On Small Deformation of Sub-Spaces of a Flat Space

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The object of this paper is to introduce the differential operator, ∇, generalised for a Riemannian space V_n immersed in a flat space V_p, and then to discuss the general small deformation of V_n.

§ 1. *Notation.*

We shall use the notation of vector analysis in the flat space, and tensor calculus in the Riemannian space. Consider a Riemannian space V_n immersed in a flat space V_p, $p > n$. Let $\mathbf{r} = (z^1, z^2, \ldots . z^p)$ be the position vector of a point of V_p, the fundamental form of V_p being

$$\phi = \sum_{a=1}^{p} e_a (dz^a)^2, \quad e_a = \pm 1. \tag{1.1}$$

The scalar product of two vectors $\mathbf{a}$, $\mathbf{b}$ in V_p is defined to be

$$\mathbf{a} \cdot \mathbf{b} = \sum_{a=1}^{p} e_a a^a b^a. \tag{1.2}$$

The space V_n is given by equations of the form $z^a = z^a(x)$, where x^i $(i = 1, 2, \ldots, n)$ are the coordinates of V_n, and, substituting for the z's, we have $\mathbf{r}$ as a function of x for points of V_n. From the form (1.1), which can now be written $\phi = (d\mathbf{r})^2$, we find that the fundamental tensor of V_n is given by

$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j, \qquad \mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial x^i}. \tag{1.3}$$

We may consider $\mathbf{r}$ as an invariant in V_n, and we can differentiate the vector covariantly with respect to g_{ij}, obtaining vectors in V_p which have tensor forms in V_n.

By considering a small displacement in V_n of a point of V_n, we find that the n vectors $\mathbf{r}_i$ are tangent to V_n; they must also be independent in order that the coordinates x^i should be independent.

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Hence a vector tangent to V_n may be written in the form

$$\mathbf{t} = \lambda^i \mathbf{r}_i \tag{1.4}$$

where λ^i are the components of a contravariant vector in V_n. Thus a vector tangent to V_n can be defined either by a vector in V_p, or by a contravariant vector in V_n. It can easily be verified that these definitions define the same magnitude of such a vector and also the same angle between two such vectors. These results are important as showing some of the relations between the two methods of discussing a Riemannian space.

Differentiating (1.3) covariantly, we get $\mathbf{r}_{,ik} \cdot \mathbf{r}_j + \mathbf{r}_i \cdot \mathbf{r}_{,jk} = 0$, where $\mathbf{r}_{,ij}$ is the second covariant derivative[1] of $\mathbf{r}$. Permuting i, j, k, we at once find that

$$\mathbf{r}_i \cdot \mathbf{r}_{,jk} = 0, \quad (i, j, k = 1, 2, \ldots, n). \tag{1.5}$$

Hence $\mathbf{r}_{,jk}$ is orthogonal to every direction tangent to V_n, that is, is normal to V_n.

The normals to V_n are given by $\mathbf{N} \cdot \mathbf{r}_i = 0$, $(i = 1, 2, \ldots, n)$. There are $p - n$ independent normals, and these can be chosen to be mutually orthogonal, such a set of unit normals being written $\mathbf{N}_{\sigma|}$, $(\sigma = 1, 2, \ldots, p - n)$.

We can define tensors $\Omega_{\sigma|ij}$, $\mu_{\sigma\nu|i}$ by the equations

$$\begin{aligned} \Omega_{\sigma|ij} &= \mathbf{N}_{\sigma|} \cdot \mathbf{r}_{,ij} = -\mathbf{N}_{\sigma|,i} \cdot \mathbf{r}_j = -\mathbf{N}_{\sigma|,j} \cdot \mathbf{r}_i, \\ \mu_{\sigma\nu|i} &= \mathbf{N}_{\sigma|} \cdot \mathbf{N}_{\nu|,i} = -\mathbf{N}_{\nu|} \cdot \mathbf{N}_{\sigma|,i}. \end{aligned} \tag{1.6}$$

These tensors can easily be identified with the second fundamental tensors[2].

