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INDUCTIVE LIMITS OF BANACH SPACES

BY
STEVEN F. BELLENOT

ABSTRACT. If X and Y are infinite-dimensional Banach spaces, then Y is the inductive limit of Banach spaces each isomorphic to X.

Several authors ([1], [2], and [3]) have obtained the result that every Banach space is an inductive limit of Hilbert spaces. Valdivia [5] has shown, that if the Banach space E has a weak-star separable dual, then every Banach space is an inductive limit of spaces isomorphic to E. The purpose of this note is to remove the restriction that E has a weak-star separable dual. We need the following facts.

Fact 1. If X is an infinite-dimensional Banach space, then there are

$$\{x_n\} \subset X, \qquad \{f_n\} \subset X^* \quad \text{with} \quad \|f_n\| = 1, \qquad f_n(x_m) = \delta_{nm}$$

and $\{\|x_n\|\}$ bounded. ([4], p. 10.)

Fact 2. A linear functional F, on Y, is in Y^* if and only if $\{F(y_n)\}$ is bounded for each $\{y_n\} \subset Y$ with $\sum_n \|y_n\| < \infty$. (If $\|F\|$ is not finite one can find $(\hat{y}_n)$ with $\|\hat{y}_n\| \le 1$ and $|F(y_n)| \ge 4^n$, from which the Fact follows with $y_n = \hat{y}_n/2^n$.)

THEOREM. *If X is an infinite-dimensional Banach space and Y is a Banach space, then Y is the inductive limit of spaces isomorphic to X.*

Proof. Let A be the set of continuous linear maps from X to Y. Let ξ (respectively η) be the norm topology (respectively, the inductive limit topology for the maps $T: X \to Y$, $T \in A$) on Y. By definition of η, $\xi \le \eta$. To show the reverse inequality, we show (Y, η) has the same dual as (Y, ξ) and invoke the barreledness of ξ.

Suppose F is a linear function on Y which is not in Y^*. By Fact 2, there is $\{y_n\} \subset Y$ with $\sum_n \|y_n\| < \infty$, and $F(y_n) \to \infty$. Let $\{x_n\}$, $\{f_n\}$ be as in Fact 1, then the map $T: X \to Y$ given by $T(x) = \sum_n f_n(x) y_n$ is in A. Furthermore, $FT(x_n) = F(y_n) \to \infty$, and F does not belong to the dual of (Y, η).

REMARK. It suffices to use the nuclear maps for A, rather than all of the continuous linear maps.

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Department of Mathematics
Florida State University
Tallahassee, Florida 32306
U.S.A.

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