FUNCTIONAL CALCULUS OF ELLIPTIC OPERATORS
WITH $C^\infty$ COEFFICIENTS
ON $L^p$ SPACES OF SMOOTH DOMAINS

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Abstract

The purpose of this paper is to show that higher order elliptic partial differential operators on smooth domains have an $H_\infty$ functional calculus and satisfy quadratic estimates in $L^p$ spaces on these domains.


1. Introduction

Let $A_B$ be the realization of an elliptic operator $A$ which acts on an $L^p$ space (with $1 < p < \infty$) and whose domain is defined by the boundary condition $B u = 0$. In [4] and [5], Seeley obtained an asymptotic expansion of the resolvent $R_\lambda = (A_B - \lambda)^{-1}$ which he used to prove the boundedness of purely imaginary powers of $A_B$:

$$\|A_B^\gamma\| \leq C e^{\gamma|\gamma|}$$

where $\gamma$ is real and $C$ and $\gamma$ are positive constants.

An operator is of type $\omega$ if its spectrum is contained in the sector $S_\omega = \{z \in \mathbb{C} : \text{arg} z \leq \omega\}$ and its resolvent satisfies certain bounds (a precise definition will be given in Section 2). In [3], McIntosh defined a functional calculus for such operators for functions which are analytic (but not necessarily bounded) on $S_\mu^0$ with $\mu > \omega$. He then established the equivalence of the following two properties for an operator $T$ defined on a Hilbert space:
\( (i) \| f(T) \| \leq C\| f \|_\infty, \ f \in H_\infty(S^0_\mu) \) where \( H_\infty(S^0_\mu) = \{ g: S^0_\mu \to \mathbb{C} | g \) is analytic and \( \| g \|_\infty < \infty \} \) and \( \| f \|_\infty = \sup\{|f(z)| |z \in S^0_\mu\} \).

\( (ii) \{ T^{iy} | y \in \mathbb{R} \} \) is a \( C^0 \) group and \( \| T^{iy} \| \leq C_\mu e^{\mu|y|} \) where \( \mu \) and \( C_\mu \) are positive constants.

Note that \( (i) \) is not equivalent to, and is in fact stronger than, \( (ii) \) for operators defined on Banach spaces. A natural question is whether the elliptic operators considered by Seeley satisfy property \( (i) \) in \( L^p \) spaces. The main result of this paper is to give an affirmative answer to this question.

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2. Operators of type \( \omega \) and their functional calculus

We first give some notation and definitions:

\[
S_\theta = \{ z \in \mathbb{C} | z = 0 \text{ or } |\arg z| \leq \theta \}, \\
S^0_\theta = \{ z \in \mathbb{C} | z \neq 0 \text{ and } |\arg z| < \theta \}.
\]

If \( 0 < \omega < \pi \), then an operator \( T \) in a Banach space \( X \) is said to be of type \( \omega \) if \( T \) is closed and densely defined, \( \sigma(T) \subset S_\omega \cup \{ \infty \} \) and for each \( \theta \in (\omega, \pi) \) there exists \( C_\theta < \infty \) such that \( \|(T - zI)^{-1}\| \leq C_\theta |z|^{-1} \) for all non-zero \( z \notin S^0_\theta \).

If \( 0 < \omega \leq \pi \), then

\[ H_\infty(S^0_\mu) = \{ f: S^0_\mu \to \mathbb{C} | f \) is analytic and \( \| f \|_\infty < \infty \} \]

where \( \| f \|_\infty = \sup\{|f(z)| |z \in S^0_\mu\} \)

\[ \Psi(S^0_\mu) = \left\{ f \in H_\infty(S^0_\mu) | s > 0, \ c \geq 0 \text{ and } |f(z)| \leq \frac{c|z|^s}{1 + |z|^{2s}} \text{ for all } z \in S^0_\mu \right\}. \]

Let \( \omega < \theta < \mu \) and \( \Gamma \) be the contour defined by the function

\[ g(t) = \begin{cases} -te^{-i\theta}, & -\infty < t \leq 0, \\ te^{i\theta}, & 0 \leq t < \infty. \end{cases} \]