From (1.5) and (1.6), it follows that $\mathbf{r}_{,ij}$, $\mathbf{N}_{\sigma|,i}$ can be written in the forms

$$\begin{aligned} \mathbf{r}_{,ij} &= \sum_{\sigma=1}^{p-n} e_\sigma \Omega_{\sigma|ij} \mathbf{N}_{\sigma|}, \\ \mathbf{N}_{\sigma|,i} &= -\Omega_{\sigma|ij} g^{jk} \mathbf{r}_k - \sum_{\nu=1}^{p-n} e_\nu \mu_{\sigma\nu|i} \mathbf{N}_{\nu|}, \end{aligned} \tag{1.6}$$

where $e_\sigma = \mathbf{N}_{\sigma|}{}^2 = \pm 1$.

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[1] This is the usual notation for covariant derivatives. With this notation, we could write $\mathbf{r}_{,i}$ for $\mathbf{r}_i$.

[2] Eisenhart, *Riemannian Geometry*, § 47. The notation used by Eisenhart will be used throughout the paper.

§ 2. *Differential Operators.*

Generalising the operator, ∇, we define

$$\nabla = \sum_{h=1}^{n} e_h \, \mathbf{t}_{h|} \frac{\partial}{\partial s_h} \tag{2.1}$$

where $\mathbf{t}_{h|}$, $(h = 1, 2, \ldots, n)$ are the vectors of an orthogonal ennuple in V_n, $e_h = \mathbf{t}_{h|}{}^2 = \pm 1$, and $\partial f / \partial s_h$ is the intrinsic derivative of f in the direction $\mathbf{t}_{h|}$. From (1.4), using the usual notation for orthogonal ennuples, we have $\mathbf{t}_{h|} = \lambda_{h|}{}^i \, \mathbf{r}_i$, where $\lambda_{h|}{}^i$ are the contravariant components of the vectors in V_n. With this notation, we have $\partial / \partial s_h = \lambda_{h|}{}^i \, \partial / \partial x^i$; hence, using the equation

$$\sum_{h=1}^{n} e_h \, \lambda_{h|}{}^i \, \lambda_{h|}{}^j = g^{ij},$$

(2.1) becomes

$$\nabla = g^{ij} \, \mathbf{r}_i \frac{\partial}{\partial x^j}. \tag{2.2}$$

It is evident that this operator is independent of the ennuple chosen in the definition.

Operating on a scalar function, f, we get a vector $\nabla f = g^{ij} f_{,j} \, \mathbf{r}^i$ called the *gradient* of f. This vector is tangent to V_n, and is in the direction of critical variation of f, the magnitude being the variation.

Operating with closed product on a vector $\mathbf{R}$, we get a scalar, $\nabla \cdot \mathbf{R} = g^{ij} \mathbf{r}_i \cdot \mathbf{R}_{,j}$, called the *divergence* of $\mathbf{R}$. For $\mathbf{t} = \lambda^i \mathbf{r}_i$, we have, from (1.5),

$$\operatorname{div} \mathbf{t} = \lambda^i_{\ ,i}.$$

Operating with open product on a vector $\mathbf{R}$, we get a dyadic, $\nabla \mathbf{R} = g^{ij} \mathbf{r}_i \, \mathbf{R}_{,j}$.

It is easily shown that, if $\mathbf{s}$, $\mathbf{t}$ are unit vectors tangent to V_n at points of V_n, *the necessary and sufficient condition that the vectors* $\mathbf{s}$ *should be parallel in* V_n *along the curves of congruence defined by* $\mathbf{t}$, *is that* $\mathbf{t} \cdot \nabla \mathbf{s}$ *should be normal to* V_n. An equivalent condition is that $(\nabla \mathbf{s}) \cdot \mathbf{R}$ should be orthogonal to $\mathbf{t}$ for all vectors $\mathbf{R}$. In particular, $\mathbf{t}$ *defines a geodesic congruence if* $\mathbf{t} \cdot \nabla \mathbf{t}$ *is normal to* V_n.

