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H-SEMIDIRECT PRODUCTS

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ABSTRACT. The concept of H-semidirect product structure on a group-like space is introduced. It is shown that the loop space ΩX of any based CW-complex X is the H-semidirect product of the identity path-component of ΩX with $\pi_1 X$. The set of free homotopy classes of maps into a H-semidirect product inherits the structure of a semidirect product. This leads to new results concerning the nilpotency of homotopy classes of maps into a group-like space.

Introduction. Let G_0 be a path connected group-like space. The group $\mathscr{E}(G_0)$ of free homotopy classes of self homotopy equivalences of G_0 has the subgroup $H\mathscr{E}(G_0)$ consisting of those elements of $\mathscr{E}(G_0)$ (all of) whose representatives are at the same time H-maps. Given a (discrete) group Π and a homomorphism $\Phi: \Pi \to H\mathscr{E}(G_0)$, we shall say that Π acts on G_0 by classes of H-self homotopy equivalences.

If Π acts on G_0 by classes of H-self homotopy equivalences, we may mimic the construction of a semidirect product of discrete groups to obtain a new group-like space which we shall refer to as the H-semidirect product of G_0 with Π under Φ, denoted by $G_0 \rtimes \Pi$ (Definition 1.3).

Given any group-like space G, the set of path components Π of G has a canonical group structure, and Π acts on the identity component G_0 of G by classes of "inner" H-automorphisms. We may then form $G_0 \rtimes \Pi$ and obtain a canonical H-map $h: G_0 \rtimes \Pi \to G$, which turns out to be an H-equivalence if and only if all path components of G are open in G (Lemma 1.6). As a consequence the loop space ΩX of any based CW-complex X is the H-semidirect product of the identity component of ΩX with $\pi_1 X$ (Theorem 1.7).

The H-semidirect product structure on a group-like space can be utilized in various ways. If Y is a path-connected space, the group of free homotopy classes $[Y, G]$ inherits from G the structure of a semidirect product $[Y, G_0] \rtimes \Pi$, which is nilpotent if and only if Π is nilpotent and Π acts nilpotently on $[Y, G_0]$. This leads to conditions under which $[K, G]$ is nilpotent, where K is an iterated mapping cone. In particular, $[K, G]$ is nilpotent for every finite dimensional CW-complex K if and only if $[S^n, G]$ is nilpotent for all $n \geq 0$ (Corollary 2.11). Thus §2 establishes a simultaneous generalization of

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G. W. Whitehead's work on "Nilpotency of mappings into group-like spaces" [6], where G was required to be path-connected, and Theorem (3.1) of Roitberg [3]. Corollary (2.11) has also been obtained by Scheerer [4] by direct inspection of the groups $[S^n, G]$.

The present paper evolved out of unpublished results of K. Varadarajan [5] (obtained in 1976) stating that for a locally path-connected topological group G, the group $[Y, G]$ (Y path-connected) is a semidirect product. Furthermore, the earlier mentioned relation between the nilpotency of a semidirect product and the nilpotency of its factors is proven there. The algebraic result concerning the nilpotency of a semidirect product of groups was also obtained by P. J. Hilton [1].

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§1. Construction of H-semidirect Products

Let G_0 be a path-connected group-like space with H-multiplication μ, H-inverse i and (homotopy) identity e. We write gg' for $\mu(g, g')$ and g^{-1} for $i(g)$ if there is no danger of confusion. Let Π be a group and $\Phi : \Pi \to H\mathscr{E}(G_0)$ a group homomorphism. We define the H-semidirect product of G_0 with Π under Φ and then give conditions for a group-like space to be a H-semidirect product.

