SHARP ESTIMATES OF APPROXIMATION BY SOME
POSITIVE LINEAR OPERATORS

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Recently, Varshney and Singh [Rend. Mat. (6) 2 (1982), 219-225]
have given sharper quantitative estimates of convergence for
Bernstein polynomials, Szasz and Meyer-Konig-Zeller operators.
We have achieved improvement over these estimates by taking
moments of higher order. For example, in case of the Meyer-
Konig-Zeller operator, they gave the following estimate
\[ \| L_n(f) - f \| \leq \left( \frac{2}{3\sqrt{3}} + \frac{2}{27} \right) \left( \frac{1}{\sqrt{n}} \right) \omega \left( f'; \frac{1}{\sqrt{n}} \right) \]
wherein \( \| \cdot \| \) stands for sup norm. We have improved this
result to
\[ \| L_n(f) - f \| \leq \left( \frac{2}{3\sqrt{3}} + \frac{10^2}{8} \right) \left( \frac{1}{27} \right) \left( \frac{1}{\sqrt{n}} \right) \omega \left( f'; \frac{1}{\sqrt{n}} \right). \]

We may remark here that for this modulus of continuity
\( \omega(f'; 1/\sqrt{n}) \) our result cannot be sharpened further by taking
higher order moments.

1. Introduction

Varshney and Singh [2] have proved the following theorem.

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THEOREM. Let $-\infty < a < b < \infty$, $p$ be a fixed positive integer. Let $K_n$ be a sequence of positive numbers and \{L_n\}_{n=1}^\infty be a sequence of positive linear operators, all having the same domain $D$ which contains the restrictions of $1, t, t^2, \ldots, t^{2p}$ to $[a, b]$. Suppose that \{L_n(1)\} is bounded. Let $f' \in D$ be continuous in $[a, b]$ with modulus of continuity $\omega(f'; \cdot)$. Then, for $n = 1, 2, \ldots$,

$$(1.1) \quad \|L_n(f) - f\| \leq \|f\| \cdot \|L_n(1) - 1\| + \|f'\| \cdot \mu_n^{(1)} \cdot \|L_n(1)\|^\frac{1}{2}$$

$$+ \omega\left(f'; K_n\mu_n^{(1)}\right)\left(\mu_n^{(1)} \|L_n(1)\|^\frac{1}{2} + \left(\mu_n^{(p)}\right)^2 / 2p \cdot \left(K_n\mu_n^{(1)}\right)^{2p-1}\right)$$

where $\mu_n^{(r)} = \|L_n(t-x)^{2r}(x)\|^\frac{1}{2}$ for $r = 1, p$ and $\|\cdot\|$ norm being sup norm over $[a, b]$.

However, if in addition $L_n(1)(x) = 1$ and $L_n(t)(x) = x$, then

$$(1.2) \quad \|L_n(f) - f\| \leq \omega\left(f'; K_n\mu_n^{(1)}\right)\left(\mu_n^{(1)} \|L_n(1)\|^\frac{1}{2} + \left(\mu_n^{(p)}\right)^2 / 2p \cdot \left(K_n\mu_n^{(1)}\right)^{2p-1}\right)$$

Using the above results, the best possible quantitative estimates of convergence for Bernstein polynomials, Szász and Meyer-Konig-Zeller operators are achieved in the following sections.

2. Bernstein polynomials

For $f \in C[0, 1]$, let the Bernstein operator of order $n$ be

$$L_n(f)(x) = \sum_{k=0}^{n} B_{n,k}(x)f(k/n)$$

where

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$  

We prove the following lemma.

