 2, 3, \ldots n$

(i)

$$(10.2) \quad \frac{1}{\pi^n} \int_{\mathbf{R}^n} \varphi(t) \frac{y_1}{(t_1 - x_1)^2 + y_1^2} \frac{y_2}{(t_2 - x_2)^2 + y_2^2} \cdots \frac{y_n}{(t_n - x_n)^2 + y_n^2} \, dt \longrightarrow \varphi(x) \text{ in } D_{L^p}(\mathbf{R}^n), p > 1.$$

(ii)

$$(10.3) \quad \int_{\mathbf{R}^n} \varphi(t) \prod_{i=1}^{n} \left[\frac{(t_i - x_i)}{(t_i - x_i)^2 + y_i^2} \right] dt \longrightarrow P \int_{\mathbf{R}^n} \frac{\varphi(t)}{\prod_{i=1}^{n} (t_i - x_i)} \, dt \text{ in } D_{L^p}(\mathbf{R}^n), p > 1.$$

(iii)

$$(10.4) \quad \frac{1}{\pi^{n-m}} \int_{\mathbf{R}^n} \varphi(t) \prod_{i=1}^{m} \left[\frac{(t_i - x_i)}{(t_i - x_i)^2 + y_i^2} \right] \times \prod_{i=m+1}^{n} \left[\frac{y_i}{(t_i - x_i)^2 + y_i^2} \right] dt \longrightarrow (H_m \cdots H_3 H_2 H_1 \varphi)(x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n) \text{ in } D_{L^p}(\mathbf{R}^n), \quad m = 1, 2, \ldots, n, (p > 1).$$

(iv)

$$(10.5) \quad \frac{1}{\pi^{n-m}} \int_{\mathbf{R}^n} \varphi(t) \prod_{i=1}^{m} \left[\frac{(t_{l_i} - x_{l_i})}{(t_{l_i} - x_{l_i})^2 + y_{l_i}^2} \right] \times \prod_{i=m+1}^{n} \left[\frac{y_{l_i}}{(t_{l_i} - x_{l_i})^2 + y_{l_i}^2} \right] dt \longrightarrow (H_{l_m} H_{l_{m-1}} \cdots H_{l_1} \varphi)(\cdots x_{l_1} \cdots x_{l_2} \cdots x_{l_m} \cdots) \text{ in } D_{L^p}(\mathbf{R}^n), p > 1, \quad m = 1, 2, 3, \ldots, n.$$

Proof. (i) For the proof see [**7**, p. 400].

(ii) Denoting the L.H.S. expression in (10.3) by $\pi^n F(x)$, we see that

$$F^{(k)}(x) = \frac{1}{\pi^n}\int_{\mathbf{R}^n} \varphi^{(k)}(t)\prod_{i=1}^{n}\frac{(t_i - x_i)}{(t_i - x_i)^2 + y_i^2}\,dt.$$

By successive application of Fubini's Theorem and [**17**, Theorem 101, p. 132], it follows that

$$\|F^{(k)}(x)\|_p \leqq C_p^n \|\varphi^{(k)}(x)\|_p,$$

where C_p is a constant independent of φ and $y_i, y_2, \ldots, y_n$.

Since the space $X(\mathbf{R}^n)$ is dense in $D_{L^p}(\mathbf{R}^n)$, $p > 1$, it is easy to show that

$$(10.6)\qquad \|F^{(k)}(x) - H\varphi^{(k)}(x)\|_p \longrightarrow 0 \quad \text{as } y_1, y_2, \ldots, y_n \longrightarrow 0.$$

A much more general result is proved in [**15**, Theorem 3.2].

(iii) follows as a result of (i) and (ii) and (iv) is only an elementary variation of (iii) and can be proved similarly.

LEMMA 10.3. *Let $z_j \in \mathbf{C}$ for $j = 1, 2, 3, \ldots, n$ where $z_j = x_j + iy_j$ and $x_j, y_j \in \mathbf{R}$. For $\varphi(t) \in D(\mathbf{R}^n)$, define a function F as a mapping from $\mathbf{C}^n$ to $\mathbf{C}$ by*

$$(10.7)\qquad F(z) = \int_{\mathbf{R}^n} \frac{\varphi(t)}{\prod_{i=1}^{n}(t_i - z_i)}\,dt,$$

if $y_i \neq 0\ \forall i = 1, 2, \ldots, n$; and

$$\begin{aligned}(10.8)\qquad & F(z_1, z_2, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n)\\ &= \frac{1}{2}[F(z_1, \ldots, z_{i-1}, x_i^+, z_{i+1}, \ldots, z_n)\\ &+ F(z_1, \ldots, z_{i-1}, x_i^-, z_{i+1}, \ldots, z_n)],\end{aligned}$$

if $y_i = 0$, for some i, $1 \leqq i \leqq n$.

