*Bull. Aust. Math. Soc.* **102** (2020), 77–90
doi:10.1017/S0004972719001060

# ON A CONJECTURE ON THE PERMUTATION CHARACTERS OF FINITE PRIMITIVE GROUPS

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(Received 10 August 2019; accepted 20 August 2019; first published online 22 October 2019)

**Abstract**

Let $G$ be a finite group with two primitive permutation representations on the sets $\Omega_1$ and $\Omega_2$ and let $\pi_1$ and $\pi_2$ be the corresponding permutation characters. We consider the case in which the set of fixed-point-free elements of $G$ on $\Omega_1$ coincides with the set of fixed-point-free elements of $G$ on $\Omega_2$, that is, for every $g \in G$, $\pi_1(g) = 0$ if and only if $\pi_2(g) = 0$. We have conjectured in Spiga ['Permutation characters and fixed-point-free elements in permutation groups', *J. Algebra* **299**(1) (2006), 1–7] that under this hypothesis either $\pi_1 = \pi_2$ or one of $\pi_1 - \pi_2$ and $\pi_2 - \pi_1$ is a genuine character. In this paper we give evidence towards the veracity of this conjecture when the socle of $G$ is a sporadic simple group or an alternating group. In particular, the conjecture is reduced to the case of almost simple groups of Lie type.

2010 *Mathematics subject classification*: primary 20B15.

*Keywords and phrases*: finite primitive groups, fixed-point-free elements.

## 1. Introduction

A permutation $g$ on $\Omega$ is said to be *fixed-point-free* (or a *derangement*) if $g$ fixes no point of $\Omega$, that is, $\omega^g \neq \omega$ for every $\omega \in \Omega$. Fixed-point-free elements have always attracted attention. In a finite permutation group $G$, the fixed-point-free elements can be detected using the permutation character $\pi$ of $G$. Indeed, $\pi(g) = 0$ if and only if $g \in G$ is a fixed-point-free element.

In general, we may ask: *what sort of information can be deduced from the fixed-point-free elements of a finite group?* With this formulation, the question seems too vague to hope for reasonable answers and deep mathematics. Even for primitive permutation groups, the permutation character does not bring much information. For instance, it is remarkable that although the permutation character can detect whether the action is transitive (or 2-transitive), the same is no longer true for primitivity. Disproving a conjecture of Wielandt [19, Problem 6.6], Guralnick and Saxl [13] have constructed two distinct permutation representations, one primitive and one imprimitive, both yielding the same permutation character. This result leaves little hope for extracting detailed information on a permutation representation only from its permutation character, let alone from its set of fixed-point-free elements.

Despite the amazing construction of Guralnick and Saxl, there are strong motivations arising from number theory for pursuing research on permutation representations yielding the same permutation character. First Perlis [20] and then Klingen [16] have observed that permutation representations having the same set of fixed-point-free elements can be used to construct distinct number fields with several arithmetical similarities. Since the permutation character $\pi$ of a permutation representation of a group $G$ encodes the fixed-point-free elements, stemming from algebraic number theory, we may ask: *what sort of information can be deduced from a finite group having two permutation characters with the same zeroes?* This is the question we address here.

In [21], we proposed the following conjecture.

CONJECTURE 1.1. Let $G$ be a group with two primitive actions on $\Omega_1$ and $\Omega_2$ having the same set of fixed-point-free elements in both actions and let $\pi_1$ and $\pi_2$ be the corresponding permutation characters. Then either $\pi_1 = \pi_2$ or one of $\pi_1 - \pi_2$ and $\pi_2 - \pi_1$ is a genuine character.

A *genuine* character of $G$ is a linear combination of complex irreducible characters with positive integer coefficients, that is, an element in the positive cone in the lattice of virtual characters. Although the conclusion proposed by this conjecture only relies on the permutation characters, the hypothesis does not: the permutation characters $\pi_1$ and $\pi_2$ cannot be used to test whether the actions are primitive (see [13]).

It is not difficult to show (see also [21]) that if $G$ is a group with two primitive actions on $\Omega_1$ and $\Omega_2$ having the same set of fixed-point-free elements in both actions, then the kernel of the action of $G$ on $\Omega_1$ equals the kernel of the action of $G$ on $\Omega_2$. Therefore, without loss of generality, in Conjecture 1.1, we may assume that $G$ acts faithfully on both $\Omega_1$ and $\Omega_2$. In what follows, we only consider faithful actions.

Conjecture 1.1 is based on computer evidence and on the preliminary investigation in [21]. In particular, [21, Theorem 10] reduces Conjecture 1.1 to the case that $G$ is an almost simple group. This reduction is very similar to the reduction of Förster and Kovács in [4, Theorem 1] for the problem of Wielandt [19, Problem 6.6] that we mentioned above. Indeed, although not explicitly mentioned in [21], the author proved the following result.

THEOREM 1.2. *Let $G$ be a group with two primitive faithful actions on $\Omega_1$ and $\Omega_2$ having the same set of fixed-point-free elements in both actions. Suppose that $G$ is a counterexample to Conjecture 1.1. Then $G$ can be embedded in a wreath product $H \operatorname{wr} \operatorname{Sym}(\ell)$, where:*

(i) *$\ell \geq 1$;*
(ii) *$H$ is an almost simple group endowed with two primitive faithful actions on $\Delta_1$ and $\Delta_2$ having the same set of fixed-point-free elements;*
(iii) *in the embedding of $G$ in $H \operatorname{wr} \operatorname{Sym}(\ell)$, the actions of $G$ on $\Omega_1$ and $\Omega_2$ are permutation isomorphic to the actions on the cartesian powers $\Delta_1^\ell$ and $\Delta_2^\ell$;*
(iv) *$H$ is a counterexample to Conjecture 1.1.*

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TABLE 1. Exceptional cases in Theorems 1.3 and 1.4 (notation as in [2]).

| Group $G$ | Stabiliser of a point of $\Omega_1$ | Stabiliser of a point of $\Omega_2$ | Comments |
|---|---|---|---|
| $M_{11}$ | $3^2 : Q_8.2$ | $2.S_4$ | Actions of degrees 55 and 165 |
| $M_{22}$ | $2^4 : A_6$ | $2^4 : S_5$ | Actions of degrees 77 and 231 |
| $M_{23}$ | $L_3(4).2_2$ | $2^4 : A_7$ | Both actions of degree 253 |
| $Mcl$ | $L_3(4).2_2$ | $2^4 : A_7$ | Both actions of degree 22275 |
| $J_1$ | $2^3.7.3$ | $7 : 6$ | Actions of degrees 1045 and 4180 |
| Alt(6) | Alt(4) | $\langle(1,2)(3,4),(3,4,5)\rangle$ | Actions of degrees 5 and 10 |

In the light of Theorem 1.2, Conjecture 1.1 is reduced to the case of almost simple groups. In this paper we give considerable further evidence towards the veracity of Conjecture 1.1. In particular, we completely solve it for almost simple groups with socle a sporadic simple group.

