# ON A THEOREM OF ISEKI

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**1.** The purpose of this paper is to generalize a result of K. Iseki [**1**]. In his note, Iseki proves that, in a normal space $S$, for every countable discrete collection $\mathscr{H} = \{H_1, H_2, \ldots\}$ of sets from $S$, there exists a countable collection $\mathscr{U} = \{U_1, U_2, \ldots\}$ of mutually disjoint open sets from $S$ such that $\bar{H}_i \subset U_i$ for every $i$.

In this paper we consider almost discrete, separated and completely separated collections of sets from a topological space. It is shown that the analogous property holds for almost discrete collections in a normal space, and for separated collections in a completely normal space. The well-known property that a topological space is normal if and only if any two closed disjoint sets are completely separated, is here expressed in terms of completely separated discrete collections.

**2. Definitions.** Let $S$ be a topological space. We denote by $\mathscr{H} = \{H_\alpha : \alpha \in \Omega\}$ an arbitrary collection of sets from $S$. For a collection $\mathscr{H}$, we denote by $\bar{\mathscr{H}}$ the collection of closures of sets from $\mathscr{H}$. Open collections, i.e. collections of open sets, will be denoted by $\mathscr{U}$, $\mathscr{V}$.

A collection $\mathscr{H} = \{H_\alpha : \alpha \in \Omega\}$ is called *discrete* if it satisfies the following two conditions:

(1) $\bar{H}_\alpha \cap \bar{H}_\beta = \emptyset$ for every $\alpha$, $\beta \in \Omega$, $\alpha \neq \beta$;

(2) for an arbitrary subset $\Gamma$ of $\Omega$, $\bigcup_{\alpha \in \Gamma} \bar{H}_\alpha = \overline{\bigcup_{\alpha \in \Gamma} H_\alpha}$.

If $\mathscr{H}$ is discrete, then $\bar{\mathscr{H}}$ is also discrete. For every discrete collection $\mathscr{H}$, $\bigcup_{\alpha \in \Omega} \bar{H}_\alpha$ is a closed set in $S$.

From the above definition, with some modifications, we get the following definition for almost discrete collections:

A collection $\mathscr{H} = \{H_\alpha : \alpha \in \Omega\}$ is *almost discrete* if, for every $\alpha \in \Omega$,

$$\bar{H}_\alpha \cap \overline{\bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} H_\beta} = \emptyset.$$

It follows that, for such a collection, for every $\alpha', \alpha'' \in \Omega$ with $\alpha' \neq \alpha''$, we have $\bar{H}_{\alpha'} \cap \bar{H}_{\alpha''} = \emptyset$. Every discrete collection is almost discrete, but it is very easy to verify that the converse is not true. From the definition it follows that, if $\mathscr{H}$ is almost discrete, then $\bar{\mathscr{H}}$ is almost discrete.

A collection $\mathscr{H} = \{H_\alpha : \alpha \in \Omega\}$ is *separated* if, for every $\alpha \in \Omega$,

$$H_\alpha \cap \overline{\bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} H_\beta} = \emptyset.$$

Every almost discrete collection is separated, but the converse is not true. Every collection of mutually disjoint open sets is separated. If $\mathscr{H} = \{H_\alpha : \alpha \in \Omega\}$ is separated, then, for every $\alpha \in \Omega$, we have $\bar{H}_\alpha \cap \bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} H_\beta = \emptyset$, i.e. the sets $H_\alpha$ and $\bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} H_\beta$ are separated in the usual sense for

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every $\alpha \in \Omega$. The condition $\bar{H}_\alpha \cap \bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} H_\beta = \emptyset$ for every $\alpha \in \Omega$, does not imply that the collection is separated.

For countable collections we introduce the concept of completely separated collection in the following way:

A countable collection $\mathscr{H} = \{H_1, H_2, \ldots\}$ is *completely separated* if, for every sequence $a_1, a_2, \ldots$ of real numbers, there exists a continuous function $f$ on $S$ such that $f(H_i) = a_i$ for every $i$.

Two completely separated sets in the usual sense form a completely separated collection. Every countable completely separated collection is discrete.

