DENJOY-BOCHNER ALMOST PERIODIC FUNCTIONS

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Abstract

The special Denjoy-Bochner integral (the D*B-integral) which are generalisations of Lebesgue-Bochner integral are discussed in [7, 6, 5]. Just as the concept of numerical almost periodicity was extended by Burkill [3] to numerically valued D*- or D-integrable function, we extend the concept of almost periodicity for Banach valued function to Banach valued D*B-integrable function. For this purpose we introduce as in [3] a distance in the space of all D*B-integrable functions with respect to which the D*B-almost periodicity is defined. It is shown that the D*B-almost periodicity shares many of the known properties of the almost periodic Banach valued function [1, 4].


1. Definitions and terminology

For the definition of almost periodicity for numerical valued and Banach valued functions we refer to [2] and [1, 4] respectively. Throughout the paper \( \mathbb{R} \) and \( \mathbb{C} \) will denote the real line and the complex plane and \( X \) will denote a fixed complex Banach space with norm \( \| \cdot \| \). For a function \( f \) defined on \( \mathbb{R} \), \( f_\eta \) will denote the translation of \( f \) by the number \( \eta \); that is, \( f_\eta(x) = f(x + \eta) \).

**Definition 1.1** [3]. Let \( \mathcal{D}^* \) be the class of all functions \( f: \mathbb{R} \to \mathbb{C} \) such that \( f \) is D*-integrable on each closed interval \( [a, b] \subset \mathbb{R} \). For \( f, g \in \mathcal{D}^* \) the D* distance
between \( f \) and \( g \) is defined to be
\[
\rho_{D^*}(f, g) = \sup_{0 < h \leq 1} \left| (D^*) \int_x^{x+h} \{ f(t) - g(t) \} \, dt \right|.
\]

A function \( f \in \mathcal{D}^* \) is almost periodic in the sense of the \( D^* \) distance (or simply \( D^* \) a.p.) if, given \( \varepsilon > 0 \) there is a relatively dense set \( \{\tau\} \) such that
\[
\rho_{D^*}(f, f_\tau) < \varepsilon
\]
for all \( \tau \in \{\tau\} \).

**DEFINITION 1.2** [7, 6, 5]. A function \( f: [a, b] \to X \) is said to be special Denjoy-Bochner integrable or \( D^*B \)-integrable in \([a, b]\) if there is a function \( F: [a, b] \to X \) such that \( F \) is strongly \( AC_{g*} \) on \([a, b]\) and \( AD_\delta F = f \) almost everywhere in \([a, b]\) where \( AD_\delta F \) stands for the strong approximate derivative of \( F \). The function \( F \) is then called an indefinite \( D^*B \)-integral of \( f \) on \([a, b]\) and \( F(b) - F(a) \) is called its definite \( D^*B \)-integral on \([a, b]\) and is denoted by
\[
(D^*B) \int_a^b f(\xi) \, d\xi.
\]

**DEFINITION 1.3.** Let \( \mathcal{D}^* \mathcal{B} \) be the class of all functions \( f: \mathbb{R} \to X \) such that \( f \) is \( D^*B \)-integrable on each closed interval \([a, b] \subseteq \mathbb{R} \). For \( f, g \in \mathcal{D}^* \mathcal{B} \) the \( D^*B \) distance between \( f \) and \( g \) is defined to be
\[
\rho_{D^*B}(f, g) = \sup_{0 < h \leq 1} \left\| (D^*B) \int_x^{x+h} \{ f(t) - g(t) \} \, dt \right\|.
\]

A function \( f \in \mathcal{D}^* \mathcal{B} \) is said to be almost periodic in the sense of the \( D^*B \)-distance (or, simply \( D^*B \) a.p.) if, given \( \varepsilon > 0 \) there is a relatively dense set \( \{\tau\} = \{\tau; f, \varepsilon\} \) such that
\[
\rho_{D^*B}(f_\tau, f) < \varepsilon
\]
for all \( \tau \in \{\tau\} \). Clearly every almost periodic function \( f: \mathbb{R} \to X \) is \( D^*B \) a.p.

**REMARK.** This definition of the \( D^*B \)-distance, of course, does not guarantee that
\[
\rho_{D^*B}(f, g) < \infty
\]
for all \( f, g \in \mathcal{D}^* \mathcal{B} \). We shall, however, prove that every \( D^*B \) a.p. function \( f \) is \( D^*B \)-bounded, that is
\[
\rho_{D^*B}(f) = \rho_{D^*B}(f, \theta) < \infty
\]
from which it will follow that for all $\mathbb{D}^*\mathbb{B}$ a.p. functions $f$ and $g$

$$
\rho_{\mathbb{D}^*\mathbb{B}}(f, g) < \infty.
$$

**Definition 1.4.** A continuous function $\phi: \mathbb{R} \times [0, 1] \to X$ is called *almost periodic in $x \in \mathbb{R}$ uniformly with respect to $h \in [0, 1]$* if to arbitrary $\varepsilon > 0$ corresponds a relatively dense set $\{\tau\}$ such that

$$
\sup_{0 \leq h \leq 1, -\infty < x < \infty} ||\phi(x + \tau, h) - \phi(x, h)|| < \varepsilon
$$

for all $\tau \in \{\tau\}$.

The following result for integration by parts for the $\mathbb{D}^*\mathbb{B}$-integral, which will be needed later, is proved in [5].