If $\mathbf{t}_{h|}$ $(h = 1, 2, \ldots, n)$ are the vectors of an orthogonal ennuple in V_n, we find that the coefficients of rotation are given by

$$\gamma_{hkl} = \mathbf{t}_{l|} \cdot \nabla \, \mathbf{t}_{h|} \cdot \mathbf{t}_{k|}. \tag{2.3}$$

Hence, if we define *normal coefficients of rotation* by

$$I_{hk\sigma} = \mathbf{t}_{k|} \cdot \nabla \mathbf{t}_{h|} \cdot \mathbf{N}_{\sigma|} = \Omega_{\sigma|ij} \lambda_{h|}{}^{i} \lambda_{k|}{}^{j},$$

we have

$$\nabla \mathbf{t}_{h|} = \sum_{\theta,\phi} e_\theta e_\phi \gamma_{h\phi\theta} \mathbf{t}_{\theta|} \mathbf{t}_{\phi|} + \Sigma e_\theta e_\sigma I_{h\theta\sigma} \mathbf{t}_{\theta|} \mathbf{N}_{\sigma|}, \tag{2.4}$$

where $e_\theta = \mathbf{t}_{\theta|}{}^2 = \pm 1$; $e_\sigma = \mathbf{N}_{\sigma|}{}^2 = \pm 1$.

Prof. C. E. Weatherburn[1] has introduced an operator $\bar{\nabla}$, similar to ∇, in the study of a surface V_2. This can be generalised by considering some normal $\mathbf{N}$ of V_n, and defining

$$\bar{\nabla} = \sum_h e_h \kappa_h \mathbf{t}_{h|} \frac{\partial}{\partial s_h}, \tag{2.5}$$

where the ennuple $\mathbf{t}_{h|}$ is the principal ennuple for the normal $\mathbf{N}$, and κ_h are the corresponding principal curvatures. From the theory of principal directions, we have

$$\Sigma e_h \kappa_h \lambda_{h|}{}^{i} \lambda_{h|}{}^{j} = g^{il} g^{jm} \Omega_{lm} = \Omega^{ij},$$

where Ω_{ij} is the tensor associate to the normal $\mathbf{N}$.
Hence we have

$$\bar{\nabla} = \Omega^{ij} \mathbf{r}_i \frac{\partial}{\partial x^j}. \tag{2.51}$$

It can easily be verified that

$$\bar{\nabla} = -(\nabla \mathbf{N}) \cdot \nabla. \tag{2.52}$$

A second order operator may be defined by $\nabla^2 = \nabla \cdot \nabla$. For an invariant V, we have

$$\nabla^2 V = g^{ij} V_{,ij}. \tag{2.6}$$

Thus ∇^2 is the Beltrami operator Δ_2.

We see that

$$\nabla^2 \mathbf{r} = g^{ij} \mathbf{r}_{,ij} = M\mathbf{N}, \tag{2.7}$$

where $\mathbf{N}$ is the mean curvature normal[2], and M is the mean curvature of V_n. This shows that *the mean curvature normal, and the mean curvature are generalisations of the principal normal and curvature of a curve*, for we have, for a curve, $\nabla = \mathbf{t}\, d/ds$ where $\mathbf{t}$ is the unit tangent,

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[1] *Quart. Journ. of Maths.*, **50** (1927), 277.

[2] *Cf.* Eisenhart, *loc. cit.*, p. **169**.

and hence

$$\nabla^2 \mathbf{r} = \kappa \mathbf{n} \tag{2.71}$$

where $\mathbf{n}$ is the principle normal, and κ is the curvature.

Another second order operator is $\overline{\nabla} \cdot \nabla$. For an invariant V, we have

$$\nabla \cdot \nabla V = \Omega^{ij} V_{,ij}. \tag{2.8}$$

§ 3. *The general small deformation.*

We shall now examine the space V'_n obtained by deforming V_n in V_k.

Let ϵ be a constant of the order of magnitude of the greatest displacement of points of V_n, and let the deformation be such that ϵ^2 may be neglected. Then the position vector of a point of V'_n is given by

$$\mathbf{r}' = \mathbf{r} + \epsilon \mathbf{s} \tag{3.1}$$

where $\epsilon\mathbf{s}$ is the displacement vector of the point $\mathbf{r}$, $\mathbf{s}$ being a finite function of position on V_n. Let dashes refer to V'_n.