For each $p \in \Pi$, fix a H-self homotopy equivalence $\varphi_p \in \Phi(p) \in H\mathscr{E}(G_0)$. Define

$$m : (G_0 \times \Pi) \times (G_0 \times \Pi) \ni ((g, p), (g', p')) \mapsto (\mu(g, \varphi_p(g')), pp') \in G_0 \times \Pi$$

$$j : G_0 \times \Pi \ni (g, p) \mapsto (\varphi_{p^{-1}}(i(g)), p^{-1}) \in G_0 \times \Pi.$$

(1.1) Proposition. *$(G, m) := (G_0 \times \Pi, m)$ is a group-like space with H-inverse map j.*

Proof. *Step 1.* m restricted to $G \vee G$ is homotopic to the folding map.

For $\xi \in G \vee G$,

$$m(\xi) = \begin{cases} (e\varphi_1(g'), p') & \text{if} \quad \xi = ((e, 1), (g', p')) \\ (g\varphi_P(e), p) & \text{if} \quad \xi = ((g, p), (e, 1)) \end{cases}$$

Let F be a homotopy of φ_1 into Id_{G_0} and for each $p \in \Pi$, let α_p be a path in G_0 joining $\varphi_p(e)$ to e (we must take $\alpha_1 := F_{|\{e\} \times I}$). This yields a homotopy

$$A : (G \vee G) \times I \ni (\xi, t) \mapsto \left\{ \begin{matrix} (eF(g', t), p') & \text{if} \quad \xi = ((e, 1), (g', p')) \\ (g\alpha_p(t), p) & \text{if} \quad \xi = ((g, p), (e, 1)) \end{matrix} \right\} \in G$$

satisfying

$$A(\xi, 1) = \begin{cases} (\mu(e, g'), p') & \text{if} \quad \xi = ((e, 1), (g', p')) \\ (\mu(g, e), p) & \text{if} \quad \xi = ((g, p), (e, 1)) \end{cases}.$$

Thus a homotopy of $\mu_{|G_0 \vee G_0}$ into the folding map of G_0 induces a homotopy of $A(\,.\,, 1)$, and hence of $m_{|G \vee G}$, into the folding map of G.

Step 2. Homotopy associativity of m. We must show that the maps $m(m \times Id)$, $m(Id \times m)$: $G \times G \times G \to G$ are homotopic.

Computing

$$m(m((g_1,p_1),(g_2,p_2)),(g_3,p_3)) = ((g_1\varphi_{p_1}(g_2))\varphi_{p_1p_2}(g_3),p_1p_2p_3)$$

$$m((g_1,p_1),m((g_2,p_2),(g_3,p_3))) = (g_1\varphi_{p_1}(g_2\varphi_{p_2}(g_3)),p_1p_2p_3)$$

we see that such a homotopy can be obtained by going through the following succession of deformations.

Since Φ is a homomorphism, $\varphi_{p_1p_2} \approx \varphi_{p_1} \circ \varphi_{p_2}$. This yields a homotopy of $m(m \times Id)$ into

$$((g_1,p_1),(g_2,p_2),(g_3,p_3)) \to ((g_1\varphi_{p_1}(g_2))\varphi_{p_1} \circ \varphi_{p_2}(g_3),p_1p_2p_3).$$

Using homotopy associativity in G_0, we see that this map is homotopic to

$$((g_1,p_1)(g_2,p_2),(g_3,p_3)) \to (g_1(\varphi_{p_1}(g_2)\varphi_{p_1} \circ \varphi_{p_2}(g_3)),p_1p_2p_3).$$

Since φ_{p_1} is an H-map, this map is homotopic to $m(Id \times m)$.

Step 3. j is a homotopy inverse. We must show that the maps

$$G \xrightarrow{\Delta} G \times G \xrightarrow{Id \times j} G \times G \xrightarrow{m} G$$

$$G \xrightarrow{\Delta} G \times G \xrightarrow{j \times Id} G \times G \xrightarrow{m} G$$

are homotopic to the constant map $G \to \{(e,1)\}$ (Δ denotes the diagonal map).