**Theorem 1.5.** Let $f: [a, b] \to X$ be $\mathbb{D}^*\mathbb{B}$-integrable and

$$
F(\xi) = \int_a^\xi f(t) \, dt.
$$

Let $g: [a, b] \to \mathbb{R}$ be $L$-integrable and let

$$
G(\xi) = \int_a^\xi g(t) \, dt.
$$

Then $fG$ is $\mathbb{D}^*\mathbb{B}$-integrable over $[a, b]$ and

$$
\int_a^b fG = [FG]_b^a - \int_a^b Fg.
$$

2. Properties of $\mathbb{D}^*\mathbb{B}$ a.p. functions

**Theorem 2.1.** If a function $f$ is $\mathbb{D}^*\mathbb{B}$ a.p. then

$$
F(x) = \int_0^x f(t) \, dt
$$

is uniformly continuous.

Since the $\mathbb{D}^*\mathbb{B}$-integral,

$$
F(x) = \int_0^x f(t) \, dt,
$$

is continuous and since a continuous Banach valued function is uniformly continuous on a closed interval the theorem can be proved by the usual process.

**Theorem 2.2.** If $f$ is $\mathbb{D}^*\mathbb{B}$ a.p. then the function $\phi: \mathbb{R} \times [0, 1] \to X$ defined by

$$
\phi(x, h) = \int_x^{x+h} f(t) \, dt
$$

is almost periodic in $x \in \mathbb{R}$, uniformly with respect to $h \in [0, 1]$.  

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PROOF. We first show that the function $\phi$ is continuous. Let $\epsilon > 0$ be arbitrary. Since by Theorem 2.1 $F(x) = \int_0^x f(t) \, dt$ is uniformly continuous, there is a $\delta > 0$ such that $\|F(x_1) - F(x_2)\| < \epsilon/2$ whenever $|x_1 - x_2| < \delta$ for all $x_1, x_2 \in \mathbb{R}$. Now, let $(x_0, h_0) \in \mathbb{R} \times [0, 1]$ be arbitrary. Then

$$
\|\phi(x_0, h_0) - \phi(x, h)\| = \left\| \int_{x_0}^{x_0+h_0} f(t) \, dt - \int_x^{x+h} f(t) \, dt \right\|
$$

$$
= \left\| F(x_0 + h_0) - F(x_0) - F(x + h) + F(x) \right\|
$$

$$
\leq \left\| F(x_0 + h_0) - F(x + h) \right\| + \left\| F(x) - F(x_0) \right\|
$$

$$
< \frac{\epsilon}{2} + \frac{\epsilon}{2}
$$

whenever $|x - x_0| < \delta/2$, $|h - h_0| < \delta/2$. Hence $\phi(x, h)$ is continuous on $\mathbb{R} \times [0, 1]$. 

Now, since $f$ is $D*B$ a.p., corresponding to $\epsilon > 0$ there is a relatively dense set $\{\tau\}$ such that $\rho_{D*B}(f, \tau) < \epsilon$ for all $\tau \in \{\tau\}$. Hence

$$
\sup_{0 \leq h \leq 1} \left\| \int_x^{x+h} f(t + \tau) \, dt - \int_x^{x+h} f(t) \, dt \right\| < \epsilon,
$$

which completes the proof.

**Lemma 2.3.** Let $C_X[0, 1]$ be the Banach space of all continuous functions $y: [0, 1] \to X$ with norm

$$
\|y\|_{C_X} = \sup_{0 \leq h \leq 1} \|y(h)\|
$$

and let $\phi: \mathbb{R} \times [0, 1] \to X$ be a continuous function. Then the function $\Phi: \mathbb{R} \to C_X[0, 1]$ defined by

$$
\Phi(x) = \phi(x, \cdot)
$$

is almost periodic if and only if the function $\phi$ is almost periodic in $x \in \mathbb{R}$, uniformly with respect to $h \in [0, 1]$. 


PROOF. Since \( \|\Phi(x)\|_{c_x} = \sup_{0 \leq h \leq 1} \|\Phi(x, h)\| \) we have
\[
\|\Phi(x + \tau) - \Phi(x)\|_{c_x} = \sup_{0 \leq h \leq 1} \|\phi(x + \tau, h) - \phi(x, h)\|
\]
and so the result follows.

**Lemma 2.4.** If the continuous functions \( \phi: \mathbb{R} \times [0, 1] \to \mathbb{X} \) and \( \psi: \mathbb{R} \times [0, 1] \to \mathbb{X} \) are almost periodic in \( x \in \mathbb{R} \) uniformly with respect to \( h \in [0, 1] \) then \( \phi + \psi \) is so.

**Proof.** Let \( \mathcal{C}_{X}[0, 1] \) be as in Lemma 2.3 and let \( \Phi: \mathbb{R} \to \mathcal{C}_{X}[0, 1] \) and \( \Psi: \mathbb{R} \to \mathcal{C}_{X}[0, 1] \) be defined by
\[
\Phi(x) = \phi(x, \cdot), \quad \Psi(x) = \psi(x, \cdot).
\]
Then by Lemma 2.3, \( \Phi \) and \( \Psi \) are almost periodic and so is the sum \( \Phi + \Psi \), and hence by Lemma 2.3, \( \phi + \psi \) is almost periodic in \( x \in \mathbb{R} \) uniformly with respect to \( h \in [0, 1] \).