Let \( T \) be of type \( \omega \). For \( \psi \in \Psi(S^0_\mu) \), we define \( \psi(T) \in L(X) \) by

\[ \psi(T) = \frac{1}{2\pi i} \int_\Gamma (T - \zeta I)^{-1} \psi(\zeta) \, d\zeta. \]  

For \( f \in H_\infty(S^0_\mu) \), \( f(T) \) can be defined by

\[ f(T) = (\psi(T))^{-1}(f\psi)(T) \text{ where } \psi(\zeta) = \frac{\zeta}{(1 + \zeta)^2}. \]
These definitions are well-defined and consistent. See [3] for details of this functional calculus.

We conclude this section with a convergence theorem which appeared in [3, Section 5]

**Theorem.** Let $0 \leq \omega < \mu \leq \pi$. Let $T$ be an operator of type $\omega$ which is one-one with dense range. Let $(f_\alpha)$ be a uniformly bounded net in $H_\infty(S^0_\mu)$. Let $f \in H_\infty(S^0_\mu)$, and suppose, for some $M < \infty$, that

1. $\|f_\alpha(T)\| \leq M$ and
2. for each $0 < \delta < \Delta < \infty$,
   $$\sup\{||f_\alpha(\zeta) - f(\zeta)|| \zeta \in S^0_\mu and \delta \leq |\zeta| \leq \Delta\} \to 0.$$

Then $f(T) \in L(X)$ and $f_\alpha(T)u \to f(T)u$ for all $u \in X$. So $\|f(T)\| \leq M$.

### 3. Elliptic boundary value problems

In this paper, we investigate the same elliptic boundary value problems that Seeley studied in [5].

Let $A$ be an elliptic operator of order $m$, with $C^\infty$ coefficients, acting on $q$-tuples of functions (or sections of a vector bundle) on a compact manifold $G \ (\dim G = n)$ with $C^\infty$ boundary $X$. An example is $A$ acting on the Banach space $L^p(G)$ with $1 < p < \infty$ where $G$ is a bounded, open subset of $\mathbb{R}^n$ with smooth boundary. Near the boundary, $A$ can be written in local coordinates as

$$A = \sum_{|\alpha| \leq m} a_\alpha(x, t)(D_{x,t})^\alpha = \sum_{j=0}^m A_j(t)D_t^{m-j}$$

where $(x_1, \ldots, x_{n-1})$ are local coordinates on $X$, $t$ is a normal variable with $t \geq 0$ in $G$, the $a_\alpha$ are $C^\infty$ square matrices, and

$$(D_{x,t})^\alpha = (-i\partial / \partial x_1)^{\alpha_1} \cdots (-i\partial / \partial t)^{\alpha_n}.$$

The principal symbol of $A$ is

$$\sigma_m(A)(x, t; \xi, \tau) = \sum_{|\alpha| = m} a_\alpha(x, t)(\xi, \tau)^\alpha.$$

A complex number $\lambda$ is a symbolic eigenvalue of $A$ if it is an eigenvalue of $\sigma_m(A)$ for some $(x, t; \xi, \tau)$. We assume that

1. $A$ is elliptic, that is, $\sigma_m(A)$ is nonsingular for $|\xi|^2 + \tau^2 > 0$,
2. $A$ has no symbolic eigenvalues $\lambda$ with $|\arg \lambda| \geq \gamma$ where $\gamma < \pi$. 
Consider the boundary operators

\[ B_j = \sum_{k=1}^{m} B_{jk} D_t^{m-k}, \quad j = 1, \ldots, mq/2, \]

where \( B_{jk} \) is a system of \( C^\infty \) differential operators on \( X \) of order at most \( m_j + k - m \); thus \( B \) has order at most \( m_j \). We denote

\[
\sigma_0'(B_j) = \sum_{k=1}^{m} \sigma_{m_j+k-m}(B_{jk}) D_t, \tag{3.5}
\]

\[
\sigma_m'(A) = \sum_{j=0}^{m} \sigma_j(A_j(0)) D_t^{m-j}. \tag{3.6}
\]