Then $\lim_{y\to 0^+} F(z)$ *converges uniformly to*

$$\sum (i\pi)^{n-l} H_{j_1} H_{j_2} \cdots H_{j_l}\varphi, \quad \forall x \in \mathbf{R}^n.$$

Proof. Since $z_j = x_j + iy_j, \forall j = 1, 2, \ldots, n$;

$$F(z) = \int_{\mathbf{R}^n} \varphi(t)\prod_{j=1}^{n}\left[\frac{(t_j - x_j) + iy_j}{(t_j - x_j)^2 + y_j^2}\right] dt,$$

as $y \longrightarrow 0^+$, (in view of Lemma 2 (ii)),

$$F(z) = \sum \int_{\mathbf{R}^n} \varphi(t) \left[\prod_{m=1}^{l} \frac{(t_{j_m} - x_{j_m})}{(t_{j_m} - x_{j_m})^2 + y_{j_m}^2} \right] \times \left[\prod_{m=1}^{n-l} \frac{i^{n-l} y_{j'_m}}{(t_{j'_m} - x_{j'_m})^2 + y_{j'_m}^2} \right] dt$$

and the result follows in view of Lemma 2 (iv). Now we come to our central problem of defining the space $H(D(\mathbf{R}^n))$ intrinsically and we need the following

Definition 10.1. A holomorphic function $\psi(z)$ defined on the complex n-space $\mathbf{C}^n$ belongs to the space Ψ if and only if the following properties hold:

(P_1): $\psi(z)$ is holomorphic outside the intervals $a_i \leqq x_i \leqq b_i, i = 1, 2, 3, \ldots, n$ (the interval depending upon $\psi(z)$).

$$(P_2): \qquad \psi^{(k)}(z) = O\left(\frac{1}{|z_1||z_2| \cdots |z_n|} \right)$$

as $|z_i| \longrightarrow \infty, \forall i$, for each fixed k satisfying $|k| = 0, 1, 2, 3, \ldots$.

(P_3): (a) For each fixed $|k| = 0, 1, 2, 3, \ldots, \psi^{(k)}(z)$ converges uniformly $\forall x \in \mathbf{R}^n$ as $y \longrightarrow 0^+$.

(b) For each fixed $|k| = 0, 1, 2, \ldots, \psi^{(k)}(z)$ converges uniformly $\forall x \in \mathbf{R}^n$ as $y \longrightarrow 0^-$.

$$(P_4): \qquad \psi(z_1, z_2, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n) = \frac{1}{2} [\psi(z_1, z_2, \ldots, z_{i-1}, x_i^+, z_{i+1}, \ldots, z_n) + \psi(z_1, z_2, \ldots, z_{i-1}, x_i^-, z_{i+1}, \ldots, z_n)], \quad i = 1, 2, 3, \ldots, n;$$

where

$$\psi(z_1, z_2, \ldots, z_{i-1}, x_i^{\pm}, z_{i+1}, \ldots, z_n). = \lim_{y_i \to 0^{\pm}} \psi(z_1, z_2, \ldots, z_i, \ldots, z_n).$$

THEOREM 10.1. *A necessary and sufficient condition that a function $\psi(z)$ defined on the complex n-space $\mathbf{C}^n$ belongs to the space Ψ is that there exists a $\varphi(t) \in D(\mathbf{R}^n)$ satisfying*

$$(10.9) \qquad \psi(z) = \int_{\mathbf{R}^n} \frac{\varphi(t)}{\prod_{i=1}^{n} (t_i - z_i)} dt, \quad \operatorname{Im} z_i \neq 0, \forall i = 1, 2, 3, \ldots, n,$$

$$(10.10) \qquad = pv \int_{\mathbf{R}^n} \frac{\varphi(t)}{(t_1 - z_1) \cdots (t_i - x_i) \cdots (t_n - z_n)} dt,$$

when Im $z_i = 0$ *for some* i, $1 \leqq i \leqq n$.