We have at once

$$\mathbf{r}'_i = \mathbf{r}_i + \epsilon \mathbf{s}_i \tag{3.11}$$

and hence,

$$g'_{ij} = \mathbf{r}'_i \cdot \mathbf{r}'_j = g_{ij} + \epsilon c_{ij} \tag{3.12}$$

where

$$c_{ij} = \mathbf{r}_i \cdot \mathbf{s}_j + \mathbf{r}_j \cdot \mathbf{s}_i. \tag{3.13}$$

From (3.12) and the identities $g'_{ij}\, g'^{jk} = \delta^k_i$, we get

$$g'^{ij} = g^{ij} - \epsilon c^{ij} \tag{3.14}$$

where

$$c^{ij} = g^{il}\, g^{jm}\, c_{lm}.$$

From (3.12) we have

$$g' = |g'_{ij}| = g\,(1 + \epsilon g^{ij} c_{ij})$$

$$i.e. \quad \sqrt{g'} = \sqrt{g}\,(1 + \epsilon \nabla \cdot \mathbf{s}). \tag{3.15}$$

If dV, dV' are corresponding elements of volume of V_n, V'_n respectively, the *dilation* is defined to be the ratio $(dV' - dV)/dV$. Hence, from (3.15), the dilation is given by

$$\frac{dV' - dV}{dV} = \epsilon \nabla \cdot \mathbf{s}, \tag{3.16}$$

i.e. the dilation is the divergence of the displacement vector.

Writing

$$2C_{ijk} = (c_{ij,k} + c_{ik,j} - c_{jk,i}); \quad C^h_{jk} = g^{ih} C_{ijk}, \tag{3.2}$$

we have

$$\Gamma'^h_{jk} = \Gamma^h_{jk} + \epsilon C^h_{jk} \tag{3.21}$$

where Γ^h_{jk}, Γ'^h_{jk} are the Christoffel symbols of the second kind. Hence the curvature tensor is given by

$$R'^h_{ijk} = R^h_{ijk} + \epsilon(C^h_{ik,j} - C^h_{ij,k}) \tag{3.22}$$

and from (3.12), we have

$$R'_{hijk} = R_{hijk} + \epsilon(c_{hl} R^l_{ijk} + C_{hik,j} - C_{hij,k}). \tag{3.23}$$

From this equation and (3.14), we get

$$R' = R - \epsilon(c^{ij} R_{ij} + c^{ij}{}_{,ij} - g^{ij} c_{,ij}) \tag{3.24}$$

where R_{ij} is the Ricci tensor, and $c = g^{ij} c_{ij} = 2 \nabla \cdot \mathbf{s}$.

Let $\mathbf{N}$ be a unit normal of V_n, and let $\mathbf{N}'$ be a corresponding unit normal of V'_n. We have $\mathbf{N}' \cdot \mathbf{r}'_i = 0$ $(i = 1, 2, \ldots., n)$, and writing $\mathbf{N}' = \mathbf{N} + \epsilon \bar{\mathbf{N}}$, we find

$$\bar{\mathbf{N}} = -(\nabla \mathbf{s}) \cdot \mathbf{N} \tag{3.3}$$

where $\bar{\mathbf{N}}$ is taken to be tangent[1] to V_n. Hence

$$\mathbf{N}' = \mathbf{N} - \epsilon(\nabla \mathbf{s}) \cdot \mathbf{N}. \tag{3.31}$$

If Ω_{ij} is the second fundamental tensor in V_n associate to the normal $\mathbf{N}$, and Ω'_{ij} the corresponding tensor for $\mathbf{N}'$, we have

$$\Omega'_{ij} = -\mathbf{N}'_{,i} \cdot \mathbf{r}'_j = \Omega_{ij} + \epsilon \mathbf{N} \cdot \mathbf{s}_{,ij}, \tag{3.32}$$

and hence, the mean curvature for the normal $\mathbf{N}'$ is given by

$$\Omega' = g'^{ij} \Omega'_{ij} = \Omega + \epsilon(\mathbf{N} \cdot \nabla^2 \mathbf{s} - 2 \bar{\nabla} \cdot \mathbf{s}) \tag{3.33}$$

where $\bar{\nabla}$ is the operator given by the normal $\mathbf{N}$.