**Theorem 2.5.** If \( f \) and \( g \) are \( D^*B \) a.p. then so is \( f + g \).

**Proof.** By Theorem 2.2, the functions \( \phi(x, h) = \int_{x}^{x+h} f(t) \, dt \) and \( \psi(x, h) = \int_{x}^{x+h} g(t) \, dt \) are almost periodic in \( x \in \mathbb{R} \) uniformly with respect to \( h \in [0, 1] \). Hence by Lemma 2.4, \( \phi(x, h) + \psi(x, h) \) is almost periodic in \( x \in \mathbb{R} \) uniformly with respect to \( h \in [0, 1] \). So, given \( \varepsilon > 0 \), there is a relatively dense set \( \{\tau\} \) such that
\[
\sup_{-\infty < x < \infty} \|\phi(x + \tau, h) + \psi(x + \tau, h) - \phi(x, h) - \psi(x, h)\| < \varepsilon
\]
for all \( \tau \in \{\tau\} \). Hence
\[
\sup_{-\infty < x < \infty} \left\| \int_{x+\tau}^{x+h} \{ f(t) + g(t) \} \, dt - \int_{x}^{x+h} \{ f(t) + g(t) \} \, dt \right\| < \varepsilon,
\]
that is,
\[
\sup_{-\infty < x < \infty} \left\| \int_{x}^{x+h} \left[ \{ f(t + \tau) + g(t + \tau) \} - \{ f(t) + g(t) \} \right] \, dt \right\| < \varepsilon,
\]
that is,
\[
\rho_{D^*B}(f + g, f + g) < \varepsilon
\]
for all \( \tau \in \{\tau\} \). Hence \( f + g \) is \( D^*B \) a.p.

**Theorem 2.6.** If \( f \) is \( D^*B \) a.p. then \( f \) is \( D^*B \) bounded, that is,
\[
\rho_{D^*B}[f] = \rho_{D^*B}(f, \theta) < \infty.
\]
PROOF. Letting \( \phi(x, h) = \int_x^{x+h} f(t) \, dt \) and constructing the function \( \Phi: \mathbb{R} \to \mathbb{C} \times [0, 1] \) as in Lemma 2.3 we see \( \Phi \) is almost periodic. Then by [1, page 5, property IV], the range of \( \Phi \) is relatively compact and hence

\[
\sup_{-\infty < x < \infty} \| \Phi(x) \|_{\mathbb{C} \times [0, 1]} < \infty.
\]

Hence by the definition of \( \| \cdot \|_{\mathbb{C} \times [0, 1]} \)

\[
\sup_{0 \leq h \leq 1} \| \phi(x, h) \| < \infty,
\]

that is,

\[
\sup_{0 \leq h \leq 1} \left\| \int_x^{x+h} f(t) \, dt \right\| < \infty,
\]

that is,

\[
\rho_{D*B}(f, \theta) < \infty.
\]

**Theorem 2.7.** If \( f \) is \( D*B \) a.p. then \( f \) is uniformly continuous with respect to the metric \( \rho_{D*B} \); that is, for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

\[
\rho_{D*B}(f, \eta) < \varepsilon
\]

for all \( \eta \) satisfying \( |\eta| < \delta \).

**Proof.** Since \( f \) is \( D*B \) a.p. by Theorem 2.2 and Lemma 2.3 the function \( \Phi: \mathbb{R} \to \mathbb{C} \times [0, 1] \) defined by \( \Phi(x) = \phi(x, \cdot) \) is almost periodic, where \( \phi(x, h) = \int_x^{x+h} f(t) \, dt \). By [1, page 5, property III], \( \Phi \) is uniformly continuous. So, for arbitrary \( \varepsilon > 0 \) there is \( \delta > 0 \) such that

\[
\sup_{-\infty < x < \infty} \| \Phi(x + \eta) - \Phi(x) \|_{\mathbb{C} \times [0, 1]} < \varepsilon
\]

for all \( \eta \) satisfying \( |\eta| < \delta \). That is,

\[
\sup_{0 \leq h \leq 1} \| \phi(x + \eta, h) - \phi(x, h) \| < \varepsilon,
\]

that is,

\[
\sup_{0 \leq h \leq 1} \left\| \int_x^{x+h} \{ f(t + \eta) - f(t) \} \, dt \right\| < \varepsilon,
\]
that is,
\[ \rho_{D^*B}(f_\eta, f) < \epsilon \]
whenever \(|\eta| < \delta\).

**Theorem 2.8.** If \( \{f_n\} \) is a sequence of \( D^*B \) a.p. functions such that \( f_n \to f \) with respect to the metric \( \rho_{D^*B} \) then \( f \) is \( D^*B \) a.p.

**Proof.** Let \( \epsilon > 0 \) be arbitrary. Then there is \( N \) such that \( \rho_{D^*B}(f_n, f) < \epsilon/3 \) for all \( n \geq N \). Since \( f_N \) is \( D^*B \) a.p. so there is a relatively dense set \( \{\tau\} \) for which \( \rho_{D^*B}(f_N, f_N) < \epsilon/3 \). Hence
\[
\rho_{D^*B}(f_\tau, f) \leq \rho_{D^*B}(f_\tau, (f_N)_\tau) + \rho_{D^*B}((f_N)_\tau, f_N) + \rho_{D^*B}(f_N, f) \\
= \rho_{D^*B}(f, f_N) + \rho_{D^*B}((f_N)_\tau, f_N) + \rho_{D^*B}(f_N, f) \\
< \epsilon.
\]
Thus \( f \) is \( D^*B \) a.p.

**Theorem 2.9.** If \( f \) is \( D^*B \) a.p. and \( u(x) \) is almost periodic numerical valued function with its derivative \( u'(x) \) uniformly continuous then \( f(x)u(x) \) is \( D^*B \) a.p.

The proof of the theorem is similar to that of the corresponding theorem of [3]. In fact all the arguments of [3] will apply in this case taking into account the fact that the integration by parts formula for integral is given in Theorem 1.5.