Proof. Necessity: If $\psi(z) \in \Psi$ then in view of the properties (P_1), $\psi(z)$ as a function of $x \in \mathbf{R}^n$ is a member of $D_{L^p}(\mathbf{R}^n)$ for a fixed $y \neq 0$ (i.e., for each component of $y \in \mathbf{R}^n$ non-zero). Now from (P_1) and (P_2) it follows that if $\{y_m\}_{m=1}^{\infty}$ is an arbitrary sequence in $\mathbf{R}^n$ such that $\|y_m\| \longrightarrow 0$ as $m \longrightarrow \infty$ then

$$\|\psi^{(k)}(x+iy_m) - \psi^{(k)}(x+iy_l)\|_p \longrightarrow 0$$

as $l, m \longrightarrow \infty$ independently of each other. Therefore $\{\psi(x+iy_m)\}_{m=1}^{\infty}$ is a Cauchy sequence in $D_{L^p}(\mathbf{R}^n), p > 1$. Since $D_{L^p}(\mathbf{R}^n)$ is sequentially complete there exists a function $\psi_+(x)$ in $D_{L^p}(\mathbf{R}^n)$ such that

$$\lim_{m\longrightarrow\infty} \psi(x+iy_m) = \psi_+(x) \quad \text{in } D_{L^p}(\mathbf{R}^n), p > 1.$$

Since $\{y_m\}$ is an arbitrary sequence in $\mathbf{R}^n$ tending absolutely to zero it follows that

$$(10.11) \quad \lim_{y\longrightarrow 0^+} \psi(x+iy) = \psi_+(x) \quad \text{in } D_{L^p}(\mathbf{R}^n).$$

Similar arguments show the existence of a function $\psi_-(x)$ in $D_{L^p}(\mathbf{R}^n)$ satisfying

$$(10.12) \quad \lim_{y\longrightarrow 0^-} \psi(x+iy) = \psi_-(x) \quad \text{in } D_{L^p}(\mathbf{R}^n), p > 1$$

and hence is the uniform limit (from Lemma 10.1) with respect to every $x \in \mathbf{R}^n$.

In quite a similar way it can be shown that

$$\psi(z_1, z_2, \ldots, z_{i-1}, x_i^{\pm}, z_{i+1}, \ldots, z_n) \in D_{L^p}(\mathbf{R}^n)$$

for each fixed $z_j \in \mathbf{C}$, $1 \leqq j \leqq n$ and $j \neq i$. Therefore

$$(10.13) \quad \psi(z_1, z_2, \ldots, z_{i-1}, x_i, z_{i+1}, \ldots, z_n) = \frac{1}{2}[\psi(z_1, z_2, \ldots, x_i^+, \ldots, z_n) + \psi(z_1, z_2, \ldots, x_i^-, \ldots, z_n)]$$

belongs to $D_{L^p}(\mathbf{R}^n), p > 1$ for fixed $y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n \neq 0$, where $y_j = \text{Im } z_j$, $1 \leqq j \leqq n, j \neq i$. Since $\psi(z)$ is analytic outside the interval $[a_i, b_i]$ on the X_i-axis, hence

$$\psi(z_1, z_2, \ldots, x_i^+, \ldots, z_n) - \psi(z_1, z_2, \ldots, x_i^-, \ldots, z_n) = 0$$

outside $[a_i, b_i]$ on the X_i real line, $\forall i = 1, 2, \ldots, n$. Using Cauchy's integral theorem it can be shown that

$$(10.14) \quad \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{t_j - z_j}[\psi(z_1, z_2, \ldots, z_{j-1}, (t_j + i\epsilon_j), z_{j+1}, \ldots, z_n)]dt_j$$
$$= \psi(z_1, z_2, \ldots, z_{j-1}, z_j + i\epsilon_j, \ldots, z_n), \text{Im } z_j > 0$$
$$= 0 \quad \text{Im } z_j < 0.$$

Letting $\epsilon_j \longrightarrow 0^+$ in (10.14), we have

$$(10.15)\quad \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{t_j - z_j}\,\psi(z_1,\ldots,z_{j-1},t_j^+,z_{j+1},\ldots,z_n)dt_j$$
$$= \psi(z_1,z_2,\ldots,z_j,\ldots,z_n),\quad \text{Im } z_j > 0$$
$$= 0,\quad \text{Im } z_j < 0.$$