The linear element of V'_n is given by

$$e ds'^2 = e ds^2 + \epsilon c_{ij} dx^i dx^j; \quad e ds^2 = g_{ij} dx^i dx^j. \tag{3.4}$$

Hence, the *extension* for the direction $\mathbf{t} = \lambda^i \mathbf{r}_i$ is given by

$$\epsilon E = \frac{ds' - ds}{ds} = \tfrac{1}{2} e \epsilon \lambda^i \lambda^j c_{ij} = e \epsilon \, \mathbf{t} \cdot \nabla \mathbf{s} \cdot \mathbf{t} \tag{3.41}$$

where $e = \mathbf{t}^2 = \pm 1$.

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[1] We need not take $\bar{\mathbf{N}}$ tangent to V_n, but we do so to define the particular normal $\mathbf{N}'$. All we actually know is that $\bar{\mathbf{N}}$ is orthogonal to $\mathbf{N}$, and satisfies $\bar{\mathbf{N}} \cdot \mathbf{r}_i + \mathbf{N} \cdot \mathbf{s}_i = 0$.

If ϵE_h $(h = 1, 2, \ldots, n)$ are the extensions for the directions of an orthogonal ennuple, we have $E_h = e_h \lambda_{h|}{}^i \lambda_{h|}{}^j \mathbf{r}_i \cdot \mathbf{s}_j$, and hence

$$\sum_h E_h = \nabla \cdot \mathbf{s}. \tag{3.42}$$

Thus *the sum of the extensions for n mutually orthogonal directions is independent of these directions and is equal to the dilation.*

From (3.41), we see that *the extension has critical values for the principal directions*[1] *determined by the tensor c_{ij}, and if ρ_h are the corresponding invariants, then* $2E_h = \rho_h$.

Writing

$$\begin{aligned} E_{hk} &= \lambda_{h|}{}^i \lambda_{k|}{}^j \mathbf{r}_i \cdot \mathbf{s}_j = \mathbf{t}_{k|} \cdot \nabla \mathbf{s} \cdot \mathbf{t}_{h|}, \\ \overline{E}_{h\sigma} &= \lambda_{h|}{}^i \mathbf{s}_i \cdot \mathbf{N}_{\sigma|} \quad = \mathbf{t}_{h|} \cdot \nabla \mathbf{s} \cdot \mathbf{N}_{\sigma|}, \end{aligned} \tag{3.43}$$

where $\mathbf{t}_{h|}$ are the vectors of any orthogonal ennuple, we have $E_{hh} = e_h E_h$, and

$$\nabla \mathbf{s} = \sum_{\theta, \phi} e_\theta e_\phi E_{\phi\theta} \mathbf{t}_{\theta|} \mathbf{t}_{\phi|} + \sum_{\theta, \sigma} e_\theta e_\sigma \overline{E}_{\theta\sigma} \mathbf{t}_{\theta|} \mathbf{N}_{\sigma|}. \tag{3.44}$$

From (3.11), it is easily shown that a direction $\mathbf{t}$ tangent to V_n becomes the direction $\mathbf{t}'$ tangent to V_n' where

$$\mathbf{t}' = \mathbf{t} + \epsilon(\mathbf{t} \cdot \nabla \mathbf{s} - E\mathbf{t}), \tag{3.5}$$

ϵE being the extension in the direction $\mathbf{t}$.

Hence, for two directions $\mathbf{t}_{1|}$, $\mathbf{t}_{2|}$ making an angle ω, the angle between the new directions is $\omega + \epsilon\,\theta$ where

$$\theta \sin \omega = \lambda_{1|}{}^i \lambda_{2|}{}^j c_{ij} - (E_1 + E_2) \cos \omega. \tag{3.51}$$

In particular, if $\omega = \pi/2$, we have

$$\theta = \lambda_{1|}{}^i \lambda_{2|}{}^j c_{ij} = \mathbf{t}_{1|} \cdot (\nabla \mathbf{s} + \mathbf{s} \nabla) \cdot \mathbf{t}_{2|}, \tag{3.52}$$

where $\mathbf{s}\nabla$ is the dyadic conjugate to $\nabla \mathbf{s}$, and hence, two orthogonal directions remain orthogonal if they satisfy

$$\lambda_{1|}{}^i \lambda_{2|}{}^j c_{ij} = 0. \tag{3.53}$$

From this condition, we see that *if two directions are orthogonal, and if one of them is a principal direction of c_{ij}, the directions remain orthogonal.*

Also *the only orthogonal ennuple remaining orthogonal is the principal ennuple of c_{ij}.*

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[1] An account of the principal directions of a tensor is given by Eisenhart, *loc. cit.* § 33.