THEOREM 1.3. *Let $G$ be an almost simple group with socle a sporadic simple group and with two primitive faithful actions on $\Omega_1$ and $\Omega_2$ having the same set of fixed-point-free elements and let $\pi_1$ and $\pi_2$ be the corresponding permutation characters. Then either:*

(1) *the actions of $G$ on $\Omega_1$ and $\Omega_2$ are permutation isomorphic; or*
(2) *replacing $\Omega_1$ by $\Omega_2$ if necessary, the triple $(G,\Omega_1,\Omega_2)$ is in Table 1.*

*In particular, either $\pi_1 = \pi_2$ or one of $\pi_1 - \pi_2$ and $\pi_2 - \pi_1$ is a genuine character.*

Moreover, we give partial evidence for almost simple groups with socle an alternating group.

THEOREM 1.4. *Let $G$ be either the alternating group* Alt($n$) *or the symmetric group* Sym($n$) *with $n \geq 5$. Suppose that $G$ admits two primitive faithful actions on $\Omega_1$ and $\Omega_2$ having the same set of fixed-point-free elements and let $\pi_1$ and $\pi_2$ be the corresponding permutation characters. Then:*

(1) *the actions of $G$ on $\Omega_1$ and $\Omega_2$ are permutation isomorphic; or*
(2) *replacing $\Omega_1$ by $\Omega_2$ if necessary, the triple $(G,\Omega_1,\Omega_2)$ is in Table 1; or*
(3) *given $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the point stabilisers $G_{\omega_1}$ and $G_{\omega_2}$ are almost simple primitive subgroups of* Sym($n$).

*In particular, in parts (1) and (2), either $\pi_1 = \pi_2$ or one of $\pi_1 - \pi_2$ and $\pi_2 - \pi_1$ is a genuine character.*

We do not have any example for part (3) when $\pi_1 \neq \pi_2$. Indeed, this situation is quite peculiar. The groups $G_{\omega_1}$ and $G_{\omega_2}$ are almost simple and primitive and are maximal subgroups of Sym($n$) or Alt($n$). Moreover, since $\pi_1$ and $\pi_2$ have the same set of zeroes, we deduce that every permutation in $G_{\omega_1}$ is $G$-conjugate to a permutation in $G_{\omega_2}$, and *vice versa*. In particular, if we do not count multiplicities, the permutations in $G_{\omega_1}$ and in $G_{\omega_2}$ have the same cycle structures. We expect that, if this happens, then $\pi_1 = \pi_2$.

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(This case for example arises by taking the primitive action of the projective general linear group on projective points and on projective hyperplanes.) At the moment, we have no idea how to handle part (3), but a preliminary goal would be to prove that the socles of $G_{\omega_1}$ and of $G_{\omega_2}$ are isomorphic. This is very likely true because, among other things, $G_{\omega_1}$ and $G_{\omega_2}$ are *isospectral*, that is, $\{\text{order } g \mid g \in G_{\omega_1}\} = \{\text{order } g \mid g \in G_{\omega_2}\}$.

The structure of this paper is straightforward. In Section 2, we prove Theorem 1.3. In Section 3, we prove Theorem 1.4.

Notation 1.5. In what follows we let $G$ be a finite group with two primitive faithful actions on $\Omega_1$ and $\Omega_2$ having the same set of fixed-point-free elements. We let $M_1$ and $M_2$ be the stabilisers in $G$ of a point of $\Omega_1$ and $\Omega_2$ and we let $\pi_1$ and $\pi_2$ be the permutation characters. In particular, $M_1$ and $M_2$ are core-free maximal subgroups of $G$ with

$$\bigcup_{g\in G} M_1^g = \bigcup_{g\in G} M_2^g. \tag{1.1}$$

## 2. Sporadic simple groups

In this section we prove Theorem 1.3 when the socle of $G$ is a sporadic simple group. The proof is entirely computational and uses the astonishing package 'The GAP Character Table Library' [1] implemented in the computer algebra system `GAP` [5]. Apart from

- the Monster and
- the action of the Baby Monster on the cosets of a maximal subgroup of type $(2^2 \times F_4(2)) : 2$,

each permutation character of each primitive permutation representation of an almost simple group with socle a sporadic simple group is available in `GAP` via the package 'The GAP Character Table Library'. Therefore, except for the two cases mentioned above, we can quickly and easily use `GAP` to test the veracity of Theorem 1.3 because it is straightforward to check when two permutation characters have the same zeroes. In particular, in what follows we may assume that $G$ is the Baby Monster or the Monster.

Suppose first that $G$ is the Baby Monster. Then $G$ has 30 conjugacy classes of maximal subgroups. If both $\pi_1$ and $\pi_2$ are not the permutation characters of the action of $G$ on the cosets of a maximal subgroup of type $(2^2 \times F_4(2)) : 2$, then Theorem 1.3 follows again with a computation using the invaluable help of 'The GAP Character Table Library'. Suppose then that $\pi_1$ (say) is the permutation character of the action of $G$ on the cosets of a maximal subgroup of type $(2^2 \times F_4(2)) : 2$. Let $\omega_1 \in \Omega_1$, let $\omega_2 \in \Omega_2$ and let $G_{\omega_1}$ and $G_{\omega_2}$ be the point stabilisers of $\omega_1$ and $\omega_2$, respectively. Now, the prime divisors of the order of $G_{\omega_1}$ are $2, 3, 5, 13, 17$. In particular, by (1.1), $2, 3, 5, 13, 17$ are the only prime divisors of $G_{\omega_2}$. A quick inspection on the order of the maximal subgroups of $G$ shows that $G_{\omega_2}$ is also of type $(2^2 \times F_4(2)) : 2$ and hence $G_{\omega_1}$ and $G_{\omega_2}$ are conjugate. Therefore, $\pi_1 = \pi_2$.