Let $(g,p) \in G$; then

$$m \circ (Id \times j) \circ \Delta(g,p) = (g\varphi_p(\varphi_{p^{-1}}(g^{-1})), 1) = (g(\varphi_p \circ \varphi_{p^{-1}})(g^{-1}), 1).$$

Since $\varphi_p \circ \varphi_{p^{-1}} \approx \varphi_{pp^{-1}} \approx \varphi_1 \approx Id_{G_0}$, we get a homotopy of $m \circ (Id \times j) \circ \Delta$ into $(g,p) \mapsto (gg^{-1},1)$, which is homotopic to the constant map $G \to \{(e,1)\}$ using the property of the H-inverse i in G_0.

Similarly, we obtain a homotopy of $m \circ (j \times Id) \circ \Delta$ to the constant map. □

(1.2) LEMMA. *For $p \in \Pi$, let $\varphi_p, \varphi'_p \in \Phi(p)$ and denote by $(G,m) := (G_0 \times \Pi, m)$, $(G',m') := (G_0 \times \Pi, m')$ the corresponding group-like spaces. Then, the identity map $G \to G'$ is an H-equivalence.*

PROOF. Since φ_p and φ'_p are homotopic self equivalences of G_0, the H-multiplications m, m' of G, G' are homotopic. □

Proposition (1.1) and Lemma (1.2) suggest the following.

(1.3) DEFINITION. *Let G_0 be a path connected group-like space, Π a discrete group, $\Phi : \Pi \to H\mathscr{E}(G_0)$ a homomorphism. A group-like space X is a H-semidirect product of G_0 and Π under Φ if and only if X is H-equivalent to the space G constructed in (1.1). In this case, we write $X \approx G_0 \rtimes_\Phi \Pi$. The subscript "$\Phi$" may be deleted if the context is clear.*

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Turning to the question as to whether or not a given group-like space (G, M) is a H-semidirect product, let us denote by G_0 the identity path component of G and by Π the set of path connected components of G. It is known that Π is a group under the multiplication $\bar{x}\bar{y} = \overline{xy}$, where $\bar{x}$ denotes the path component of $x \in G$.

(1.4) PROPOSITION. *Π acts on G_0 by classes of H-equivalences.*

PROOF. For $\bar{x} \in \Pi$, let $\Phi(\bar{x}) \in H\mathscr{E}(G_0)$ be represented by φ_x

$$\varphi_x(g) := xgx^{-1} \quad g \in G_0$$

φ_x is a H-equivalence with H-inverse $\varphi_{x^{-1}}$ and, since $\varphi_x(e) \in G_0$, takes values in G_0. If $\bar{x} = \bar{x}'$, a path in G joining x to x' yields a homotopy of φ_x into $\varphi_{x'}$. Thus $\Phi(\bar{x})$ is well defined. The same technique shows that $\varphi_{xy} \approx \varphi_x \circ \varphi_y$. Hence, Φ is a homomorphism. □

Now fix an element x for each path-component $\bar{x} \in \Pi$. From the data in (1.4), we may then form the H-semidirect product $G_0 \rtimes_\Phi \Pi$ in accordance with (1.1). We obtain a canonical continuous map

$$h : G_0 \rtimes \Pi \ni (g, \bar{x}) \mapsto gx \in G$$

(1.5) LEMMA. *(i) h is a H-map.*
(ii) Taking different choices $x' \in \bar{x}' = \bar{x}$ in the various path components of G yields a H-map $h' : G_0 \rtimes \Pi \to G$ with $h \approx h'$.