If the principal ennuple of c_{ij} is also the principal ennuple given by a normal $\mathbf{N}$, it becomes the principal ennuple in V_n' of the normal $\mathbf{N}'$ if

$$(\Omega'_{ij} - \kappa'_h g'_{ij})\, \lambda_{h|}{}^i = 0. \tag{3.54}$$

Writing $\kappa'_h = \kappa_h + \epsilon\bar{\kappa}_h$, and substituting from (3.12), (3.32), these conditions become

$$\{(2\kappa_h E_h + \bar{\kappa}_h)\, g_{ij} - k_{ij}\}\, \lambda_{h|}{}^i = 0 \tag{3.55}$$

where $k_{ij} = \mathbf{N} \cdot \mathbf{s}_{,ij}$. Hence the ennuple must also be the principal ennuple of the tensor k_{ij}, and if ρ_h are the principal invariants for this tensor, the principal curvatures for the normal $\mathbf{N}'$ are $\kappa_h + \epsilon\kappa_h$ where

$$\bar{\kappa}_h = \rho_h - 2\kappa_h E_h. \tag{3.56}$$

Let us now find the conditions that a geodesic congruence λ^i in V_n becomes geodesic in V_n'. We have

$$\lambda^i \mathbf{r}_i \rightarrow \lambda'^i \mathbf{r}'_i, \quad \lambda'^i = (1 - \epsilon E)\, \lambda^i, \quad E = e\lambda^i \lambda^j c_{ij}. \tag{3.6}$$

Differentiating covariantly with respect to g'_{ij}, and substituting $\lambda^i_{,j} \lambda^j = 0$ in V_n, we find

$$\lambda'^i_{,j}\, \lambda'^j = \epsilon\lambda^j \lambda^k (C^i_{jk} - 2e\lambda^i \lambda^l C_{ljk}). \tag{3.61}$$

Hence the congruence remains geodesic if

$$\lambda^j \lambda^k (C^i_{jk} - 2e\lambda^i \lambda^l C_{ljk}) = 0. \tag{3.62}$$

Multiplying by λ_i and summing, we get

$$\lambda^l \lambda^j \lambda^k C_{ljk} = 0. \tag{3.63}$$

Substituting in (3.62), we have *the necessary and sufficient conditions that the geodesic congruence λ^i should remain geodesic are*

$$\lambda^j \lambda^k C^i_{jk} = 0, \quad (i = 1, 2, \ldots, n). \tag{3.64}$$

We at once see that *the necessary and sufficient conditions that all geodesics of V_n should become geodesics of V_n' are*

$$c_{ij,k} = 0 \quad (i, j, k = 1, 2, \ldots, n). \tag{3.65}$$

A more general theorem is as follows.

If the vectors μ^i are parallel along the curves of the congruence λ^i in V_n, the corresponding vectors are parallel along the corresponding curves in V_n' if

$$\lambda^j \mu^k C^i_{jk} = 0. \tag{3.66}$$

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The differential equations (3.65) have been studied by Eisenhart[1] and Levy[2]. A particular result is that when V_n has constant Riemannian curvature, the tensor c_{ij} must be a constant multiple of the fundamental tensor g_{ij}. In this case V_n, V_n' are conformal, and the extension is constant in all directions and at all points of V_n, being $\epsilon\rho$ where $c_{ij} = \rho\, g_{ij}$.