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Suppose next that $G$ is the Monster. A recent account on the classification of the maximal subgroups of the sporadic simple groups is in [22]. From [22, Section 3.6], we see that the classification of the maximal subgroups of the Monster $G$ is complete except for a few small open cases. In particular, if $M$ is a maximal subgroup of $G$, then either:

(a) $M$ is in [22, Section 3.6]; or
(b) $M$ is almost simple with socle isomorphic to $L_2(8)$, $L_2(13)$, $L_2(16)$, $U_3(4)$ or $U_3(8)$.

(From [14], we have learned that the problem has been reduced to $\mathrm{PSL}_2(13)$, but we do not need this information here.) Now, we argue as in the case above. We let $\omega_1 \in \Omega_1$, let $\omega_2 \in \Omega_2$ and let $G_{\omega_1}$ and $G_{\omega_2}$ be the point stabilisers of $\omega_1$ and $\omega_2$, respectively. For each known maximal subgroup of $G$ and for each putative maximal subgroup of $G$, we compute the set of prime divisors of its order. Observe that if $\pi_1$ and $\pi_2$ have the same zeroes, then $|G_{\omega_1}|$ and $|G_{\omega_2}|$ have the same set of prime divisors. A direct inspection shows that this happens only when $G_{\omega_1}$ is conjugate to $G_{\omega_2}$ or when the two maximal subgroups under consideration are one of type $13^2 : (2L_2(13)).4$ and the other of type $\mathrm{PSL}_2(13)$ or $\mathrm{PGL}_2(13)$. In the first case, $\pi_1 = \pi_2$. In the second case, we observe that $13^2 : (2L_2(13)).4$ has elements of order 26 but neither $\mathrm{PSL}_2(13)$ nor $\mathrm{PGL}_2(13)$ does.

## 3. Almost simple groups with socle an alternating group: the setup

In this section we assume that $G$ is an almost simple group with socle an alternating group $\mathrm{Alt}(n)$, with $n \geq 5$. In particular, except when $n = 6$, we have either $G = \mathrm{Alt}(n)$ or $G = \mathrm{Sym}(n)$. To avoid some cumbersome divisions later in our argument, we deal with the case $n = 6$ right away. A computer-aided computation reveals that $M_1$ and $M_2$ are conjugate in $G$. In particular, from now on, we may assume that $n \neq 6$ and hence $G = \mathrm{Alt}(n)$ or $G = \mathrm{Sym}(n)$.

A *partition* of $n$ is an unordered tuple $[x_1, \ldots, x_k]$, where we have $n = \sum_{j=1}^{k} x_j$ and $x_j \in \mathbb{N}\setminus\{0\}$ for $j \in \{1, \ldots, k\}$. Given $\sigma \in \mathrm{Sym}(n)$, the *type* of $\sigma$ is the partition $p(\sigma) := [x_1, \ldots, x_k]$, where $x_1, \ldots, x_k$ are the lengths of the orbits on $\{1, \ldots, n\}$ of the subgroup $\langle \sigma \rangle$ generated by $\sigma$.

For $\sigma, \tau \in \mathrm{Sym}(n)$, we say that $\sigma$ is $\mathrm{Sym}(n)$-conjugate to $\tau$ if and only if $p(\sigma) = p(\tau)$. Therefore, when $G = \mathrm{Sym}(n)$, (1.1) is equivalent to

$$\{p(\sigma) \mid \sigma \in M_1\} = \{p(\sigma) \mid \sigma \in M_2\}.$$

However, when $G = \mathrm{Alt}(n)$, this latter condition is only necessary for guaranteeing that (1.1) holds true.

Since $M_1$ is a maximal subgroup of $G$, there is a natural division into three cases:

(1) $M_1$ is intransitive on $\{1, \ldots, n\}$ and hence $M_1$ is the setwise stabiliser in $G$ of a $k$-subset of $\{1, \ldots, n\}$ with $1 \leq k < n/2$;
(2) $M_1$ is imprimitive on $\{1, \ldots, n\}$ and hence $M_1$ is the stabiliser in $G$ of a uniform partition of $\{1, \ldots, n\}$ into $b$ parts each of cardinality $a$;
(3) $M_1$ is primitive on $\{1, \ldots, n\}$.

Clearly, a similar division holds for the subgroup $M_2$. In what follows we consider these possibilities in turn.

## 4. $M_1$ or $M_2$ is intransitive on $\{1, \ldots, n\}$

Replacing $M_1$ with $M_2$ if necessary, we assume that $M_1$ is intransitive. Here

$$M_1 = G \cap (\mathrm{Sym}(\{1, \ldots, k\}) \times \mathrm{Sym}(\{k+1, \ldots, n\}))$$

for some $k \in \{1, \ldots, n\}$ with $1 \leq k < n/2$.

As $n \geq 5$ and $k < n/2$, we have $n - k \geq 3$. Thus, $M_1$ contains a 3-cycle and hence so does $M_2$. In particular, if $M_2$ is primitive on $\{1, \ldots, n\}$, then $M_2 \geq \mathrm{Alt}(n)$ by [3, Theorem 3.3A], contradicting the fact that $M_2$ is core-free in $G$.

Suppose that $M_2$ is imprimitive on $\{1, \ldots, n\}$. In particular, $M_2$ preserves a uniform partition of $\{1, \ldots, n\}$ into $b$ parts each of cardinality $a$ with $n = ab$ and $1 < a, b < n$. Thus,

$$M_2 = G \cap (\mathrm{Sym}(a) \,\mathrm{wr}\, \mathrm{Sym}(b)).$$

Observe that $\mathrm{Sym}(a) \,\mathrm{wr}\, \mathrm{Sym}(b)$ contains an $n$-cycle in its imprimitive action on $\{1, \ldots, ab\}$. In particular, $M_2$ contains a permutation $g$ with $p(g) = [n]$ when $n$ is odd or when $G = \mathrm{Sym}(n)$, and with $p(g) = [n/2, n/2]$ when $n$ is even and $G = \mathrm{Alt}(n)$. Therefore, $M_1$ contains a permutation $g'$ with $p(g') = p(g)$. However, this is a contradiction, because $M_1$ fixes a $k$-subset with $1 \leq k < n/2$, but $g'$ fixes no such $k$-subset.