**Lemma 2.10.** If \( f \) is \( D^*B \) a.p. then \( x*f \) is \( D^* \) a.p. for every \( x^* \in X^* \), where \( X^* \) is the conjugate space of the Banach space \( X \).

**Proof.** Take any \( x^* \in X^* \) and \( \epsilon > 0 \). Then there corresponds a relatively dense set \( \{\tau\} = \{\tau; f, \epsilon(||x^*|| + 1)^{-1}\} \) such that
\[
\sup_{0 < h \leq \infty \leq \infty} \left\| \int_x^{x+h} \{ f(t + \tau) - f(t) \} \, dt \right\| < \epsilon(||x^*|| + 1)^{-1}
\]
for all \( \tau \in \{\tau\} \). Now since \( f \) is \( D^*B \) a.p., \( f \) is \( D^*B \)-integrable on each closed interval \([a, b]\) and so by a result of [5] \( x^*f \) is \( D^* \)-integrable on each \([a, b]\) and therefore \( x^*f \in D^* \). Moreover
\[
x^* \int_x^{x+h} f(t) \, dt = \int_x^{x+h} x^*f(t) \, dt
\]
for all $x \in \mathbb{R}$ and $h \in [0, 1]$. Hence for all $\tau \in \{\tau\}$

$$
\sup_{0 \leq h \leq 1} \left| \int_{-\infty}^{x+h} \{x^* f(t + \tau) - x^* f(t)\} \, dt \right|
= \sup_{0 \leq h \leq 1} \left| \int_{-\infty}^{x+h} x^* \{f(t + \tau) - f(t)\} \, dt \right|
= \sup_{0 \leq h \leq 1} \left| x^* \int_{-\infty}^{x+h} \{f(t + \tau) - f(t)\} \, dt \right|
\leq \sup_{0 \leq h \leq 1} \|x^*\| \left\| \int_{-\infty}^{x+h} \{f(t + \tau) - f(t)\} \, dt \right\|
= \|x^*\| \sup_{0 \leq h \leq 1} \left\| \int_{-\infty}^{x+h} \{f(t + \tau) - f(t)\} \, dt \right\|
< \|x^*\|e(||x^*|| + 1)^{-1} < \epsilon.
$$

This completes the proof of the lemma.

**Lemma 2.11.** If $x^* f$ is $D^*$ a.p. for all $x^* \in X^*$ and if

$$
F(t) = \int f(x) \, dx
$$

is bounded then $F$ is weakly almost periodic (that is, $x^* F$ is almost periodic for all $x^* \in X^*$).

**Proof.** The function $F(t)$ being bounded $x^* F(t)$ is also bounded for all $x^* \in X^*$ and since

$$
x^* F(t) = (D^*) \int_0^t x^* f(x) \, dx,
$$

$x^* F$ is almost periodic by [3], that is, $F$ is weakly almost periodic.

**Theorem 2.12.** If $f$ is $D^* B$ a.p. and if

$$
F(t) = \int f(x) \, dx
$$

is such that the range of $F$ is relatively compact then $F$ is almost periodic.

**Proof.** By Lemma 2.10, $x^* f$ is $D^*$ a.p. for all $x^* \in X^*$. The range of $F$ being relatively compact (that is, its closure being compact) $F$ is bounded. Hence by Lemma 2.11, $F$ is weakly almost periodic. So by [1, page 45, property X] $F$ is almost periodic.
THEOREM 2.13. The class of all D*B a.p. functions is identical with the D*B-closure of the set of all trigonometric polynomials

\[ P(t) = \sum_{r=1}^{n} a_r e^{i\lambda_r t} \]

where \( a_r \in \mathbb{X}, \lambda_r \in \mathbb{R} \).

The theorem can be proved in the same way as the corresponding theorem of D a.p. functions of [3].

THEOREM 2.14. If \( f \) is D*B a.p. and is uniformly continuous then \( f \) is almost periodic.

PROOF. Let \( \phi: \mathbb{R} \to \mathbb{R} \) be a nonnegative function with support \([0, 1]\) having continuous derivative \( \phi' \) such that \( \int_0^1 \phi(t) \, dt = 1 \). For a fixed \( n \) let \( \phi_n(x) = n \phi(nx) \). Then \( \phi_n \) is a nonnegative function with support \([0, 1/n]\) having continuous derivative \( \phi'_n \) and \( \int_0^{1/n} \phi_n(t) \, dt = 1 \). Let

\[ f_n(x) = \int_0^{1/n} f(t + x) \phi_n(t) \, dt. \]

Then we shall show that \( f_n \) is almost periodic for each \( n \). Let \( n \) be fixed and let \( \epsilon > 0 \) be arbitrary. Let

\[ M = \sup_{0 \leq x \leq 1} |\phi_n(x)|, \quad M' = \sup_{0 \leq x \leq 1} |\phi'_n(x)|. \]