The operator \( A_B \) is defined as the closure in \( L^2 \) of the operator \( A \) acting on the domain

\[ C_B^\infty = \{ u \in C^\infty(G) | B_j u = 0 \text{ on } X, \text{ for } j = 1, \ldots, q/2 \}. \]

A complex number \( \lambda \neq 0 \) is a symbolic eigenvalue of \( A_B \) unless, for every \( x \) and \( \zeta \) and every choice of the constants \( g_j \), there is a unique solution \( u \) of the system of ordinary differential equations

\[
\begin{aligned}
\sigma_m'(A)u &= \lambda u \quad \text{for } t > 0, \\
\sigma_0'(B_j)u &= g_j \quad \text{for } t = 0, \quad j = 1, \ldots, mq/2, \\
u(+\infty) &= 0.
\end{aligned} \tag{3.7}
\]

In addition, the symbolic eigenvalues of \( A \) are symbolic eigenvalues of \( A_B \). We assume that

\[
A_B \text{ is elliptic, that is (3.7) has a unique solution when } \lambda = 0 \text{ and } \xi \neq 0. \tag{3.8}
\]

\[
A_B \text{ has no symbolic eigenvalues } \lambda \text{ with } |\arg \lambda| \geq \gamma \text{ where } \gamma \text{ is independent of the choice of coordinate system.} \tag{3.9}
\]

Under these conditions and by adding a constant \( C_0 \) if necessary, it can be verified that \( (A_B + C_0) \) is an operator of type \( \omega \) and the functional calculus of \( (A_B + C_0) \) can be defined by (2.1) or (2.2). The estimate of the \( L^p \) norm \( \| \psi(A_B + C_0) \|_p \) is based on the expansion of the resolvent \( (A_B + C_0 - \lambda)^{-1} \) in the next section. From now on, for convenience we denote \( (A_B + C_0) \) by \( A_B \).
4. The expansion of \((A_B - \lambda)^{-1}\)

Under the conditions of Section 3, Seeley proved in [4] that the resolvent \((A_B - \lambda)^{-1}\) exists for \(|\arg \lambda| \geq \gamma\) and can be written, for any integer \(K\), in the form

\[
(A_B - \lambda)^{-1} = \sum_k \chi_k \left( \sum_{j=0}^K \mathcal{O}_k(\theta_2 c_{-m-j}) + \sum \mathcal{O}_k'(\theta_1 d_{-m-j}) \right) \Phi_k + R_k
\]

where \(R_k\) is an integral operator

\[
R_k f(x) = \int_G r_k(x,y)f(y)\,dy \quad \text{and} \quad |D_x^\alpha D_{\xi}^\beta r_k(x,y)| \leq C_{\alpha \beta}(1 + |\lambda|^{1/m})^n + |\alpha| + |\beta| - K - m,
\]

\(\varphi_k\) is a partition of unity in \(X\), \(\varphi_k \chi_k = \varphi_k\) and \(\chi_k\) has support in a neighborhood \(U_k\) of the support of \(\varphi_k\).

The functions \(c_{-m-j}\) and \(d_{-m-j}\) are defined for \(|\arg \lambda| \geq \gamma\) and for \(|\lambda|/|\xi|\) sufficiently small, and satisfy the conditions

\[
c_{-m} = (\sigma_m(A) - \lambda)^{-1},
\]

\[
|D_x^\alpha D_{\xi}^\beta c_{-m-j}(x, t, \xi, \tau, \lambda)| \leq C_{\alpha \beta}(|\xi| + |\tau| + |\lambda|^{1/m})^{-m-j-|\beta|},
\]

\[
|D_x^\alpha D_{-m-j}(x, t, \xi, \tau, s, \lambda)| \leq C_{\alpha \beta}e^{-c(|t+s|)(|\xi| + |\lambda|^{1/m})(|\xi| + |\lambda|^{1/m})^{-m-j-|\beta|}}.
\]