Suppose that $M_2$ is intransitive on $\{1, \ldots, n\}$. Therefore, replacing $M_2$ by a suitable $G$-conjugate if necessary,

$$M_2 := G \cap (\mathrm{Sym}(\{1, \ldots, k'\}) \times \mathrm{Sym}(\{k'+1, \ldots, n\}))$$

for some $1 \leq k' < n/2$. Suppose first that $n$ is even or that $G = \mathrm{Sym}(n)$. Then $M_1$ contains a permutation of type $[k, n-k]$. Therefore, $M_2$ also contains a permutation $g$ with $p(g) = [k, n-k]$; however, the only proper subsets of $\{1, \ldots, n\}$ fixed by $g$ have cardinalities $k$ and $n - k$. Therefore, $k' = k$ and $M_1 = M_2$. Suppose next that $n$ is odd and $G = \mathrm{Alt}(n)$. In particular, $M_1$ contains a permutation having type $[1, k, n-k-1]$ and therefore so does $M_2$; let $g \in M_2$ with $p(g) = [1, k, n-k-1]$. The subsets of $\{1, \ldots, n\}$ of cardinality less then $n/2$ fixed by $g$ have sizes 1, $k$ and $k+1$ (where the latter case occurs only when $k + 1 < n/2$). We deduce that

$$k' \in \{1, k, k+1\}.$$

An entirely symmetric argument (interchanging the roles of $M_1$ and $M_2$) yields

$$k \in \{1, k', k'+1\}.$$

If $k' = k$, then $M_2 = M_1$. Interchanging the roles of $k$ and $k'$ if necessary, we may suppose that $k < k'$. As $k \in \{1, k', k'+1\}$, we deduce that $k = 1$. Moreover, as $k' \in \{1, k, k+1\} = \{1, 2\}$, we deduce that $k' = 2$. Now, $M_1$ contains a permutation with

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cycle type $[1,(n-1)/2,(n-1)/2]$ and hence so does $M_2$. However, since the elements in $M_2$ fix setwise a 2-subset, we deduce that $(n-1)/2 = 2$, that is, $n = 5$, $G = \mathrm{Alt}(5)$, $M_1 = \mathrm{Alt}(4)$ and $M_2 = \mathrm{Alt}(5) \cap (\mathrm{Sym}(\{1,2\}) \times \mathrm{Sym}(\{3,4,5\}))$. In this case, we see that (1.1) is satisfied. Moreover, an explicit computation shows that $\pi_2 - \pi_1$ is the irreducible character of $G = \mathrm{Alt}(5)$ of degree five. This exception is listed in Table 1.

For the rest of our argument we may assume that neither $M_1$ nor $M_2$ is intransitive.

## 5. $M_1$ or $M_2$ is imprimitive on $\{1,\ldots,n\}$

Replacing $M_1$ with $M_2$ if necessary, we assume that $M_1$ is imprimitive. Here

$$M_1 = G \cap (\mathrm{Sym}(a) \,\mathrm{wr}\, \mathrm{Sym}(b))$$

for some divisors $a$ and $b$ of $n$ with $n = ab$ and $1 < a, b < n$.

Suppose that $M_2$ is primitive on $\{1,\ldots,n\}$. If $a \geq 3$, then $M_1$ contains a 3-cycle and hence so does $M_2$. Then $M_2 \geq \mathrm{Alt}(n)$ by [3, Theorem 3.3A], contradicting the fact that $M_2$ is core-free in $G$. If $a = 2$, then $M_1$ contains a permutation having type $[2,2,1,\ldots,1]$. Therefore, $M_2$ contains a double transposition. By [3, Theorem 3.3D and Example 3.3.1], we have $n \leq 8$. Thus, $n \in \{6,8\}$. These cases can be handled with a computer-aided computation; no example arises.

Suppose that $M_2$ is imprimitive on $\{1,\ldots,n\}$. Here

$$M_2 = G \cap (\mathrm{Sym}(a') \,\mathrm{wr}\, \mathrm{Sym}(b'))$$

for some divisors $a'$ and $b'$ of $n$ with $n = a'b'$ and $1 < a', b' < n$.

Assume that $G = \mathrm{Sym}(n)$. In particular, for every positive integer $k$ with $k \leq a$, the group $M_1$ contains a permutation having type $[k,1,\ldots,1]$ in its action on $\{1,\ldots,n\}$. Therefore, $M_2$ contains a permutation $g$ with $p(g) = [k,1,\ldots,1]$. As $g \in M_2$, $g$ fixes a uniform partition having $b'$ parts of cardinality $a'$. Clearly, this happens only when $k \leq a'$ or $a'$ divides $k$. Applying this argument with $k = a-1$ and $k = a$:

- $a - 1 \leq a'$ or $a'$ divides $a - 1$; and
- $a \leq a'$ or $a'$ divides $a$.

Since $a$ and $a-1$ are relatively prime and since $a, a' \geq 2$, we deduce that $a \leq a'$. An entirely symmetric argument (interchanging the roles of $M_1$ and $M_2$) yields $a' \leq a$. Thus, $a = a'$ and hence $M_1$ and $M_2$ are conjugate in $G$.

Assume that $G = \mathrm{Alt}(n)$. In particular, for every odd number $k$ with $k \leq a$, the group $M_1$ contains a permutation having type $[k,1,\ldots,1]$ in its action on $\{1,\ldots,n\}$. Therefore, $M_2$ contains permutations of type $[k,1,\ldots,1]$ on $\{1,\ldots,n\}$. In particular, when $a$ is odd, arguing as in the paragraph above with $k \in \{a-2, a\}$:

- $a - 2 \leq a'$ or $a'$ divides $a - 2$; and
- $a \leq a'$ or $a'$ divides $a$.

Since $a$ and $a-2$ are relatively prime and since $a, a' \geq 2$, we deduce that $a \leq a'$. Similarly, when $a$ is even, arguing as in the paragraph above with $k \in \{a-3, a-1\}$:

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- $a - 3 \leq a'$ or $a'$ divides $a - 3$; and
- $a - 1 \leq a'$ or $a'$ divides $a - 1$.