PROOF. (i) The techniques of constructing homotopies that have been developed up to here show that the following diagram commutes up to homotopy.

$$\begin{array}{ccccc}
(G_0 \rtimes \Pi) \times (G_0 \rtimes \Pi) & \ni & ((g_1, \bar{x}_1), (g_2, \bar{x}_2)) \overset{m}{\longmapsto} & (g_1\varphi_{x_1}(g_2), \overline{x_1x_2}) & \in G_0 \rtimes \Pi \\
h \times h \Big\downarrow & & \Big\downarrow & \Big\downarrow h & \Big\downarrow \\
 & & & g_1\varphi_{x_1}(g_2) \cdot [x_1x_2] & \\
 & & & \int\int & \\
 & & & g_1(x_1g_2x_1^{-1})x_1x_2 & \\
 & & & \int\int & \\
G \times G & \ni & (g_1x_1, g_2x_2) \overset{M}{\longmapsto} & g_1x_1g_2x_2 & \in G
\end{array}$$

Here $[x_1x_2]$ denotes the fixed representative for the path component $\overline{x_1x_2} \in \Pi$. M and m denote the H-multiplications of G and $G_0 \rtimes \Pi$ respectively.
(ii) is clear. □

Representing the path component of the identity G_0 of Π by the identity e itself, we see that the restriction

$$h_{|G_0 \times \{\bar{e}\}} : G_0 \times \{\bar{e}\} \to G_0$$

is homotopic to Id_{G_0} (if we identify $G_0 \times \{\bar{e}\}$ with G_0). Therefore, for any $\bar{x} \in \Pi$,

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$$h_{|G_0 \times \{\bar{x}\}}: G_0 \times \{\bar{x}\} \to \bar{x}$$

is also a homotopy equivalence. Since h establishes a bijection between the path components of $G_0 \rtimes \Pi$ and those of G, we would like to assert that the path componentwise homotopy inverses of h combine to a homotopy inverse k of h and are confronted with the question whether the topology of G is fine enough to admit this. Elementary point set topology yields.

(1.6) LEMMA. *The path componentwise homotopy inverses of h combine to a homotopy inverse k of h if and only if the path components of G are open in G.* □

This result can be utilized as follows. If X is a based CW-complex, Milnor [2] shows that ΩX has the homotopy type of CW-complex. The connected components of a CW-complex are open. Therefore, the path components of ΩX are open. Thus

(1.7) THEOREM. *Let X be a based CW-complex,* $(\Omega X)_0$ *the identity component of* ΩX *and* $\Pi := \pi_1 X$. *As in (1.4), let* $\Phi: \Pi \to H\mathscr{E}(\Omega X)_0$ *denote the action of* Π *on* $(\Omega X)_0$ *by classes of H-self homotopy equivalences. Then there is a H-equivalence* $h: (\Omega X)_0 \rtimes_\Phi \Pi \to \Omega X$.

PROOF. Apply (1.4), (1.5), (1.6). □

Since the path components of a locally path connected space are open, we see that locally path connected group-like spaces are H-semidirect products. Of particular interest here is the case where G is a locally path connected topological group with identity component G_0. In this case, G_0 is a normal subgroup of G and easy checking shows that the quotient group $\Pi := G/G_0$ is the same as the group constructed on the path components of G in accordance with (1.4).

(1.8) THEOREM. *(i) The action of* Π *on* G_0 *as defined in (1.4) is by homotopy classes of inner automorphisms of G restricted to* G_0.
(ii) There is a H-homeomorphism $h: G_0 \rtimes \Pi \to G$.

PROOF. Apply the ideas contained in (1.4), (1.5), (1.6) to the present situation. □

§2. Nilpotency of mappings into group-like spaces.

Throughout this section G will denote a group-like space obtained by taking the H-semidirect product of a path connected group-like space G_0 with a discrete group Π under $\Phi: \Pi \to H\mathscr{E}(G_0)$.

In [6], [7] G. W. Whitehead showed that for a path-connected space X the group $[X, G_0]$ is nilpotent if X has finite category. Here, we give conditions for $[X, G]$ (free homotopy classes) to be nilpotent. We begin by stipulating some notation.