Since \( f \) is D*B a.p. there is a relatively dense set \( \{\tau\} \) such that

\[ \rho_{D*B}(f, f) < \epsilon(M + M')^{-1} \]

for all \( \tau \in \{\tau\} \). Let \( \tau \in \{\tau\} \). Then writing \( F(x) = \int_0^x f(t) \, dt \) and \( \psi(x) = F(x + \tau) - F(x) \) we have employing Theorem 1.5

\[ \|f_n(x + \tau) - f_n(x)\| \]

\[ = \left\| \int_0^{1/n} \{f(t + x + \tau) - f(t + x)\} \phi_n(t) \, dt \right\| \]

\[ = \left\| \left[ \phi_n(t) \{F(t + x + \tau) - F(t + x)\} \right]_{t=0}^{1/n} \right\| \]

\[ = \left\| \int_0^{1/n} \{F(t + x + \tau) - F(t + x)\} \phi'_n(t) \, dt \right\| \]

\[ = \left\| \left[ \phi_n(t) \psi(x + t) \right]_{t=0}^{1/n} - \int_0^{1/n} \psi(x + t) \phi'_n(t) \, dt \right\| \]

\[ = \left\| \left[ \phi_n(t) \{\psi(x + t) - \psi(x)\} \right]_{t=0}^{1/n} - \int_0^{1/n} \{\psi(x + t) - \psi(x)\} \phi'_n(t) \, dt \right\| \]

\[ = \left\| \left[ \phi_n \left( \frac{1}{n} \right) \left\{ \psi \left( x + \frac{1}{n} \right) - \psi(x) \right\} \right]_{t=0}^{1/n} - \int_0^{1/n} \{\psi(x + t) - \psi(x)\} \phi'_n(t) \, dt \right\|. \]
Now let $t \in [0, 1/n]$. Then from (2.1)
\[ \|\psi(x + t) - \psi(x)\| = \left\| \int_x^{x+t} (f(\xi + t) - f(\xi)) \, d\xi \right\| \leq \rho_{D*B}(f, _) < \epsilon (M + M')^{-1}. \]

Hence from (2.2)
\[ \|f_n(x + \tau) - f_n(x)\| < Me(M + M')^{-1} + M'e(M + M')^{-1} = \epsilon. \]
Since $\tau \in \{\tau\}$ is arbitrary, $f_n$ is almost periodic for each $n$.

Now since $f$ is uniformly continuous, for every $\epsilon > 0$ there is $\delta > 0$ such that
\[ \|f(t + x) - f(x)\| < \epsilon \]
whenever $|t| < \delta$. Choose $N$ such that $1/N < \delta$. Then when $n \geq N$ we have
\[ \|f_n(x) - f(x)\| = \left\| \int_0^{1/n} f(t + x)\phi_n(t) \, dt - \int_0^{1/n} f(x)\phi_n(t) \, dt \right\| \]
\[ = \left\| \int_0^{1/n} \{f(t + x) - f(x)\}\phi_n(t) \, dt \right\| \leq \int_0^{1/n} \|f(t + x) - f(x)\|\phi_n(t) \, dt < \epsilon. \]
Thus $\{f_n\}$ converges uniformly to $f$. Since each $f_n$ is almost periodic, by [1, page 6, property V] $f$ is almost periodic.

3. Mean values and Fourier series

**Theorem 3.1.** If $f$ is $D*B$ a.p. then the mean value
\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) \, dt = M(f) \]
exists; further
\[ \lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} f(t) \, dt = M(f) \]
uniformly with respect to $a \in \mathbb{R}$.

**Proof.** Since
\[ \frac{1}{T} \int_a^{a+T} e^{i\lambda t} \, dt = \begin{cases} 1 \\ \frac{1}{i\lambda T} [e^{i\lambda(a+T)} - e^{i\lambda a}] \end{cases} \]
if $\lambda = 0$, if $\lambda \neq 0$,
it is clear that
\[
\lim_{T \to \infty} \frac{1}{T} \int_{a}^{a+T} e^{i\lambda t} \, dt = \begin{cases} 
1 & \text{if } \lambda = 0, \\
0 & \text{if } \lambda \neq 0
\end{cases}
\]
uniformly with respect to \(a \in \mathbb{R}\) and hence for any trigonometric polynomial \(P\),
\[
P(t) = \sum_{r=1}^{n} a_r e^{i\lambda_r t} \quad (a_r \in X, \lambda_r \in \mathbb{R}),
\]
\[
\lim_{T \to \infty} \frac{1}{T} \int_{a}^{a+T} P(t) \, dt = M(P)
\]
exists uniformly with respect to \(a \in \mathbb{R}\). Let \(\varepsilon > 0\) be arbitrary. By Theorem 2.13 there is a trigonometric polynomial \(P\) such that \(\rho_{D,B}(f, P) < \varepsilon\). Hence
\[
(3.1) \quad \sup_{-\infty < x < \infty} \left\| \int_{x}^{x+h} \{f(t) - P(t)\} \, dt \right\| < \varepsilon.
\]
Now corresponding to \(\varepsilon\) there is \(T_\varepsilon\) which is independent of \(a\), such that
\[
(3.2) \quad \left\| \frac{1}{T'} \int_{a}^{a+T'} P(t) \, dt - \frac{1}{T''} \int_{a}^{a+T''} P(t) \, dt \right\| < \varepsilon
\]
for all \(T', T'' > T_\varepsilon\).