§ 4. *Some particular types of deformation.*

An *inextensible deformation* is such that all lengths remain unaltered. For this, we must have $g'_{ij} = g_{ij}$. Hence, the necessary and sufficient conditions for an inextensible deformation are

$$c_{ij} = 0 \quad (i, j = 1, 2, \ldots, n). \tag{4.1}$$

In this case, we have $\nabla \cdot \mathbf{s} = 0$, $\overline{\nabla} \cdot \mathbf{s} = 0$, and (3.33) reduces to

$$\Omega' = \Omega + \epsilon\, \mathbf{N} \cdot \nabla^2 \mathbf{s}. \tag{4.11}$$

From the definition, the curvature tensors remain unaltered.

A *normal deformation,* is such that all points of V_n are displaced in directions normal to V_n. If $\mathbf{N}$ is the normal direction of displacement of the point $\mathbf{r}$, the deformation is given by

$$\mathbf{s} = s\,\mathbf{N} \tag{4.2}$$

where s is a function of position of V_n.

If Ω_{ij} is the tensor associate to $\mathbf{N}$, and Ω the corresponding mean curvature, we have

$$c_{ij} = -\, 2s\, \Omega_{ij}, \tag{4.21}$$

and

$$\nabla \cdot \mathbf{s} = -\, s\, \Omega. \tag{4.22}$$

The normal to V_n' corresponding to $\mathbf{N}$ is now

$$\mathbf{N}' = \mathbf{N} - \epsilon \nabla s. \tag{4.23}$$

If $p = n + 1$, (3.24), (3.32), and (3.33) reduce to

$$R' = R + 2\epsilon\, (s\, \Omega^{ij} R_{ij} + \overline{\nabla} \cdot \nabla s - \Omega\, \nabla^2 s), \tag{4.24}$$

$$\Omega'_{ij} = \Omega_{ij} + \epsilon\, \{s_{,\, ij} - s\, (\Omega\, \Omega_{ij} + eR_{ij})\}, \tag{4.25}$$

$$\Omega' = \Omega + \epsilon\, \{\nabla^2 s + s\, (\Omega^2 + eR)\}, \tag{4.26}$$

where $e = \mathbf{N}^2 = \pm 1$.

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[1] *Trans. of the Amer. Math. Soc.*, 25 (1923), 297.

[2] *Annals of Math.*, 27 (1926), 91.

The only orthogonal ennuple remaining orthogonal for a normal deformation is now the principal ennuple of the normal **N**, and substituting for k_{ij} in (3.55), we find that this becomes the principal ennuple of **N**$'$ if it is also the ennuple given by the tensor $s_{,ij} - \sum_{\sigma} e_{\sigma} \mu_{\sigma|i} \mu_{\sigma|j}$ where $\mu_{\sigma|i} = \mathbf{N} \cdot \mathbf{N}_{\sigma|i}$, $\mathbf{N}_{\sigma|}$ $(\sigma = 1, 2, \ldots, p - n - 1)$ being orthogonal to **N**, and $e_{\sigma} = \mathbf{N}_{\sigma|}{}^2 = \pm 1$. Also, if ρ_h are the invariants of this tensor, the principal curvatures κ_h of **N** become $\kappa_h + \epsilon \bar{\kappa}_h$ where

$$\bar{\kappa}_h = \rho_h + s\kappa_h^2. \tag{4.27}$$

A *tangent deformation* is such that all points of V_n are displaced in directions tangent to V_n. Writing

$$\mathbf{s} = \lambda^i \mathbf{r}_i \tag{4.3}$$

we have

$$\mathbf{r}' = \mathbf{r} + \epsilon \lambda^i \mathbf{r}_i = \mathbf{r}(x^i + \epsilon \lambda^i) \tag{4.31}$$

to the first order of approximation.
Hence this *tangent deformation is equivalent to a point transformation of* V_n, *given by*

$$x'^i - x^i = \epsilon \lambda^i. \tag{4.32}$$

Tangent deformations have been discussed intrinsically from this point of view by McConnell.[1]

In concluding, we may remark that, writing δt for ϵ, and considering **s** also as a function of the parameter t, the spaces V_n, V_n' may be considered as members of a family of such spaces in V_p, *i.e.* hypersurfaces of a V_{n+1}. Many of the above results may then be interpreted as giving the variation with respect to t of the tensors, etc., connected with V_n.

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[1] *Annali di Mat.*, 6 (1928-1929), 207.