Notation. (i) $\Upsilon = \Upsilon^1: G \times G \ni (g,h) \mapsto ghg^{-1}h^{-1} \in G$ and for $n \geq 1$,

$$\Upsilon^{n+1}: G \times G^{n+1} \ni (g_0, \ldots, g_{n+1}) \mapsto \Upsilon(g_0, \Upsilon^n(g_1, \ldots, g_{n+1})) \in G$$

denote the commutator maps of G.

(ii) Let $\Upsilon_\Phi = \Upsilon^1_\Phi : (G_0 \times \Pi) \times G_0 \ni (g, p, h) \mapsto g\varphi_p(h)g^{-1}h^{-1} \in G_0$ and, for $n \geq 1$, $\Upsilon^{n+1}_\Phi : (G_0 \times \Pi) \times (G_0 \times \Pi)^n \times G_0 \to G_0$

$$(g_0, p_1, g_1, \ldots, p_{n+1}, g_{n+1}) \mapsto \Upsilon_\Phi(g_0, p_1, \Upsilon^n_\Phi(g_1, \ldots, p_{n+1}, g_{n+1}))$$

denote the Φ-commutator maps of G_0.

(2.1) DEFINITION. *(i) G is H-nilpotent of nilpotency index* nil $G \leq c$: $\langle = \rangle$ Υ^c *is homotopic to a constant map.*

(ii) G_0 is ΦH-nilpotent of nilpotency index $\text{nil}_\Phi\, G_0 \leq c$: $\langle = \rangle$ Υ^c_Φ *is homotopic to a constant map.*

Thus, $\text{nil}_\Phi\, G_0 \leq c$ implies nil $G_0 \leq c$.

Note that the above commutators, when formally applied to a group H (resp. Π acting on H by $\psi : \Pi \to \text{Aut}\, H$), agree with the usual commutators (ψ-commutators). Here for Υ^c (resp Υ^c_ψ) to be homotopically trivial means that all c-fold commutators (ψ-commutators) of H are equal to the identity element of H, and the image set of Υ^n (resp Υ^n_ψ) generates the usual n-th central series group $\Gamma^n H$ (resp $\Gamma^n_\psi H$). Thus the H-nilpotency indices of (2.1) agree with the usual group theoretic nilpotency indices.

(2.2) PROPOSITION. *If* nil $G \leq c$, *then* nil $[X, G] \leq c$.

To obtain more information on the structure of the group $[X, G]$, note first that Π acts on $[X, G_0]$ by composition, $\psi : \Pi \to \text{Aut}[X, G_0]$ being defined by

$$\psi_p(f) := \Phi(p) \circ f \quad \text{for} \quad p \in \Pi, \quad f \in [X, G_0].$$

A routine check yields

(2.3) THEOREM. *The function*

$$R : [X, G_0] \rtimes {}_\psi\Pi \ni (f, p) \mapsto (f, 1)(e, p) \in [X, G_0 \rtimes {}_\Phi\Pi]$$

is an isomorphism. □

It is known [1], [5], that $[X, G_0] \rtimes \Pi$ is nilpotent if and only if $[X, G_0]$ is ψ-nilpotent and Π is nilpotent. There is a H-analogue.

(2.4) THEOREM. $G = G_0 \rtimes {}_\Phi\Pi$ *is H-nilpotent if and only if G_0 is ΦH-nilpotent and Π is nilpotent.*

PROOF OF PROPOSITION (2.2). Let $f_0, \ldots, f_c \in [X, G]$ be represented by maps $\alpha_0, \ldots, \alpha_c : X \to G$. The commutator of $f_0, \ldots, f_c$ is represented by the composite

$$X \xrightarrow{\Delta} X^{c+1} \xrightarrow{\alpha_0 x \ldots x\alpha_c} G^{c+1} \xrightarrow{\Upsilon^c} G$$

which is homotopically trivial, because Υ^c is homotopically trivial. □

PROOF OF THEOREM (2.4). Suppose nil G = nil $G_0 \rtimes \Pi = c$. Then Υ^c is null homotopic so that the image of Υ^c is contained in the single path component of $G_0 \rtimes \Pi$ containing $\Upsilon^c((e, 1), \ldots, (e, 1))$. Hence Υ^c takes values in $G_0 \times \{1\}$. Inspection of the H-multiplication of G shows that for any $((g_0, p_0), \ldots, (g_c, p_c)) \in G^{c+1}$, the