Similarly we can show that

$$(10.16)\quad \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{t_j - z_j}\,\psi(z_1,z_2,\ldots,z_{j-1},t_j^-,z_{j+1},\ldots,z_n)dt_j$$
$$= -\psi(z_1,z_2,\ldots,z_j,\ldots,z_n),\quad \text{Im } z_j < 0$$
$$= 0,\quad \text{Im } z_j > 0.$$

Therefore, combining (10.15) and (10.16) we get

$$(10.17)\quad \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{t_j - z_j}\,[\psi(z_1,z_2,\ldots,z_{j-1},t_j^+,z_{j+1},\ldots,z_n)$$
$$-\psi(z_1,z_2,\ldots,z_{j-1},t_j^-,z_{j+1},\ldots,z_n)]dt_j$$
$$= \psi(z_1,z_2,\ldots,z_n),\quad \text{Im } z_j \neq 0;\quad 1 \leqq j \leqq n.$$

In view of Lemmas 2 and 3 and (P_4) it follows that

$$(10.18)\quad \psi(z_1,z_2,\ldots,z_{j-1},x_j,z_{j+1},\ldots,z_n)$$
$$= \frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{1}{t_j - x_j}\,[\psi(z_1,z_2,\ldots,z_{j-1},t_j^+,$$
$$z_{j+1},\ldots,z_n) - \psi(z_1,z_2,\ldots,z_{j-1},t_j^-,z_{j+1},\ldots,z_n)]dt_j$$

$$(10.19)\quad = p.v.\int_{-\infty}^{\infty}\frac{\theta(z_1,z_2,\ldots,z_{j-1},t_j,z_{j+1},\ldots,z_n)}{x_j - t_j}\,dt_j,$$

where

$$(10.20)\quad -2\pi i\theta(z_1,z_2,\ldots,t_j,\ldots,z_n)$$
$$= \psi(z_1,z_2,\ldots,t_j^+,\ldots,z_n) - \psi(z_1,\ldots,t_j^-,\ldots,z_n).$$

Clearly $\theta(z_1,\ldots,t_j,\ldots,z_n) = 0$ when $t_j \notin [a_j,b_j]$. Exploiting the Lemmas 2, 3 and (P_4) once again it can be proved that

$$(10.21)\quad \psi(z_1,z_2,\ldots,z_{j-1},x_j,z_{j+1},\ldots,z_{l-1},x_l,z_{l+1},\ldots,z_n)$$
$$= p.v.\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\eta(z_1,z_2,\ldots,z_{i-1},t_j,z_{j+1},\ldots,z_{l-1},t_l,z_{l+1},\ldots,z_n)}{(x_j - t_j)(x_l - t_l)}\,dt_j dt_l\right],$$

for a suitable $\eta(z_1, z_2, \ldots, t_j, \ldots, t_l, \ldots, z_n)$ vanishing whenever $t_j \notin [a_j, b_j]$ and $t_l \notin [a_l, b_l]$. Carrying similar arguments one can show that there exists $\varphi(t) \in D(\mathbf{R}^n)$ with support contained in $a_i \leqq t_i \leqq b_i\ \forall i = 1, 2, \ldots, n$; such that

$$(10.22)\quad \psi(x_1, x_2, \ldots, x_n) = p.v. \int_{\mathbf{R}^n} \frac{\varphi(t)}{\prod_{i=1}^{n}(x_i - t_i)}\, dt.$$

Now using (10.17) and repeating the technique of contour integration etc. (as used in deducing (10.17)) it can be shown that there exists $\varphi(t) \in D(\mathbf{R}^n)$ satisfying

$$(10.23)\quad \psi(z) = \int_{\mathbf{R}^n} \frac{\varphi(t)dt}{\prod_{j=1}^{n}(z_j - t_j)},$$

when

$$\operatorname{Im} z_j \neq 0\ \forall_j,\ 1 \leqq j \leqq n.$$

It can easily be seen during the course of derivation that φ's used in (10.21) and (10.22) are the same. This completes the proof of necessity.

Sufficiency. Assume that $\varphi(t) \in D(\mathbf{R}^n)$ and define a function $\psi(z)$ and a mapping from $\mathbf{C}^n$ to $\mathbf{C}$ by the relation

$$(10.24)\quad \psi(z) = \int_{\mathbf{R}^n} \frac{\varphi(t)}{\prod_{j=1}^{n}(t_j - z_j)}\, dt,$$

when

$$\operatorname{Im} z_j \neq 0\ \forall j,\ 1 \leqq j \leqq n;$$

$$(10.25)\quad = p.v. \int_{\mathbf{R}^n} \frac{\varphi(t)}{(t_1 - z_1)\ldots(t_{j-1} - z_{j-1})(t_j - x_j)\ldots(t_n - z_n)}\, dt$$

when $\operatorname{Im} z_j = 0$ for some j, $1 \leqq j \leqq n$.