**3. Normal spaces.** In a normal space, the following additional properties hold for almost discrete collections:

LEMMA. 1. *Let $S$ be a normal space. If, for every almost discrete collection $\mathscr{H} = \{H_\alpha : \alpha \in \Omega\}$, there exists a collection $\mathscr{V} = \{V_\alpha : \alpha \in \Omega\}$ of mutually disjoint open sets such that, for every $\alpha \in \Omega$, we have $\bar{H}_\alpha \subset V_\alpha$, then there exists an almost discrete collection $\mathscr{U} = \{U_\alpha : \alpha \in \Omega\}$ of open sets such that $\bar{H}_\alpha \subset U_\alpha$ for every $\alpha \in \Omega$.*

*Proof.* For every $\alpha \in \Omega$, let $U_\alpha$ be an open set such that $\bar{H}_\alpha \subset U_\alpha \subset \bar{U}_\alpha \subset V_\alpha$; then $U_\alpha \cap \overline{\bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} U_\beta} = \emptyset$. But, since $\bar{U}_\alpha \subset V_\alpha$ and $\overline{\bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} U_\beta} \subset \overline{\bigcup_{\substack{\beta \in \Omega \\ \beta \neq \alpha}} V_\beta}$, it follows that $\bar{U}_\alpha \cap \overline{\bigcup_{\substack{\alpha \in \Omega \\ \beta \neq \alpha}} U_\beta} = \emptyset$. Hence $\mathscr{U} = \{U_\alpha : \alpha \in \Omega\}$ is almost discrete.

THEOREM 1. *A topological space $S$ is normal if and only if, for every countable almost discrete collection $\mathscr{H} = \{H_1, H_2, \ldots\}$, there exists an almost discrete collection $\mathscr{V} = \{V_1, V_2, \ldots\}$ of open sets such that $\bar{H}_i \subset V_i$ for every $i$.*

*Proof.* The condition of the theorem is evidently sufficient for normality of $S$.

For necessity, let $S$ be a normal space and let $\mathscr{H} = \{H_1, H_2, \ldots\}$ be a countable almost discrete collection of sets from $S$. We prove the following assertion $P(n)$:

$P(n)$: There exist open sets $V_1, V_2, \ldots, V_n, V^{n+1}$ such that

(*a*) $\bar{H}_i \subset V_i$ for $i = 1, 2, \ldots, n$,

(*b*) $\overline{\bigcup_{k=n+1}^{\infty} H_k} \subset V^{n+1}$,

(*c*) $V_k \cap V_i = \emptyset$ for $i < k \leqq n$ and

(*d*) $\bigcup_{i=1}^{n} V_i \cap V^{n+1} = \emptyset$.

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For $n = 1$, the sets $\overline{H}_1$ and $\overline{\bigcup_{k=2}^{\infty} H_k}$ are closed and disjoint in $S$. Hence there exist open sets $V_1$ and $V^2$ such that (*a*) $\overline{H}_1 \subset V_1$, (*b*) $\overline{\bigcup_{k=2}^{\infty} H_k} \subset V^2$ and (*d*) $V_1 \cap V^2 = \emptyset$.

We assume that $P(n)$ is valid for $n$. There exists an open set $U^{n+1}$ such that

$$\overline{\bigcup_{k=n+1}^{\infty} H_k} \subset U^{n+1} \subset \overline{U^{n+1}} \subset V^{n+1}.$$

Then $\overline{U^{n+1}}$ is a normal subspace of $S$ and $\mathscr{H}_{n+1} = \{H_{n+1}, H_{n+2}, \ldots\}$ is almost discrete in $\overline{U^{n+1}}$. There exist the sets $V_1^{n+1}$ and $V_2^{n+1}$, open in $\overline{U^{n+1}}$, such that $\overline{H_{n+1}} \subset V_1^{n+1}$, $\overline{\bigcup_{k=n+2}^{\infty} H_k} \subset V_2^{n+1}$ and $V_1^{n+1} \cap V_2^{n+1} = \emptyset$. Then there exist $W_1^{n+1}$ and $W_2^{n+1}$, open in $S$, such that $V_1^{n+1} = \overline{U^{n+1}} \cap W_1^{n+1}$ and $V_2^{n+1} = \overline{U^{n+1}} \cap W_2^{n+1}$. Now let $V_{n+1} = U^{n+1} \cap W_1^{n+1}$, $V^{n+2} = U^{n+1} \cap W_2^{n+1}$. The sets $V_1, V_2, \ldots, V_{n+1}, V^{n+2}$ satisfy $P(n+1)$.