The operators \(\mathcal{O}_p(c)\) and \(\mathcal{O}_p''(d)\) are defined by

\[
\mathcal{O}_p(c)f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x,\xi) + it\tau} c(x, t, \xi, \tau, \lambda) \hat{f}(\xi, \tau) \,d\xi \,d\tau,
\]

\[
\mathcal{O}_p''(d)f(x,t) = (2\pi)^{1-n} \int_0^\infty \int_{\mathbb{R}^n} e^{i(x,\xi)} d(x, t, \xi, s, \lambda) \hat{f}(\xi, s) \,d\xi \,ds.
\]

where \(\hat{f}(\xi, \tau)\) and \(\hat{f}(\xi, s)\) are the full and tangential Fourier transforms of \(f(x, t)\), respectively, understood to be taken after we transform \(G\) to \(\mathbb{R}^n_+\) via the usual technique of local maps and partitions of unity.

\[
\begin{align*}
\theta_1(\xi, \lambda) &= C^\infty, \quad \theta_1 \equiv 0 \text{ for } (|\xi|^2 + |\lambda|^{2/\omega}) \text{ small}, \\
\theta_1 &\equiv 1 \text{ for } (|\xi|^2 + |\lambda|^{2/\omega}) \text{ large}, \\
\theta_2(\xi, \lambda) &\text{ satisfies the same condition with } (|\xi|^2 + \tau^2) \text{ in place of } |\xi|^2.
\end{align*}
\]
5. $H_\infty$ functional calculus for elliptic operators

We first need the following two lemmas 1 and 2 which are essentially due to Seeley [5].

**Lemma 1.** Let $1 < p < \infty$, $0 < R < \infty$. There is a constant $C = C(p, n, R)$ such that if $k(x, \xi)$ vanishes for $|x| \geq R$ and $|D_\xi^\alpha D_\xi^\beta k(x, \xi)| \leq 1$ for $|\alpha| \leq n + 1$, $|\beta| \leq n$ then $\| \text{Op}(k) \|_p \leq C$. (Here, $x$ and $\xi$ are in $\mathbb{R}^n$.)

**Proof.** Let

$$k_\eta(x, \xi) = \int_{\mathbb{R}^n} e^{i(x,y)} k(x, \xi) \, dx.$$  

Then

$$k(x, \xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i(x,y)} k_\eta(x, \xi) \, dy$$

and

$$\text{Op}(k) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x,y)} \text{Op}(k_\eta) \, dy.$$  

Hence

$$\| \text{Op}(k) \|_p \leq (2\pi)^{-n} \int_{\mathbb{R}^n} \| \text{Op}(k_\eta) \|_p \, dy. \tag{5.2}$$

The Mikhlin multiplier theorem gives us the estimate

$$\| \text{Op}(k_\eta) \|_p \leq C_p \sup_{\xi \in \mathbb{R}^n} \max_{|\beta| \leq n} |\xi^\beta D_\xi^\beta k_\eta(\xi)|, \quad 1 < p < \infty.$$  

Since

$$\eta^\alpha \xi^\beta D_\xi^\beta k_\eta(\xi) = \int_{\mathbb{R}^n} [D_\xi^\alpha \xi^\beta D_\xi^\beta k(x, \xi)] e^{i(x,y)} \, dx$$

and $k(x, \xi)$ vanishes for $x \geq R$, we have

$$|\eta^\alpha| |\xi^\beta D_\xi^\beta k_\eta(\xi)| \leq C_{\alpha\beta} \quad \text{for } |\alpha| \leq n + 1. \tag{5.3}$$

Thus $\| \text{Op}(k_\eta) \|_p \leq C_p (1 + |z|)^{-n-1}$ and it follows from (5.2) that

$$\| \text{Op}(k) \|_p \leq (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{C_p}{(1 + |z|)^{n+1}} \, dz = C(p, n, R). \tag{5.4}$$