LEMA. Let the function $T_{n,m}$ be defined on $[0, 1]$ for positive integers $m$ and $n$ by
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\[ T_{n,m}(x) = \sum_{k=0}^{n} B_{n,k}(x) \frac{(k/n-x)^m}{m!} \]

then

(2.1) \[ nT_{n,m+1}(x) = x \cdot (1-x) \cdot \left( nT_{n,m-1}(x) + T'_{n,m}(x) \right) \]

Proof. We have

\[ T'_{n,m}(x) = \sum_{k=0}^{n} B'_{n,k}(x) \frac{(k/n-x)^m}{m!} - mT_{n,m-1}(x) \]

Because \( x \cdot (1-x) \cdot B'_{n,k}(x) = B_{n,k}(x) \cdot (k-nx) \) for all \( 0 < k < n \), we write

\[ x \cdot (1-x) \cdot T'_{n,m}(x) \]

\[ = \sum_{k=0}^{n} x \cdot (1-x) B'_{n,k}(x) \cdot (k/n-x)^m - m \cdot x \cdot (1-x) \cdot T_{n,m-1}(x) \]

\[ = \sum_{k=0}^{n} B_{n,k}(x) \cdot n \cdot (k/n-x)^{m+1} - m \cdot x \cdot (1-x) \cdot T_{n,m-1}(x) \]

\[ = nT_{n,m+1}(x) - m \cdot x \cdot (1-x)T_{n,m-1}(x) \]

This leads to (2.1).

Using the above result we have

\[ \mu_n^{(1)} = \frac{1}{(2+\sqrt{n})}, \quad \mu_n^{(2)} = (3)^{\frac{1}{2}}/\sqrt{n} \quad \text{and} \quad \mu_n^{(3)} = (15/64n^3)^{\frac{1}{2}}. \]

Choosing \( K_n = 2n^{-\alpha+\frac{1}{2}} \) for \( 0 < \alpha \leq \frac{1}{2} \), \( p = 3 \) in (1.2), we obtain, for \( f \in C^1([0,1]) \) and \( n \geq 1 \),

(2.2) \[ \|L_n(f) - f\| \leq \left[ 1 + \frac{5}{64 \cdot (n^{-\alpha+\frac{1}{2}})} \right] \frac{1}{2 \cdot (\sqrt{n})} \omega(f'; n^{-\alpha}) \]

We note that (2.2) is sharper than the following estimate of Varshney and Singh [2]:

\[ \|L_n(f) - f\| \leq \left[ 1 + \frac{3}{32 \cdot (n^{-\alpha+\frac{1}{2}})} \right] \frac{1}{2 \cdot (\sqrt{n})} \omega(f'; n^{-\alpha}) \]
3. Szász operators

For \( f \in C[0, \infty) \), let the Szász operator of order \( n \) be

\[
L_n(f)(x) = \sum_{k=0}^{\infty} S_{n,k}(x)f(k/n)
\]

where

\[
S_{n,k}(x) = \exp \left( -nx \right) \cdot (nx)^k / k!
\]

We now prove the following lemma.

**Lemma.** Let the function \( T_{n,m} \) be defined on \([0, \infty)\) for positive integers \( m \) and \( n \) by

\[
T_{n,m}(x) = \sum_{k=0}^{\infty} S_{n,k}(x)(k/n-x)^m;
\]

then

\[
nT_{n,m+1}(x) = x \cdot \left[ mT_{n,m-1}(x) + T_{n,m}(x) \right].
\]  

**Proof.** We have

\[
T'_{n,m}(x) = \sum_{k=0}^{\infty} S'_{n,k}(x)(k/n-x)^m - mT_{n,m-1}(x).
\]

Now, because \( x \cdot S'_{n,k}(x) = (k-nx) \cdot S_{n,k}(x) \) for all \( 0 < k < n \), we write

\[
xT'_{n,m}(x) = \sum_{k=0}^{\infty} x \cdot S'_{n,k}(x)(k/n-x)^m - m \cdot xT_{n,m-1}(x)
\]

\[
= \sum_{k=0}^{\infty} n \cdot S_{n,k}(x)(k/n-x)^{m+1} - m \cdot xT_{n,m-1}(x)
\]

\[
= nT_{n,m+1}(x) - m \cdot xT_{n,m-1}(x).
\]

This leads to (3.1).