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second coordinate of $\Upsilon^c((g_0,p_0),\ldots,(g_c,p_c))$ is equal to $\Upsilon^c(p_0,\ldots,p_c) = 1$. Thus nil $\Pi \leq c$.

To see that G_0 is ΦH-nilpotent, observe that there is a homotopy $G_0 \times \Pi \times G_0 \times I \to G_0$ deforming $\Upsilon^1((g,p),(h,1))$ into $(\Upsilon^1_\Phi(g,p,h),1)$. For

$$\begin{aligned}\Upsilon^1((g,p),(h,1)) &= (g,p)(h,1)(\varphi_{p^{-1}}(g^{-1}),p^{-1})(h^{-1},1)\\ &= (g\varphi_p(h),p)(\varphi_{p^{-1}}(g^{-1}),p^{-1})(h^{-1},1)\\ &= (g\varphi_p(h)(\varphi_p\varphi_{p^{-1}}(g^{-1})),1)(h^{-1},1).\end{aligned}$$

Since $\varphi_p\varphi_{p^{-1}}$ is homotopic to the identity map on G_0, there is a homotopy deforming the latter expression into

$$(g\varphi_p(h)g^{-1},1)(h^{-1},1) = (g\varphi_p(h)g^{-1}h^{-1},1) = (\Upsilon^1_\Phi(g,p,h),1).$$

For $n \geq 1$, induction gives a homotopy $G_0 \times (\Pi \times G_0)^n \times I \to G_0$ deforming $\Upsilon^n((g_0,p_0),\ldots,(g_{n-1},p_{n-1}),(g_n,1))$ into $(\Upsilon^n_\Phi(g_0,p_0,\ldots,g_{n-1},p_{n-1},g_n),1)$. Thus a null homotopy of Υ^c yields a null homotopy of Υ^c_Φ showing that $\mathrm{nil}_\Phi G_0 \leq c$.

Conversely, suppose nil $\Pi \leq a$ and $\mathrm{nil}_\Phi\, G_0 \leq b$. Since the second coordinate of $\Upsilon^n((g_0,p_0),\ldots,(g_n,p_n))$ is equal to $\Upsilon^n(p_0,\ldots,p_n)$, it follows that for $s \geq 0$, $\Upsilon^{a+s}(G_0 \rtimes \Pi)^{a+s+1} \subset G_0 \times \{1\}$. Hence, we may proceed as above and construct a homotopy (for $s \geq 1$), $(G_0 \rtimes \Pi)^s \times (G_0 \rtimes \Pi)^{a+1} \times I \to G_0$ deforming $\Upsilon^{a+s}((g_0,p_0),\ldots,(g_{s-1},p_{s-1}),(g_s,p_s),\ldots,(g_{s+a},p_{s+a}))$ into $\Upsilon^s_\Phi(g_0,p_0,\ldots,g_{s-1},p_{s-1},\alpha((g_s,p_s),\ldots,(g_{s+a},p_{s+a})))$, where $\alpha((g_s,p_s),\ldots,(g_{s+a},p_{s+a}))$ denotes the first coordinate of $\Upsilon^a((g_s,p_s),\ldots,(g_{s+a},p_{s+a}))$.