The support of $\varphi(t)$ is contained in $a_i \leqq t_i \leqq b_i$, $i = 1, 2, \ldots, n$. Using (10.23) and (10.24) now it follows quite easily that (P_1), (P_2), (P_3) and (P_4) follow. This completes the proof of Theorem 10.1.

Theorem 10.1 demonstrates one to one correspondence between the space Ψ and $H(D(\mathbf{R}^n))$. We, therefore, can define the space $H(D(\mathbf{R}^n))$ in a genuinely intrinsic way as follows:

A C^∞ function $\psi(x)$ defined on $\mathbf{R}^n$ is said to *belong to* the space $H(D(\mathbf{R}^n))$ if and only if there exists a holomorphic function $\psi(z)$ defined on $\mathbf{C}^n$ satisfying

(P_1), (P_2), (P_3) and (P_4). In other words $\psi(x) \in H(D(\mathbf{R}^n))$ if and only if $\psi(x)$ can be extended uniquely as a holomorphic function satisfying (P_1), (P_2), (P_3) and (P_4).

The convergence of a sequence $\{\psi_m(x)\}_{m=1}^{\infty}$ to zero in the space $H(D(\mathbf{R}^n))$ can be defined in an intrinsic way as follows:

A sequence $\{\psi_m\}_{m=1}^{\infty}$ in $H(D(\mathbf{R}^n))$ *converges to zero* in $H(D(\mathbf{R}^n))$ if and only if

(i) the associated functions $\psi_m(z)$ in accordance with Theorem 10.1 are analytic outside a closed n-box $\prod_{j=1}^{n}[a_j, b_j]$ of $\mathbf{R}^n$ or else $\psi_m(x)$ is analytic outside a fixed closed n-box $\prod_{j=1}^{n}[a_j, b_j]$.

(ii) $\psi_m(x) \longrightarrow 0$ in $D_{L^p}(\mathbf{R}^n)$ as $m \longrightarrow \infty$.

Clearly if $\{\varphi_m(x)\}_{m=1}^{\infty}$ is a sequence in $D(\mathbf{R}^n)$ tending to zero in $D(\mathbf{R}^n)$ as $m \longrightarrow \infty$ and

$$(10.26)\quad \psi_m(x) = p.v. \int_{\mathbf{R}^n} \frac{\varphi_m(t)}{\prod_{j=1}^{n}(t_j - x_j)}\, dt$$

$$\psi_m(z) = \int_{\mathbf{R}^n} \frac{\varphi_m(t)}{\prod_{j=1}^{n}(t_j - z_j)}\, dt, \quad \operatorname{Im} z_i \neq 0 \quad \forall i = 1, 2, \ldots, n,$$

then $\psi_m(z)$ is analytic outside the closed intervals $a_j \leqq x_j \leqq b_j, j = 1, 2, \ldots, n$; and

$$D^k\psi_m(x) = p.v. \int_{\mathbf{R}^n} \frac{D_t^k \varphi_m(t)}{\prod_{j=1}^{n}(t_j - x_j)}\, dt.$$

Therefore

$$\|D^k\psi_m(x)\|_p \leqq C_p\|\varphi_m^{(k)}\|_p \longrightarrow 0 \quad \text{as } m \longrightarrow \infty.$$

Hence, (i) and (ii) are satisifed.

If however, (i) and (ii) are assumed then there exists closed intervals $a_j \leqq t_j \leqq b_j$ containing the supports of all

$$\varphi_m(x) = \left(-\frac{1}{\pi^2}\right)^n \int_{\mathbf{R}^n} \frac{\psi_m(x)}{\prod_{j=1}^{n}(t_j - x_j)}\, dt.$$