Set \(T_0 = \max\{T_\varepsilon, 2\}\) and let \(T_1, T_2 > T_0\). Then there is a positive integer \(N\) such that \(N - 1 < T_1 \leq N\). Putting \(h = T_1/N\), since \(N > 2\), we have \(\frac{1}{2} < h \leq 1\). Now by (3.1) we have
\[
(3.3) \quad \left\| \frac{1}{T_1} \int_{a}^{a+T_1} \{f(t) - P(t)\} \, dt \right\| = \left\| \frac{1}{Nh} \int_{a}^{a+Nh} \{f(t) - P(t)\} \, dt \right\|
\]
\[
= \left\| \frac{1}{Nh} \sum_{n=1}^{N} \int_{a+(n-1)h}^{a+nh} \{f(t) - P(t)\} \, dt \right\|
\]
\[
\leq \frac{1}{Nh} \sum_{n=1}^{N} \left\| \int_{a+(n-1)h}^{a+nh} \{f(t) - P(t)\} \, dt \right\|
\]
\[
< \frac{1}{Nh} Ne
\]
\[
< 2\varepsilon
\]
since \(1/h < 2\). Similarly for \(T_2 > T_0\),
\[
(3.4) \quad \left\| \frac{1}{T_2} \int_{a}^{a+T_2} \{f(t) - P(t)\} \, dt \right\| < 2\varepsilon.
\]
Since \( T_0 \geq T_\varepsilon \) we have from (3.2), (3.3) and (3.4) when \( T_1, T_2 > T_0 \),

\[
\left\| \frac{1}{T_1} \int_a^{a+T_1} f(t) \, dt - \frac{1}{T_2} \int_a^{a+T_2} f(t) \, dt \right\| \\
\leq \left\| \frac{1}{T_1} \int_a^{a+T_1} f(t) \, dt - \frac{1}{T_1} \int_a^{a+T_1} P(t) \, dt \right\| \\
+ \left\| \frac{1}{T_1} \int_a^{a+T_1} P(t) \, dt - \frac{1}{T_2} \int_a^{a+T_2} P(t) \, dt \right\| \\
+ \left\| \frac{1}{T_2} \int_a^{a+T_2} f(t) \, dt - \frac{1}{T_2} \int_a^{a+T_2} P(t) \, dt \right\| \\
< 5\varepsilon.
\]

Thus since \( X \) is complete and since \( T_0 \) is independent of \( a \),

\[
\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} f(t) \, dt = M(f)
\]

exists uniformly with respect to \( a \in \mathbb{R} \), completing the proof.

Now if \( f \) is D*B a.p. then since \( u(x) = e^{-i\lambda x} \) is numerically valued almost periodic function and \( u'(x) \) is uniformly continuous, by Theorem 2.9 \( f(x)e^{-i\lambda x} \) is D*B a.p. for all \( \lambda \in \mathbb{R} \) and consequently

\[
M\{ f(x)e^{-i\lambda x} \} = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x)e^{-i\lambda x} \, dx
\]

exists for every \( \lambda \in \mathbb{R} \). For a D*B a.p. function \( f \) we shall write

\[
a(\lambda) = a(\lambda; f) = M\{ f(x)e^{-i\lambda x} \}.
\]

**Theorem 3.2.** If \( f \) is D*B a.p. then \( a(\lambda; f) \) differs from the zero element \( \theta \) of \( X \) for only an enumerable set of values of \( \lambda \).

**Proof.** Let

\[
F(x) = \int_0^x f(t) \, dt.
\]

Then for a given \( h \in [0, 1] \) we have, by integrating by parts by Theorem 1.5.

\[
(3.5) \quad \frac{1}{T} \int_0^T (f(x + h) - f(x)) e^{-i\lambda x} \, dx = \frac{1}{T} \left[ \{F(x + h) - F(x)\} e^{-i\lambda x} \right]_0^T \\
+ i\lambda \int_0^T \{F(x + h) - F(x)\} e^{-i\lambda x} \, dx.
\]
Also

\begin{equation}
(3.6) \quad \frac{1}{T} \int_0^T \{ f(x + h) - f(x) \} e^{-i\lambda x} \, dx
\end{equation}

\[ = \frac{1}{T} \left[ e^{i\lambda h} \int_h^{T+h} f(t) e^{-i\lambda t} \, dt - \int_T^T f(t) e^{-i\lambda t} \, dt \right]. \]

Now by Theorem 2.2 the function \( F(x + h) - F(x) \) is almost periodic. Let its Fourier coefficients be \( a_h(\lambda) \). Then applying Theorem 3.1 we get from (3.5) and (3.6), by letting \( T \to \infty \) since \( F(x + h) - F(x) \) is bounded,

\begin{equation}
(3.7) \quad (e^{i\lambda h} - 1) a(\lambda; f) = i\lambda a_h(\lambda).
\end{equation}

So, if \( \lambda \neq 2n\pi, n = 0, \pm 1, \pm 2, \ldots \)

\[ a(\lambda; f) = \frac{i\lambda}{e^{i\lambda h} - 1} a_h(\lambda). \]

Since \( a_h(\lambda) \neq \theta \) for at most enumerable number of \( \lambda \), \( a(\lambda) \neq \theta \) for these enumerable \( \lambda \) and probably for \( \lambda = 2n\pi, n = 0, \pm 1, \pm 2, \ldots \). Thus \( a(\lambda) \) differs from \( \theta \) for at most an enumerable set of values of \( \lambda \). This completes the proof of the theorem.