Since $a - 3$ and $a - 1$ are relatively prime and since $a, a' \geq 2$, we deduce that $a - 1 \leq a'$. Summing up, either $a$ is odd and $a \leq a'$, or $a$ is even and $a - 1 \leq a'$. An entirely symmetric argument (interchanging the roles of $M_1$ and $M_2$) shows that either $a'$ is odd and $a' \leq a$, or $a'$ is even and $a' - 1 \leq a$. If $a = a'$, then $M_1$ and $M_2$ are conjugate in $G$. Therefore, we assume that $a \neq a'$; moreover, replacing $M_1$ by $M_2$ if necessary, we may assume that $a < a'$. A moment's thought shows that $a$ is odd and $a' = a + 1$. Now, $M_2$ contains a permutation of type $[a', a', 1, \ldots, 1] = [a + 1, a + 1, 1, \ldots, 1]$ and hence $M_1$ contains a permutation $g$ of the same type. Therefore, $g$ fixes a uniform partition $\mathcal{P} := \{X_1, \ldots, X_b\}$ of $\{1, \ldots, n\}$ with $b$ parts of cardinality $a$. Suppose that $\mathcal{P}$ has a part $X_i$ containing a fixed point of $g$ and also a point lying in a cycle of length $a + 1$. Then $X_i^g \in \mathcal{P}$ and, since $g$ fixes a point from $X_i$, we have $X_i^g = X_i$. This forces $X_i$ to contain an $(a + 1)$-cycle of $g$. Thus, $a = |X_i| \geq a + 1$, which is a contradiction. This proves that the fixed points of $g$ are a union of parts from $\mathcal{P}$. Thus, $a$ divides $n - 2(a + 1)$. Since $a$ divides $n$, we deduce that $a$ divides $2(a + 1)$. As $\gcd(a, a + 1) = 1$, we get $a = 2$, contradicting the fact that $a$ is odd.

For the rest of our argument, we may assume that neither $M_1$ nor $M_2$ is imprimitive.

## 6. $M_1$ and $M_2$ are primitive on $\{1, \ldots, n\}$

One of the main ingredients here is the structure of the lattice of maximal subgroups of $G$. From [18], we see that every maximal subgroup has O'Nan–Scott type:

HA holomorphic abelian; or
PA product action; or
SD simple diagonal; or
AS almost simple.

**6.1. $M_1$ or $M_2$ has O'Nan–Scott type HA.** Replacing $M_1$ with $M_2$ if necessary, we assume that $M_1$ has O'Nan–Scott type HA. Then $n = p^\ell$ for some prime number $p$ and for some positive integer $\ell$. Moreover,

$$M_1 = G \cap \mathrm{AGL}_\ell(p),$$

where $\mathrm{AGL}_\ell(p)$ is the affine general linear group for the $\ell$-dimensional vector space over the field with $p$ elements. As a Singer cycle in $\mathrm{GL}_\ell(p)$ has type $[p^\ell - 1, 1]$, we deduce that either $G$ contains a permutation of type $[p^\ell - 1, 1]$, or $G = \mathrm{Alt}(n)$ and $G$ contains a permutation of type $[(p^\ell - 1)/2, (p^\ell - 1)/2, 1]$. In particular, $M_2$ contains a permutation $g$ of the same type. In all cases, $g$ has at most three cycles in its action on $\{1, \ldots, n\}$. Finite primitive groups containing a permutation with at most four disjoint cycles have been classified in a series of papers (see [8–10]). From [10], the proper primitive group $M_2$ contains a permutation of type $[p^\ell - 1, 1]$ or $[(p^\ell - 1)/2, (p^\ell - 1)/2, 1]$ if and only if one of the following holds:

(1) $M_2 = \mathrm{PSL}_2(11)$ or $M_2 = M_{11}$ in its primitive action of degree 11; or
(2) $M_2 = M_{23}$ in its primitive action of degree 23; or

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(3) $M_2$ has socle $\mathrm{PSL}_2(16)$ in its primitive action of degree 17; or
(4) $M_2$ has socle $\mathrm{PSL}_2(q)$ in its primitive action of degree $p^\ell = n = q + 1$ and $q$ is prime; or
(5) $M_2$ has socle $\mathrm{PSL}_d(q)$ in its primitive action of degree $p^\ell = n = 2^d - 1$; or
(6) $M_2$ has abelian socle.

Cases (1) and (2) do not arise because $M_2$ contains a permutation of type $[2, 3, 6]$, but $M_1$ has no permutations of this type.

Similarly, Case (3) does not arise because $M_2$ contains a permutation of type $[15, 1, 1]$, but $M_1$ has no permutations of this type.

In Case (4), since $q$ is prime and $p^\ell = q + 1$, we have $p = 2$ and hence $q$ is a Mersenne prime. Now, a transvection of $\mathrm{GL}_\ell(2)$ lies in $M_1$ and fixes $n/2$ points. Thus, $M_2$ contains a permutation fixing half of the elements. However, this is impossible because $M_2 \leq \mathrm{Aut}(\mathrm{PSL}_2(q)) = \mathrm{PGL}_2(q)$ and no nonidentity element of $\mathrm{PGL}_2(q)$ fixes more than two points; thus, $2 \geq n/2$, which is a contradiction.

In Case (5), $p^\ell + 1 = 2^d$. In particular, $p^{2\ell} - 1$ has no primitive prime divisor. By a celebrated theorem of Zsigmondy, $2\ell = 2$ and $p + 1$ is a power of two. Therefore, $\ell = 1$ and $p = n = 2^\ell - 1$. Thus, $M_1 = G \cap \mathrm{AGL}_1(p)$ and $\mathrm{PSL}_d(2) \trianglelefteq M_2$. Therefore, $M_1$ has no nonidentity permutations fixing more than one point, whereas $M_2$ contains permutations fixing the $2^{d-1} - 1$ points of a projective hyperplane. Thus, $1 \geq 2^{d-1} - 1$ and $d \leq 2$, which is clearly a contradiction.

In Case (6), the socles of $M_2$ and $M_1$ are elementary abelian regular subgroups of $\mathrm{Sym}(n)$ and hence they are conjugate in $G$. Without loss of generality, we may assume that the socle $V$ of $M_1$ equals the socle of $M_2$. The maximality of $M_1$ and $M_2$ in $G$ yields $M_1 = \mathbf{N}_G(V) = M_2$.

For the rest of our argument, we may assume that neither $M_1$ nor $M_2$ has HA type.