Using the above result we have, for \( x \in [0, \lambda) \), \( 0 < \lambda < \infty \),

\[
\mu_n^{(1)} = (\lambda/n)^{1/2}, \quad \mu_n^{(2)} = (3)^{1/2} \cdot (\lambda/n) \quad \text{and} \quad \mu_n^{(3)} = (15 \cdot 3^n / n^3)^{1/2}.
\]

Choosing \( K_n = (1/\lambda)^{1/2} \), \( p = 2 \) in (1.2), we obtain, for
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\( f \in C^1[0, \lambda] \) and \( n \geq 1 \),

\[
\|L_n(f) - f\| \leq \left[ \sqrt{\lambda} + \frac{3\lambda^2}{4} \right] \frac{1}{\sqrt{n}} \omega(f'; n^{-\frac{1}{2}}).
\]

We note that (3.2) is sharper than the following estimate of Varshney and Singh,

\[
\|L_n(f) - f\| \leq [\sqrt{\lambda} + \lambda/2] \frac{1}{\sqrt{n}} \omega(f'; n^{-\frac{1}{2}}),
\]

for all \( \lambda < 2/3 \).

Again choosing \( K_n = (1/\lambda)^{\frac{1}{2}} \), \( p = 3 \) in (1.2), we obtain, for

\( f \in C^1[0, \lambda] \) and \( n \geq 1 \),

\[
\|L_n(f) - f\| \leq [\sqrt{\lambda} + 5\lambda^3/2] (1/\sqrt{n}) \omega(f'; n^{-\frac{1}{2}}),
\]

which is sharper than (3.2) for all \( \lambda < 3/10 \).

It may be observed that if we have to approximate \( f \in C^1[0, \lambda] \) when \( \lambda \) is very small (possibly less than \( 3/10 \)), we may have still sharper estimates of the approximation by using higher values of \( p \) in (1.2).

4. Meyer-Konig-Zeller operators

For \( f \in C[0, \infty) \) let

\[
L_n(f)(x) = \sum_{k=0}^{\infty} Z_{n,k}(x) f(k/(n+k))
\]

be the Meyer-Konig-Zeller operator of order \( n \), where

\[
Z_{n,k}(x) = \binom{n+k}{n}(1-x)^{n+1} \cdot x^k.
\]

We note the following result of Rathore [1, (2.4), p. 213]:

\[
L_n(|t-x|^{\alpha}; x) = \left[ \Gamma(\alpha+1)/\Gamma(\frac{1}{2}) \right] \cdot (2x \cdot (1-x)^2/n)^{\alpha/2}.
\]

Using (4.1) we have

\[
\mu_n^{(1)} = (2/3 \cdot \sqrt{3} \cdot \sqrt{n}), \quad \mu_n^{(2)} = (\sqrt{3} \cdot 4 / 27n), \quad \mu_n^{(3)} = (\sqrt{3} \cdot 8 / 81 \cdot n^{3/2})
\]
Choosing $K_n = 3\sqrt{3}/2$, $p = 4$ in (1.2), we obtain, for $f \in C^1[0, 1]$ and $n \geq 1$,

$$
\|L_n(f) - f\| \leq \left[ 2/3\sqrt{3} + (105/8) \cdot (4/27)^4 \right] (1/\sqrt{n}) \omega(f'; 1/\sqrt{n}).
$$

We note that this is sharper than the following estimate given by Varshney and Singh [2]:

$$
\|L_n(f) - f\| \leq [2/3\sqrt{3}+2/27](1/\sqrt{n}) \omega(f'; 1/\sqrt{n}).
$$

We may remark that the estimate in (4.2) cannot be bettered by taking higher value of $p$ in (1.2) for modulus of continuity $\omega(f'; 1/\sqrt{n})$.

References


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