**Lemma 2.** Let $1 < p < \infty$. There is a constant $C = C(p, n, R)$ such that if $k(x, t, \xi, s)$ has support in $|x| \leq R$ and satisfies

$$|\xi| |\xi^\beta D_\xi^\beta D_x^\beta k(x, t, \xi, s)| \leq \frac{1}{t + s}$$

for $s > 0$, $t > 0$, $|\beta| \leq n - 1$, $|\alpha| \leq n$, then $\| \text{Op}''(k) \|_p \leq C$. (Here, $x$ and $\xi$ are in $\mathbb{R}^{n-1}$.)
PROOF. Let \( k_{t,s}(x,\xi) = k(x, t, \xi, s) \) and \( K_{t,s} = \text{Op}(k_{t,s}) \). Then
\[
\text{Op}''(k) f(x, t) = \int_0^\infty K_{t,s} f(x, s) \, ds.
\]
Thus
(5.5)
\[
\int_0^\infty \int_{\mathbb{R}^{n-1}} \left| \int_0^\infty K_{t,s} f(x, s) \, ds \right|^p \, dx \, dt
\leq 
\int_0^\infty \left( \int_0^\infty \left( \int_{\mathbb{R}^{n-1}} |K_{t,s} f(x, s)|^p \, dx \right)^{1/p} \, ds \right)^p \, dt
\]
(Minkowski's inequality)
\[
\leq \int_0^\infty \left( \int_0^\infty \frac{C_1}{t + s} \left( \int_{\mathbb{R}^{n-1}} |f(x, s)|^p \, dx \right)^{1/p} \, ds \right)^p \, dt \quad \text{(Lemma 1)}
\]
\[
= C_1^p \int_0^\infty \left( \int_0^\infty \frac{1}{t + s} \phi(s) \, ds \right)^p \, dt
\]
where \( \phi(s) = \left( \int_{\mathbb{R}^{n-1}} |f(x, s)|^p \, dx \right)^{1/p} \).

For \( 1/p + 1/p' = 1 \) and \( 0 < q < \min\{1/p', 1/p\} \), we also have
\[
\left\| \int_0^\infty \frac{\phi(s)}{t + s} \, ds \right\|_p = \left\{ \int_0^\infty \left( \int_0^\infty \left( \frac{1}{t + s} \right)^{1/p} \left( \frac{t}{s} \right)^q \left( \frac{1}{t + s} \right)^{1/p} \phi(s) \, ds \right)^{1/p} \right\}^{1/p}
\leq \left\{ \int_0^\infty \left( \int_0^\infty \frac{1}{t + s} \left( \frac{t}{s} \right)^{qp'} \, ds \right)^{p/p'} \right. \times \left( \int_0^\infty \frac{1}{t + s} \left( \frac{s}{t} \right)^{qp} |\phi(s)|^p \, ds \right) \, dt \right\}^{1/p}
\leq \left( \sup_t \int_0^\infty \frac{1}{t + s} \left( \frac{t}{s} \right)^{qp'} \, ds \right)^{1/p'}
\times \left( \sup_s \int_0^\infty \frac{1}{t + s} \left( \frac{s}{t} \right)^{qp} \, dt \right)^{1/p} \|\phi(s)\|_p
\]
\[
= \left( \int_0^\infty \frac{1}{1 + \sigma_1} \left( \frac{1}{\sigma_1} \right)^{qp'} \, d\sigma_1 \right)^{1/p'}
\times \left( \int_0^\infty \frac{1}{1 + \sigma_2} \left( \frac{1}{\sigma_2} \right)^{qp} \, d\sigma_2 \right)^{1/p} \|\phi(s)\|_p
\]
(5.5')
\[
= C_2 \|\phi(s)\|_p.
\]
It follows from (5.5) and (5.5') that \( \|\text{Op}''(k)\|_p \leq C \).
We now come to the $L^p$ norm estimate for the $H_\infty$ functional calculus corresponding to the operator $\text{Op}''$.

**Lemma 3.** Suppose that $k(x, t, \xi, s, \lambda)$ vanishes for $|x| \geq R$ and there is a positive constant $c$ such that

\begin{equation}
|D_x^\alpha D_\xi^\beta k| \leq e^{-c(t+s)(|\xi|+|\lambda|)^{1/m}}(|\xi| + |\lambda|^{1/m})^{1-m-|\beta|}
\end{equation}

for $m \geq 1$, $|\alpha| \leq n$ and $|\beta| \leq n$. Then for each $p$, $1 < p < \infty$, there is a constant $C_p$ (depending on $p$, $c$, $r$ and $n$) such that

\[ \| \text{Op}'' \left( \int \psi(\lambda)k(x, t, \xi, s, \lambda), d\lambda \right) \|_p \leq C_p \| \psi \|_\infty, \quad \psi \in H_\infty(S^0_\mu). \]