In this way we construct a sequence of open sets $V_1, V_2, \ldots$ with the required properties, i.e. $\overline{H}_i \subset V_i$ $(i = 1, 2, \ldots)$ and $V_i \cap V_j = \emptyset$ $(i \neq j)$. The theorem follows then from Lemma 1.

As a consequence we obtain Iseki's theorem:

THEOREM 2 (Iseki). *A topological space S is normal if and only if, for every countable discrete collection* $\mathscr{H} = \{H_1, H_2, \ldots\}$, *there exists a countable discrete collection of open sets* $\mathscr{U} = \{U_1, U_2, \ldots\}$ *such that* $\overline{H}_i \subset U_i$ *for every i.*

Now let $\xi$ be the collection of countable collections of mutually disjoint sets from a topological space $S$ which satisfies the following conditions:

(*a*) If $\mathscr{H} \in \xi$, then $\overline{\mathscr{H}} \in \xi$,

(*b*) for every $\mathscr{H} = \{H_1, H_2, \ldots\}$ from $\xi$, there exists a collection of open sets $\mathscr{U} = \{U_1, U_2, \ldots\}$ from $\xi$ such that $H_i \subset U_i$ for every $i$.

Such a collection $\xi$ contains, as it is easy to verify, only almost discrete collections. The following theorem holds:

THEOREM 3. *A topological space S is normal if and only if* $\xi$ *consists exactly of all countable almost discrete collections.*

For completely separated collections the following theorem holds:

THEOREM 4. *A topological space S is normal if and only if every countable discrete collection of sets from S is completely separated.*

*Proof.* For collections of two sets, this is Urysohn's lemma. Therefore it follows immediately that the condition of the theorem is sufficient for normality of $S$.

For necessity, let $S$ be normal, and let $\mathscr{H} = \{H_1, H_2, \ldots\}$ be a discrete collection in $S$. There exists a discrete collection $\mathscr{U} = \{U_1, U_2, \ldots\}$ of open sets such that $\overline{H}_i \subset U_i$. Let $a_1, a_2, \ldots$ be an arbitrary sequence of real numbers. For every pair $(H_i, U_i)$, there exists a continuous function $f_i$ such that $f_i(H_i) = a_i$ and $f_i(S - U_i) = 0$. We consider then $f = \sum_{i=1}^{\infty} f$. Evidently $f(H_i) = a_i$. It is easy to verify that the function $f$ is continuous.

**4. Completely normal spaces.** A similar characterization with separated collections is possible for complete normality.

THEOREM 5. *A topological space S is completely normal if and only if, for every countable separated collection* $\mathscr{H} = \{H_1, H_2, \ldots\}$, *there exists a countable separated collection* $\mathscr{U} = \{U_1, U_2, \ldots\}$ *of open sets such that* $H_i \subset U_i$ *for every i.*

The proof of Theorem 5 is analogous to the proof of Theorem 1, and is here omitted.

Let $\xi'$ be the collection of all countable collections of mutually disjoint sets from a topological space $S$ which satisfies the condition:

If $\mathscr{H} = \{H_1, H_2, \ldots\}$ is from $\xi'$, then there exists a collection $\mathscr{U} = \{U_1, U_2, \ldots\}$ of open sets from $\xi'$ such that $H_i \subset U_i$ for every $i$.

Every collection from $\xi'$ is separated, and the following theorem holds:

THEOREM 6. *A topological space S is completely normal if and only if* $\xi'$ *consists exactly of all countable separated collections from S.*

## REFERENCES

1. K. Iseki, A note on normal spaces, *Math. Japon.* 3 (1953), 45.

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