**6.2. $M_1$ or $M_2$ has O'Nan–Scott type PA.** Replacing $M_1$ with $M_2$ if necessary, we assume that $M_1$ has O'Nan–Scott type PA. Then $n = a^b$ for some positive integers $a$ and $b$ with $a \geq 5$ and $b \geq 2$. Moreover,

$$M_1 = G \cap (\mathrm{Sym}(a) \,\mathrm{wr}\, \mathrm{Sym}(b)),$$

where $\mathrm{Sym}(a) \,\mathrm{wr}\, \mathrm{Sym}(b)$ is endowed with its primitive product action on a cartesian product $\Delta^b$ of degree $a^b = n$. We write the elements of $\mathrm{Sym}(a) \,\mathrm{wr}\, \mathrm{Sym}(b)$ in the form $(h_1, \ldots, h_b)\sigma$, where $h_1, \ldots, h_b \in \mathrm{Sym}(a)$ and $\sigma \in \mathrm{Sym}(b)$. Moreover, for every $(\delta_1, \ldots, \delta_b) \in \Delta^b$,

$$(\delta_1, \ldots, \delta_b)^{(h_1, \ldots, h_b)\sigma} = (\delta_{1^{\sigma^{-1}}}^{h_{1^{\sigma^{-1}}}}, \ldots, \delta_{b^{\sigma^{-1}}}^{h_{b^{\sigma^{-1}}}}).$$

Without loss of generality, $\Delta := \{1, \ldots, a\}$.

Assume first that $M_2$ is of O'Nan–Scott type PA. Thus,

$$M_2 = G \cap (\mathrm{Sym}(m) \,\mathrm{wr}\, \mathrm{Sym}(r)),$$

where $\mathrm{Sym}(m) \,\mathrm{wr}\, \mathrm{Sym}(r)$ is endowed with its primitive product action on a cartesian product of degree $m^r = n$, $m \geq 5$ and $r \geq 2$. Set $\delta := 2$ when $G = \mathrm{Sym}(n)$ and $\delta := 3$

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when $G = \mathrm{Alt}(n)$. It follows from the proof of Theorem 3.2 in [6] that the maximum number of fixed points of a nonidentity permutation in $M_1$ is $(a-\delta)a^{b-1}$. (This bound is attained by the permutations of the form $((1\cdots\delta), 1, \ldots, 1)$.) Applying this comment to $M_2$, we see that the maximum number of fixed points of a nonidentity permutation in $M_2$ is $(m-\delta)m^{r-1}$. From this,

$$(a-\delta)a^{b-1} = (m-\delta)m^{r-1}.$$

Since $a^b = n = m^r$, we deduce that $a = m$ and hence $M_1$ is $G$-conjugate to $M_2$. Now, we assume that $M_2$ is not of PA type. In particular, $M_2$ has type AS or SD.

Assume that $a \geq 7$ or that $G = \mathrm{Sym}(n)$. When $G = \mathrm{Sym}(n)$, it is easy to verify that the permutation

$$((1,2)(3,4,5), 1, 1, \ldots, 1)$$

lies in $M_1$, has order six and has no cycles of length six (the last assertion follows with a computation, see also [6, Lemma 3.1 and Theorem 3.2] for more details). Similarly, when $a \geq 7$, it is easy to verify that the permutation

$$((1,2)(3,4)(5,6,7), 1, 1, \ldots, 1)$$

lies in $M_1$, has order six and has no cycles of length six (again, the last assertion follows with a computation or consulting [6, Lemma 3.1 and Theorem 3.2]). In particular, the primitive group $M_1$ contains a permutation $g$ of order six having no cycles of length six. As $g$ is $G$-conjugate to an element of $M_2$, we deduce that also the primitive group $M_2$ contains an element of order six with no cycles of length six. From [11, Theorem 1.1] (see also [6]), we deduce that there exist integers $k \geq 1$, $r \geq 1$ and $m \geq 5$ with $1 \leq k < m/2$ such that $M_2$ preserves a product structure on $\{1, \ldots, n\} = \Lambda^r$ and $\mathrm{Alt}(m)^r \trianglelefteq G \leq \mathrm{Sym}(m) \,\mathrm{wr}\, \mathrm{Sym}(k)$, where $\Lambda$ consists of the set of all $k$-subsets of $\{1, \ldots, m\}$ and $\mathrm{Sym}(m)$ induces its natural $k$-subset action on $\Lambda$. In our case, since $M_2$ is not of PA type, we have $r = 1$, that is,

$$M_2 = G \cap \mathrm{Sym}(m),$$

where $\mathrm{Sym}(m)$ is endowed with its primitive action on the $k$-subsets of $\{1, \ldots, m\}$ of degree $\binom{m}{k} = n$ and $m/2 > k \geq 2$. Recall that $M_1$ contains a permutation $g$ having order six and with no cycles of length six on $\{1, \ldots, n\}$. Thus, the same conclusion holds for $M_2$. However, it is a computation (following the idea in the proof of Theorem 1.1 in [6]) to show that $\mathrm{Sym}(m)$ in its action on $k$-subsets has no such permutations unless $k = 1$, which is a contradiction because here $k \geq 2$. Therefore, for the rest of the proof we may assume that $a \leq 6$ and $G = \mathrm{Alt}(n)$.

In the remaining case $n = 5^b$ or $n = 6^b$, and $M_2$ is of O'Nan–Scott type SD or AS. Clearly, $M_2$ cannot be of type SD, otherwise $n = |T|^\ell$ for some nonabelian simple group $T$ and some positive integer $\ell$. Thus, $M_2$ has type AS. If $n = 5^b$, then $G$ is an almost simple primitive group of prime power degree and hence by a classical result of Guralnick [12] we deduce that no example arises in our case. Suppose then that $n = 6^b$. In [17, Theorem 1.1], the authors have classified the primitive groups having degree a product of two prime powers. We denote by $\mathrm{Soc}(M_2)$ the socle of $M_2$. Applying this classification to the primitive group $M_2$, we deduce that either:

(1) $n = 36$ and $M_2 \in \{\mathrm{PSL}_2(8), \mathrm{PSL}_2(9)\mathrm{PSU}_3(3), \mathrm{PSp}_4(3), \mathrm{PSp}_6(2), \mathrm{Alt}(9)\}$; or
(2) $n = 1296$ and $M_2 = \mathrm{PSU}_4(3)$.

We have tested these groups with the help of a computer and in no case is (1.1) satisfied.

For the rest of our argument, we may assume that neither $M_1$ nor $M_2$ has type HA or PA.