Therefore

$$\|\varphi_m^{(k)}(x)\|_p \leqq \frac{1}{\pi^{2n}} C_p\|\psi_m^{(k)}\|_p \longrightarrow 0 \quad \text{as } m \longrightarrow \infty$$

i.e., $\varphi_m(x) \longrightarrow 0$ in $D_{L^p}(\mathbf{R}^n)$ as $m \longrightarrow \infty$. Therefore, by Lemma 10.1, $\varphi_m(x) \longrightarrow 0$ uniformly $\forall x \in \mathbf{R}^n$ as $m \longrightarrow \infty$. By (i) all $\varphi_m(x)$ have supports contained in a fixed n-box $\prod_{j=1}^n [a_j, b_j]$. Therefore if $\psi_m(x) \longrightarrow 0$ in $H(D(\mathbf{R}^n))$ as $m \longrightarrow \infty$ then the associated sequence $\{\varphi_m\}_{m=1}^\infty$ tends to zero in $D(\mathbf{R}^n)$ as $m \longrightarrow \infty$. Thus we have proved that

$$\varphi_m \longrightarrow 0 \text{ in } D(\mathbf{R}^n) \text{ as } m \longrightarrow \infty \Leftrightarrow \psi_m \longrightarrow 0 \text{ in } H(D(\mathbf{R}^n)) \text{ as } m \longrightarrow \infty.$$

Thus the conditions (i) and (ii) together describe intrinsically the convergence of a sequence $\{\psi_m\}_{m=1}^\infty$ to zero in $H(D(\mathbf{R}^n))$ as $m \longrightarrow \infty$.

Acknowledgement. This work was done while the author J. N. Pandey was the visiting Professor in the Department of Applied Mathematics Institute of Technology, Banaras Hindu University, India and the work was also supported by Canadian NSERC Grant #A5298. Author J. N. Pandey would also like to express his gratitude to Professors Hans Heinig of McMaster University, Canada, C. Carton Lebrun of Université de l'état a Mons, Belgium and the late Professor P. R. Beesack of Carleton University, Ottawa for giving valuable suggestions in the preparation of the manuscript.

REFERENCES

1. H. Bremmerman, *Distributions, complex variables and Fourier transforms* (Addison-Wesley Publishing Company Inc., 1965).

2. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Mathematica, T.*51* (1974), 241–250.

3. M. Cotlar, *Some generalizations of Hardy Littlewood maximal theorem*, Rev. Math. Cuyand *1* (1955), 85–104.

4. L. Ehrenpeis, *Analytic functions and the fourier transform of distributions*, J. Ann. Math. *63* (1956), 120–159.

5. A. Freidman, *Generalized functions and partial differential equations* (Prentice-Hall, Inc., 1963).

6. I. M. Gel'fand and G. E. Shilov, *Generalized functions, Vol. II* (Academic Press, 1968).

7. E. Hewit and K. Stromberg. *Real and abstract analysis* (Springer Verlag, 1969).

8. H. A. Laurier, *The Hilbert problem for generalized functions*, Arch. Rational Mech. Anal. *13* (1963), 157–166.

9. V. M. Kokilashvili, *Singular integral operators in weighted spaces*, Colloquia Mathematica Societatis in Janos Bolyai. *35* Functions, series, operators, Budapest (Hungary), 1980.

10. M. Orton, *Hilbert transformations, Plemelj relations and Fourier transform of distributions*, SIAM J. Math. Anal. *4* (1973), 656–667.

11. D. Mitrovic, *A distributional representation of analytic functions*, Mathematica Balkanica, 4-*97* (1974), 437–440.

12. N. I. Muskhelishvili, *Singular integral equations*, Moscow (1946).

13. J. N. Pandey and M. A. Chaudhry, *The Hilbert transform of generalized functions and applications*, Can. J. Math. *35* (1983), 478–495.

14. J. N. Pandey, *The Hilbert transform of Schwartz distribution*, Proc. of Amer. Math. Soc. *89* (1983), 96–90.

15. J. N. Pandey and O.P. Singh, *Characterization of functions with Fourier transform supported on orthants*, Trans. Amer. Math. Soc. (submitted).

16. L. Schwartz, *Theorie des distributions* (Hermann, Paris, 1966).

17. E. C. Titchmarsh, *Introduction to the theory of fourier integrals* (Oxford University Press, 1967).
18. J. H. Williams, *Lebesque integration* (Holt, Rinehard and Winston, New York, 1962).
19. A. H. Zemanian, *Distribution theory and transform analysis* (McGraw-Hill Book Company, 1965).
20. ——— *Generalized integral transforms* (Interscience Publishers, New York, 1968).

Banaras Hindu University,
Varanasi, India;

Carleton University,
Ottawa, Ontario