**Proof.** Let $K(x, t, \xi, s) = \int \psi(\lambda)k(x, t, \xi, s, \lambda) d\lambda$ with the curve $\Gamma$ as in (2.1). Then

\[ (t+s)D_x^\alpha|\xi|^{|\beta|}D_\xi^\beta K \leq (t+s)|\xi|^{\beta} \int e^{-c(t+s)(|\xi|+|\lambda|)^{1/m}} \times (|\xi| + |\lambda|^{1/m})^{1-m-|\beta|} |\psi(\lambda)| d|\lambda| \]

\[ \leq m(t+s)|\xi|^{\beta}e^{-c(t+s)|\xi|} \int_0^\infty e^{-c(t+s)\lambda_1} \times (|\xi| + \lambda_1)^{1-m-|\beta|} |\psi(\lambda)| \cdot (\lambda_1)^{m-1} d\lambda_1 \]

where $\lambda_1 = |\lambda|^{1/m}$

\[ \leq m(t+s)e^{-c(t+s)|\xi|} \| \psi \|_\infty \int_0^\infty e^{-c(t+s)\lambda_1} d\lambda_1 \]

\[ = 2m(t+s)e^{-c(t+s)|\xi|} \| \psi \|_\infty \int_0^\infty \frac{1}{c(t+s)} d\lambda_1 \]

\[ \leq K \| \psi \|_\infty. \]

Hence Lemma 3 follows from Lemma 2.

The main result of this paper is the following theorem.

**Theorem.** There exists $M > 0$ such that

\begin{equation}
\| \psi(A_B) \|_p \leq M \| \psi \|_\infty, \quad \psi \in H_\infty(S^0_\mu), \quad 1 < p < \infty.
\end{equation}

**Proof.** We first consider the case $\psi \in \Psi(S^0_\mu)$. For $R_K f(x) = \int_G r_K(x, y) f(y) dy$,

we have

\[ \left\{ \int_G |R_K f(x)|^p dx \right\}^{1/p} = \left\{ \int_G \left( \int_G r_K(x, y) f(y) dy \right)^p dx \right\}^{1/p} \]
and it follows from (4.2) and Hölder’s inequality that
\[ \|R_k\|_p \leq c(1 + |\lambda|^{1/m})^{n-m-K}. \]

Thus
\[
\left\{ \int_G \left| \int_\Gamma |\psi(\lambda)[R_K f(x)]| d\lambda \right|^p dx \right\}^{1/p} \\
\leq \int_\Gamma \left\{ \int_G |\psi(\lambda)[R_K f(x)]|^p dx \right\}^{1/p} d\lambda \\
\leq \left\{ \int_\Gamma \frac{c}{(1 + |\lambda|^{1/m})^{K+m-n}} d\lambda \right\} \|f\|_p \\
= M_1 \|\psi\|_\infty\|f\|_p \quad \text{when we choose } K > n. \tag{5.8}
\]

For the term \( c_{-m} = (\sigma_m(A) - \lambda)^{-1} \), we have
\[
\text{Op} \left( \int_\Gamma \theta_2 c_{-m} \psi(\lambda) d\lambda \right) f(x, t) \\
= (2\pi)^{-n} \int \int_{\mathbb{R}^n} e^{i(x,\xi) + i\tau} \left( \int_\Gamma \frac{\theta_2 \psi(\lambda) d\lambda}{\sigma_m(A) - \lambda} \right) f(\xi, T) d\xi d\tau.
\]

Since
\[
\int_\Gamma \frac{\theta_2 \psi(\lambda) d\lambda}{\sigma_m(A) - \lambda} = \psi(\sigma_m(A)) \quad \text{for } |\xi|^2 + \tau^2 \text{ large}
\]
and the analytic function \( \psi \) satisfies the conditions of the Mikhlin multiplier theorem, we obtain
\[
\left\| \text{Op} \left( \int_\Gamma \theta_2 c_{-m} \psi(\lambda) d\lambda \right) \right\|_p \leq M_2 \|\psi\|_\infty.
\]