**6.3. $M_1$ or $M_2$ has O'Nan–Scott type SD.** Replacing $M_1$ with $M_2$ if necessary, we assume that $M_1$ has O'Nan–Scott type SD. Then $n = |T|^\ell$ for some nonabelian simple group $T$ and for some positive integer $\ell$. Suppose that $M_2$ also has type SD. Then $n = |T'|^{\ell'}$ for some nonabelian simple group $T'$ and some positive integer $\ell'$. In particular,

$$|T|^\ell = |T'|^{\ell'},$$

where $T, T'$ are nonabelian simple groups and $\ell, \ell'$ are positive integers. This equation has been studied in [15]. By [15, Theorem 6.1], we see that $\ell = \ell'$ and (replacing $T$ with $T'$ if necessary):

(1) $T \cong T'$; or
(2) $T \cong \mathrm{PSL}_3(4)$ and $T' \cong \mathrm{PSL}_4(2)$; or
(3) $T \cong \mathrm{PSp}_{2\kappa}(q)$ and $T' \cong \mathrm{P}\Omega_{2\kappa+1}(q)$ for some positive integer $\kappa \geq 3$ and for some odd prime power $q$.

In Case (1), from the structure of primitive groups of type SD, $M_1$ is $G$-conjugate to $M_2$. To conclude our analysis of Cases (2) and (3), we need some more information on the structure of primitive groups of SD type. We first set up some notation.

Let $\ell \geq 1$ and let $T$ be a nonabelian simple group. Consider the group $N = T^{\ell+1}$ and let $D = \{(t, \ldots, t) \in N \mid t \in T\}$ be the diagonal subgroup of $N$. Set $\Omega := N/D$, the set of right cosets of $D$ in $N$. Then $|\Omega| = |T|^\ell$. We may identify each element $\omega \in \Omega$ with an element of $T^\ell$ as follows: the right coset $\omega = D(\alpha_0, \alpha_1, \ldots, \alpha_\ell)$ contains a unique element whose first coordinate is 1, namely, the element $(1, \alpha_0^{-1}\alpha_1, \ldots, \alpha_0^{-1}\alpha_\ell)$. We choose this distinguished coset representative and we denote the element $D(1, \alpha_1, \ldots, \alpha_\ell)$ of $\Omega$ simply by

$$[\alpha_1, \ldots, \alpha_\ell].$$

An element $\varphi$ of $\mathrm{Aut}(T)$ acts on $\Omega$ by

$$[\alpha_1, \ldots, \alpha_\ell]^\varphi = [\alpha_1^\varphi, \ldots, \alpha_\ell^\varphi].$$

Note that this action is well defined because $D$ is $\mathrm{Aut}(T)$-invariant. Next, the element $(t_0, \ldots, t_\ell)$ of $N$ acts on $\Omega$ by

$$[\alpha_1, \ldots, \alpha_\ell]^{(t_0, \ldots, t_\ell)} = [t_0, \alpha_1 t_1, \ldots, \alpha_\ell t_\ell] = [t_0^{-1}\alpha_1 t_1, \ldots, t_0^{-1}\alpha_\ell t_\ell].$$

Observe that the action induced by $(t, \ldots, t) \in N$ on $\Omega$ is the same as the action induced by the inner automorphism corresponding to conjugation by $t$. Finally, the element $\sigma$

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in $\mathrm{Sym}(\{0,\dots,\ell\})$ acts on $\Omega$ simply by permuting the coordinates. Note that this action is well defined because $D$ is $\mathrm{Sym}(\ell+1)$-invariant.

The set of all permutations we described generates a group $W$ isomorphic to $T^{\ell+1}\cdot(\mathrm{Out}(T)\times\mathrm{Sym}(\ell+1))$. A subgroup $X$ of $W$ containing the socle $N$ is primitive if either $\ell=2$ or $X$ acts primitively by conjugation on the $\ell+1$ simple direct factors of $N$ [3, Theorem 4.5A]. Such primitive groups are the primitive groups of diagonal type. Write

$$M=\{(t_0,t_1,\dots,t_\ell)\in N\mid t_0=1\}.$$

Clearly, $M$ is a normal subgroup of $N$ acting regularly on $\Omega$. Since the stabiliser in $W$ of the point $[1,\dots,1]$ is $\mathrm{Sym}(\ell+1)\times\mathrm{Aut}(T)$,

$$W=(\mathrm{Sym}(\ell+1)\times\mathrm{Aut}(T))M.$$

Every element $x\in W$ can be written uniquely as $x=\sigma\varphi m$ with $\sigma\in\mathrm{Sym}(\ell+1)$, $\varphi\in\mathrm{Aut}(T)$ and $m\in M$.

In Case (2), let $t$ be an involution of $T$ (observe that $T\cong\mathrm{PSL}_4(3)$ has a unique class of involutions); in Case (3), let $t$ be an involution of $T$ of 'type $t_1$' according to the notation in [7, Section 4.5]. Now, let $\iota_t\in\mathrm{Aut}(T)$ be the inner automorphism of $T$ induced by $t$ viewed as a permutation in $M_1$. The points fixed by the permutation $\iota_t$ are of the form

$$[\alpha_1,\dots,\alpha_\ell],$$

where $\alpha_1,\dots,\alpha_\ell\in\mathbf{C}_T(t)$. Therefore, the number of fixed points of $\iota_t$ is

$$F:=\begin{cases}64^\ell & \text{when } T\cong\mathrm{PSL}_3(4),\\ \Big(\dfrac{1}{2}(q^2-1)q^{(\kappa-1)^2+1}\displaystyle\prod_{i=1}^{\kappa-1}(q^{2i}-1)\Big)^\ell & \text{when } T\cong\mathrm{PSp}_{2\kappa}(q).\end{cases}\tag{6.1}$$