Let \( \{\lambda_n\} \) be the enumerable set such that \( a(\lambda_n) \neq \theta \). Putting \( a_n = a(\lambda_n) \) we say that \( \sum a_n e^{i\lambda_n x} \) is the Fourier series of \( f \) and write

\[ f \sim \sum_n a_n e^{i\lambda_n x}. \]

**Lemma 3.3.** If \( f \) is \( \text{D}^*\text{B} \) a.p. and \( x^* \in X^* \) then

\[ x^* a(\lambda; f) = a(\lambda; x^* f). \]

**Proof.**

\[ x^* a(\lambda; f) = x^* M\{ f(x) e^{-i\lambda x} \} \]

\[ = x^* \lim_{T \to \infty} \frac{1}{T} (\text{D}^*\text{B}) \int_0^T f(x) e^{-i\lambda x} \, dx \]

\[ = \lim_{T \to \infty} \frac{1}{T} x^* (\text{D}^*\text{B}) \int_0^T f(x) e^{-i\lambda x} \, dx \]

since \( x^* \) is continuous. Now since a Denjoy-Bochner integrable function is Denjoy-Pettis integrable with integrals equal [5], we have

\[ x^* (\text{D}^*\text{B}) \int_0^T f(x) e^{-i\lambda x} \, dx = (\text{D}^*) \int_0^T x^* f(x) e^{-i\lambda x} \, dx \]

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and hence
\[ x^*a(\lambda; f) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^*f(x)e^{-i\lambda x} \, dx \]
\[ = M\{x^*f(x)e^{-i\lambda x}\} \]
\[ = a(\lambda; x^*f). \]

**THEOREM 3.4 (Uniqueness Theorem).** If two D*B a.p. functions \( f \) and \( g \) have same Fourier series then

\[ \rho_{D*B}(f, g) = 0. \]

**PROOF.** Let \( x^* \in X^* \) be arbitrarily chosen. By Lemma 2.10 \( x^*f \) and \( x^*g \) are D* a.p. scalar functions and by Lemma 3.3 they have same Fourier series. As the corresponding theorem of [3] it can be shown that \( \rho_{D*}(x^*f, x^*g) = 0 \), that is,

\[ \sup_{0 \leq h \leq 1} \left| (D^*) \int_x^{x+h} \{x^*f(t) - x^*g(t)\} \, dt \right| = 0. \]

Now by our previous remark

\[ x^*(D*B) \int_x^{x+h} \{f(t) - g(t)\} \, dt = (D^*) \int_x^{x+h} x^*\{f(t) - g(t)\} \, dt \]

and hence

\[ \sup_{0 \leq h \leq 1} \left| x^*(D*B) \int_x^{x+h} \{f(t) - g(t)\} \, dt \right| = 0. \]

Therefore,

\[ x^*(D*B) \int_x^{x+h} \{f(t) - g(t)\} \, dt = 0 \]

for all \( x \in \mathbb{R} \) and \( h \in [0, 1] \). Since \( x^* \) is arbitrary, by Hahn-Banach Theorem

\[ (D*B) \int_x^{x+h} \{f(t) - g(t)\} \, dt = 0 \]

for all \( x \in \mathbb{R} \) and \( h \in [0, 1] \). Therefore

\[ \sup_{0 \leq h \leq 1} \left\| \int_x^{x+h} \{f(t) - g(t)\} \, dt \right\| = 0, \]

that is,

\[ \rho_{D*B}(f, g) = 0. \]
4. Bochner-Fejer summability of Fourier series

We shall show that if \( f \) be \( D^*B \) a.p. then the Fourier series of \( f \) is Bochner-Fejer summable to \( f \) with respect to the metric \( \rho_{D^*B} \) defined on the space of all \( D^*B \) a.p. functions. For this purpose we shall use the ‘Bochner-Fejer Kernel’ and the ‘Bochner-Fejer Polynomials’ the details of which are discussed in [2, pages 46-50], [1, page 26] and [4, page 153].

Let \( f \) be \( D^*B \) a.p. and let \( f(t) \sim \sum a_k e^{i\lambda_k t} \). Let \( \beta_1, \beta_2, \ldots \) be a basis of the sequence \( \{\lambda_k\} \) of the Fourier exponents of \( f \). For each positive integer \( m \) we consider the Bochner-Fejer Kernel

\[
K_m(t) = \sum \left( 1 - \frac{|v_1|}{(m!)^2} \right) \cdots \left( 1 - \frac{|v_m|}{(m!)^2} \right) \exp \left( - \frac{it}{m!} \sum_{k=1}^{m} v_k \beta_k \right)
\]

and the Bochner-Fejer polynomial for \( f \)

\[
\sigma_m(t) = \sigma_m(t; f) = \sum \left( 1 - \frac{|v_1|}{(m!)^2} \right) \cdots \left( 1 - \frac{|v_m|}{(m!)^2} \right) \times a \left( \frac{1}{m!} \sum_{k=1}^{m} v_k \beta_k; f \right) \exp \left( \frac{it}{m!} \sum_{k=1}^{m} v_k \beta_k \right),
\]

where the first summations in (4.1) and (4.2) extend to all \( v_j, \ |v_j| \leq (m!)^2, \ j = 1, 2, \ldots, m \), and \( a(\lambda; f) \) in (4.2) is defined by

\[
a(\lambda; f) = M\{ fe^{-i\lambda x} \}.
\]

If, however, the basis contains a finite number of elements \( \beta_1, \beta_2, \ldots, \beta_p \) then we take

\[
\sigma_m(t) = \sum \left( 1 - \frac{|v_1|}{(m!)^2} \right) \cdots \left( 1 - \frac{|v_p|}{(m!)^2} \right) \times a \left( \frac{1}{m!} \sum_{k=1}^{p} v_k \beta_k; f \right) \exp \left( \frac{it}{m!} \sum_{k=1}^{p} v_k \beta_k \right),
\]

the summation being extended to \( |v_j| \leq (m!)^2, \ j = 1, 2 \cdots p \) with similar modification for \( K_m(t) \). It can be verified that

\[
\sigma_m(t; f) = \lim_{T \to \infty} \int_0^T K_m(u) f(u + t) \, du.
\]

In what follows we need the function

\[
\phi(x, h) = \int_x^{x+h} f(t) \, dt, \quad x \in \mathbb{R}, \ h \in [0, 1].
\]
For fixed $h \in [0, 1]$ this is a function of $x$ alone which is almost periodic by Theorem 2.2. Therefore for arbitrary but fixed $h \in [0, 1]$, the $\sigma_m(x; \phi)$ will have the same meaning as given in (4.2).