For the term \( c_{-m-j} \) with \( j \geq 1 \), we apply Lemma 1 with \( k = \theta_2 c_{-m-j} \), noting (4.5), and obtain
\[
(|\xi|^2 + \tau^2)^{1/2}\beta|\cdot(D_{x,\tau})^\alpha(D_{\xi,\tau})^\beta k(x, t, \xi, \tau, \lambda) | \\
\leq (|\xi|^2 + \tau^2)^{1/2}\beta|C_{\alpha\beta}(1 + |\xi| + |\tau| + |\lambda|^{1/m})^{m-j} - |\omega|) \\
\leq C'_{\alpha\beta}(1 + |\lambda|^{1/m})^{-m-j}
\]
Therefore $\|\text{Op}(\theta_2 c_{-m-j})\|_p \leq C(1 + |\lambda|^{1/m})^{-m-j}$. Thus
\[
\left\| \int_G \psi(\lambda) \text{Op}(\theta_2 c_{-m-j}) f d\lambda \right\|_p^p \leq \left\{ \int_G \left\{ \int_G |\psi(\lambda) \text{Op}(\theta_2 c_{-m-j}) f|^p \ dx \right\} d\lambda \right\}^{1/p} \leq \left\{ \int_G |\psi(\lambda)| \text{Op}(\theta_2 c_{-m-j}) \| f \|_p d\lambda \right\} \|f\|_p.
\]

Therefore
\[
(5.10) \quad \left\| \text{Op} \left( \int_G \theta_2 c_{-m-j} \psi(\lambda) \right) \right\|_p \leq M_3 \|\psi\|_\infty.
\]
The estimate
\[
(5.11) \quad \left\| \text{Op} \left( \int_G \theta_1 d_{-m-j} \psi(\lambda) \right) \right\|_p \leq M_4 \|\psi\|_\infty
\]
follows from (4.6), Lemma 3 and the inequality $\theta_1(|\xi| + |\lambda|^{1/m})^{-j} \leq \text{constant}$. It follows from (5.8), (5.9), (5.10), (5.11) and (4.1) that
\[
(5.12) \quad \|\psi(A_B)\|_p \leq M \|\psi\|_\infty, \quad \psi \in \Psi(S_0^0), \ 1 < p < \infty.
\]
To extend the result for $\psi \in H_\infty(S_0^0)$, we define $\psi_{\varepsilon,R} \in \Psi(S_0^0)$ by
\[
\psi_{\varepsilon,R}(\xi) = \int_0^R (\psi_\varepsilon(t))(\xi) \frac{dt}{t}
\]
where $\phi \in \Psi(S_0^0)$ such that $\int_0^\infty \phi(t) \ dt/t = 1$ and $\phi_\varepsilon(t) = \phi(t\xi)$. The inequality (5.12) shows that $\|\psi_{\varepsilon,R}(A_B)\|_p \leq M \|\psi\|_\infty$ where $M$ is independent of $\varepsilon$ and $R$. It is not difficult to check that all the conditions of the theorem in Section 2 are satisfied. We conclude that
\[
\|\psi(A_B)\|_p \leq M \|\psi\|_\infty, \quad \psi \in H_\infty(S_0^0), \ 1 < p < \infty.
\]
It is an interesting and still open question whether the $H_\infty$ functional calculus property still holds for elliptic operators with milder assumptions on the smoothness of the coefficients or on the boundaries of the domains.

It is also worth noting that the $H_\infty$ functional calculus property implies that the elliptic operator $A_B$ satisfies quadratic estimates, that is, there exists a positive constant $K_p$ such that
\[
\left\| \left\{ \int_0^\infty |\psi(t A_B) u(t)\|^2 \frac{dt}{t} \right\}^{1/2} \right\|_p \leq K_p \|u\|_p, \quad \text{for all } u \in L^p(G), \ 1 < p < \infty.
\]
for certain classes of functions $\psi$, including, for example

$$\psi(z) = z/(z - \alpha)(z - \alpha) \quad \text{where } \alpha \in S_\omega.$$ 

Details can be found in the joint paper of M. Cowling, A. McIntosh and A. Yagi [2].

References


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