(The size of $\mathbf{C}_T(t)$ can be inferred from [7, Section 4.5 and Table 4.5.1].) By (1.1), $M_2$ contains a permutation $g$ which is conjugate, via an element of $G$, to $\iota_t$. Since $\iota_t$ fixes some point, we may assume that $g$ fixes the point $[1,\dots,1]$ and hence $g\in\mathrm{Aut}(T')\times\mathrm{Sym}(\ell+1)$; therefore, $g=\sigma\varphi$ for some $\sigma\in\mathrm{Sym}(\ell+1)$ and for some $\varphi\in\mathrm{Aut}(T')$. Observe that $g$ has order two and that $g$ fixes exactly $F$ points. Suppose first that $\sigma\neq 1$. Then, replacing $\sigma$ by a suitable $\mathrm{Sym}(\ell+1)$-conjugate, we may assume that

$$\sigma:=(0\,1)\cdots(2x\;2x+1)$$

for some $0\le x\le(\ell-1)/2$. A computation yields

$$\begin{aligned}[t_1,\dots,t_\ell]^g&=D(1,t_1,\dots,t_\ell)^{\sigma\varphi}=D(t_1,1,t_3,t_2,\dots,t_{2x+1},t_{2x},t_{2x+2},\dots,t_\ell)^\varphi\\&=[t_1^{-1},t_1^{-1}t_3,t_1^{-1}t_2,\dots,t_1^{-1}t_{2x+1},t_1^{-1}t_{2x},t_1^{-1}t_{2x+2},\dots,t_1^{-1}t_\ell]^\varphi\\&=[(t_1^{-1})^\varphi,(t_1^{-1}t_3)^\varphi,(t_1^{-1}t_2)^\varphi,\dots,(t_1^{-1}t_{2x+1})^\varphi,(t_1^{-1}t_{2x})^\varphi,(t_1^{-1}t_{2x+2})^\varphi,\dots,(t_1^{-1}t_\ell)^\varphi].\end{aligned}$$

Assume that $2x+1<\ell$. If $[t_1,\dots,t_\ell]$ is fixed by $g$, then, by checking the first and last coordinates,

$$t_1=(t_1^{-1})^\varphi,\qquad t_\ell=(t_1^{-1}t_\ell)^\varphi.$$

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Since $g^2 = 1$, we have $\varphi^2 = 1$ and hence $t_1^{\varphi} = t_1^{-1}$ and $t_\ell^{\varphi} = t_1^{-1}t_\ell$. Now, it is easy to verify that the mapping $(t_1, t_\ell) \mapsto (t_\ell^{-1}t_1, t_\ell^{-1})$ defines a one-to-one correspondence between $\{(t_1, t_\ell) \in T'^2 \mid t_1^{\varphi} = t_1^{-1}, t_\ell^{\varphi} = t_1^{-1}t_\ell\}$ and $\{(x, x^{\varphi}) \mid x \in T, x \in \mathbf{C}_T(\varphi^2)\}$. In particular, as $\varphi^2 = 1$,

$$|\{(t_1, t_\ell) \in T'^2 \mid t_1^{\varphi} = t_1^{-1}, t_\ell^{\varphi} = t_1^{-1}t_\ell\}| = |\{(x, x^{\varphi}) \mid x \in T'\}| = |T'|$$

and hence $F$ is divisible by $|T'| = |T|$. However, by checking (6.1) and the order of $T$, we see that this is a contradiction. Assume now that $2x + 1 = \ell$. Following the computations above, we see that $[t_1, \ldots, t_\ell]$ is fixed by $g$ if and only if

$$t_1^{\varphi} = t_1^{-1}, \quad t_3 = t_1 t_2^{\varphi}, \quad t_5 = t_1 t_4^{\varphi}, \ldots, t_{2y+1} = t_1 t_{2y}^{\varphi}, \ldots, t_\ell = t_1 t_{\ell-1}^{\varphi}.$$

In particular, if we let $T_0$ denote the cardinality of the set $\{t_1 \in T' \mid t_1^{\varphi} = t_1^{-1}\}$, we deduce that $g$ fixes $F_0|T'|^{(\ell-1)/2}$ points. Thus,

$$F_0|T'|^{(\ell-1)/2} = F = |\mathbf{C}_T(t)|^{\ell}.$$

If $\ell > 1$, then $|T'| = |T|$ divides $|\mathbf{C}_T(t)|^{\ell}$, which (by checking (6.1) and the order of $T$) is a contradiction. Suppose that $\ell = 1$, that is,

$$|\{t_1 \in T' \mid t_1^{\varphi} = t_1^{-1}\}| = |\mathbf{C}_T(t)| = F.$$

When $T' = \mathrm{PSL}_4(2)$, we see with a direct inspection that $\mathrm{Aut}(T')$ contains no element $\varphi$ such that $|\{t_1 \in T' \mid t_1^{\varphi} = t_1^{-1}\}| = F = 64$. It remains to consider the case that $\ell = 1$, $T \cong \mathrm{PSp}_{2\kappa}(q)$ and $T' = \mathrm{P}\Omega_{2\kappa+1}(q)$ for some $\kappa \geq 3$ and some power $q$ of some odd prime number $p$. Let $u$ be a transvection of $T = \mathrm{PSp}_{2\kappa}(q)$; thus, $u$ has order $p$, $\mathbf{C}_T(t)$ is a maximal subgroup of $T$ and $|T : \mathbf{C}_T(t)| = (q^{2\kappa} - 1)/(q - 1)$. In particular, $\iota_u \in M_1$ and $\iota_u$ fixes $|\mathbf{C}_T(t)| = |T|(q - 1)/(q^{2\kappa} - 1)$ points. Thus, $M_2$ contains an element $g' = \sigma'\varphi'$ conjugate to $\iota_u$ via an element of $G$, where $\sigma' \in \mathrm{Sym}(2)$ and $\varphi' \in \mathrm{Aut}(T')$. Since $u$ has odd order $p$, we deduce that $\sigma' = 1$ and $\varphi' \in \mathrm{Aut}(T')$ has order $p$. A direct inspection in [7, Section 4.5] shows that $\mathrm{Aut}(T')$ contains no element $\varphi'$ of order $p$ fixing $|\mathbf{C}_T(t)| = |T'|(q - 1)/(q^{2\kappa} - 1)$ elements. This last contradiction concludes our argument when $\ell = 1$.

We have shown that $\sigma = 1$ and hence $g = \varphi \in \mathrm{Aut}(T')$. In particular, the number of fixed points of $\varphi$ is $|\mathbf{C}_{T'}(\varphi)|^{\ell}$. A direct inspection in [7, Section 4.5] shows that $\mathrm{Aut}(T')$ contains no involution $\varphi$ with $|\mathbf{C}_{T'}(\varphi)|^{\ell} = F = |\mathbf{C}_T(t)|^{\ell}$. From this last contradiction, (1.1) is never satisfied.

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https://doi.org/10.1017/S0004972719001060 Published online by Cambridge University Press