**Theorem 4.1.** Let $f$ be $D^*B$ a.p. and let

$$f(t) \sim \sum a_k e^{i\lambda_k t}.$$

Then the sequence of trigonometric polynomials $\{\sigma_m(t; f)\}$ converges to $f$ with respect to the metric $\rho_{D^*B}$ as $m \to \infty$.

We shall complete the proof of the theorem in three lemmas.

**Lemma 4.2.** If $f$ is $D^*B$ a.p. then

$$\sigma_m(x; \phi) \to \phi(x, h)$$

as $m \to \infty$ uniformly with respect to $x \in \mathbb{R}$ and $h \in [0, 1]$ where $\phi(x, h) = \int_{x + h} f(t) \, dt$.

**Proof.** By Theorem 2.2 $\phi(x, h)$ is almost periodic in $x \in \mathbb{R}$ uniformly with respect to $h \in [0, 1]$. Hence by Lemma 2.3 the Banach valued function $\Phi: \mathbb{R} \to \mathbb{C}_x[0, 1]$ defined by $\Phi(t) = \phi(t, \cdot)$ is almost periodic. If

$$\Phi(t) \sim \sum b_n e^{i\lambda_n t}$$

then $b_n \in \mathbb{C}_x[0, 1]$ and

(4.3) $$b_n = \lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} \Phi(t) e^{-i\lambda_n t} \, dt$$

uniformly with respect to $a$ (see [4, page 146]). By the definition of $\Phi$ we can write

$$\Phi(t) e^{-i\lambda_n t} = \phi(t, \cdot) e^{-i\lambda_n t}$$

and so

$$\frac{1}{T} \int_a^{a+T} \Phi(t) e^{-i\lambda_n t} \, dt = \frac{1}{T} \int_a^{a+T} \phi(t, \cdot) e^{-i\lambda_n t} \, dt. $$

Hence from (4.3)

$$\lim_{T \to \infty} \left\| \frac{1}{T} \int_a^{a+T} \Phi(t) e^{-i\lambda_n t} \, dt - b_n \right\|_{\mathbb{C}_x} = 0$$

uniformly with respect to $a$. That is

$$\lim_{T \to \infty} \sup_{0 \leq h \leq 1} \left\| \frac{1}{T} \int_a^{a+T} \phi(t, h) e^{-i\lambda_n t} \, dt - b_n(h) \right\| = 0$$
uniformly with respect to \( a \). Hence
\[
\lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \phi(t, h) e^{-i \lambda_n t} \, dt = b_n(h)
\]
uniformly with respect to \( a \) and \( h \). So, \( b_n(h) \) are the Fourier coefficients of \( \phi(t, h) \) and the Fourier exponents of \( \Phi(t) \) and \( \phi(t, h) \) will remain the same. Now it is proved in [1, page 26] that
\[
\lim_{m \to \infty} \sigma_m(t; \Phi) = \Phi(t)
\]
uniformly with respect to \( t \), where \( \sigma_m(t; \Phi) \) is defined as in (4.2) and the limit in (4.4) is taken with respect to the Banach space in which \( \Phi(t) \) lies and so (4.4) becomes
\[
\| \sigma_m(t; \Phi) - \Phi(t) \|_{C_x} \to 0
\]
as \( m \to \infty \) uniformly with respect to \( t \). That is
\[
\sup_{0 \leq h \leq 1} \| \sigma_m(t; \phi) - \phi(t, h) \| \to 0
\]
as \( m \to \infty \) uniformly with respect to \( t \). Thus
\[
\sigma_m(t; \phi) \to \phi(t, h)
\]
as \( m \to \infty \) uniformly with respect to \( t \) and \( h \).

**Lemma 4.3.** If \( f \) is D* B a.p. then for each \( h \in [0, 1] \)
\[
\int_{x}^{x+h} \sigma_m(t; f) \, dt = \sigma_m(x; \phi).
\]

Integrating (4.2) and using (3.7) the proof can be completed.

**Lemma 4.4.** If \( f \) is D* B a.p. then \( \sigma_m(t; f) \to f(t) \) as \( m \to \infty \) with respect to the metric \( \rho_{D*B} \).

**Proof.** Let \( \phi(x, h) = \int_{x}^{x+h} f(t) \, dt \). Then by Lemma 4.2
\[
\sigma_m(x; \phi) \to \phi(x, h)
\]
as \( m \to \infty \) uniformly with respect to \( x \in \mathbb{R} \) and \( h \in [0, 1] \). So,
\[
\sup_{-\infty < x < \infty} \sup_{0 \leq h \leq 1} \| \sigma_m(x; \phi) - \phi(x, h) \| \to 0
\]
as \( m \to \infty \). Hence by Lemma 4.3
\[
\sup_{-\infty < x < \infty} \left\| \int_{x}^{x+h} \sigma_m(t; f) \, dt - \int_{x}^{x+h} f(t) \, dt \right\| \to 0
\]
as $m \to \infty$. So,

$$\sup_{-\infty < x < \infty} \left\| \int_{x}^{x+h} \{ \sigma_m(t; f) - f(t) \} \, dt \right\| \to 0,$$

that is,

$$\rho_{DB}(\sigma_m(t; f), f) \to 0$$

as $m \to \infty$. This completes the proof of Theorem 4.1.

References


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