Consequently, Υ^{a+b} is homotopic to Υ^b_Φ following α, so that a null homotopy of Υ^b_Φ yields a null homotopy of the composite and hence, of Υ^{a+b}. Thus nil $G_0 \rtimes \Pi \leq a + b$. $\square$

We shall now enter a discussion of the nilpotency of the groups $[C,G_0 \rtimes {}_\Phi\Pi] \cong [C,G_0] \rtimes {}_\psi\Pi$, where C is a mapping cone. Since $[C,G_0] \rtimes \Pi$ can only be nilpotent if Π is, we shall from now on require Π to be nilpotent. Let us also agree that all spaces denoted by symbols A, Y are based path connected compactly generated Hausdorff spaces. In order to get a canonical isomorphism $[(Y,*),(G_0,e)] \to [Y,G_0]$, we require the inclusion map of a base point into its space to have the homotopy extension property and stipulate for the sequel that mappings and homotopies $A \to Y$ are to be based and mappings and homotopies into G are free.

So let $a:A \to Y$ be a map and consider the exact sequence

$$(2.5)\qquad [Y,G_0] \xleftarrow{\epsilon = i^*} [C,G_0] \xleftarrow{\nu = q^*} [SA,G_0]$$

arising from the Puppe sequence

$$A \xrightarrow{a} Y \xrightarrow{i} C := C_a \xrightarrow{q} SA.$$

We know that Π acts on $[Y,G_0],[C,G_0],[SA,G_0]$ and denote the corresponding homomorphisms into the respective automorphism groups by $\psi(Y),\psi(C),\psi(SA)$. It is clear that ϵ and ν are operator homomorphisms.

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The key is the following

(2.6) LEMMA. *$[C, G]$ is nilpotent if and only if there exists $s \in \mathbb{N}$ such that*

(i) $\Gamma^s_{\psi(C)}[C, G_0] \subset im\ \nu$

(ii) there exists $t \in \mathbb{N}$ such that $\Gamma^t_{\psi(SA)}(\nu^{-1}\Gamma^s_{\psi(C)}[C, G_0]) \subset \ker \nu$.

(2.7) COROLLARY. *Suppose $[Y, G]$ is nilpotent and there exists $t \in \mathbb{N}$ such that $\Gamma^t_{\psi(SA)}[SA, G_0] \subset \ker \nu$. Then $[C, G]$ is nilpotent.*

This yields immediately

(2.8) COROLLARY. *If $[Y, G]$ and $[SA, G]$ are nilpotent, then $[C, G]$ is nilpotent.* □

Our previous discussion can be extended to iterated mapping cones as follows. Let $\mathscr{A} = \{\mathscr{A}_\lambda\}_{\lambda \in \Lambda}$ be a family of spaces. Inspired by G. W. Whitehead [7], let us call a sequence $Y = S_0 \subset \ldots \subset S_k = K$ an $\mathscr{A}$-stratification of K if, for $n \geq 1$, S_i is obtained from S_{i-1} as the mapping cone of a wedge of spaces in $\mathscr{A}$.

(2.9) THEOREM. *Let $[Y, G]$ be nilpotent and let $Y = S_0 \subset \ldots \subset S_k = K$ be an $\mathscr{A}$-stratification for K. Suppose there exists a positive integer N such that for all λ, $\mathrm{nil}[SA_\lambda, G] \leq N$. Then $[K, G]$ is nilpotent.*

Letting $\mathscr{A}$ be the family of n-spheres S^n ($n \geq 0$) and Y the 1-point space, we obtain

(2.10) COROLLARY. *Let $Y = S_0 \subset \ldots \subset S_k = K$ be a stratification of a connected CW-complex K by connected subcomplexes S_i. Suppose there exists a positive integer N such that for every n-sphere ($n \geq 0$) $\mathrm{nil}[S^n, G] \leq N$. Then $[K, G]$ is nilpotent.*

Using the stratification of finite dimensional connected CW-complexes K by their skeleta, we obtain

(2.11) COROLLARY. *$[K, G]$ is nilpotent if and only if for all $n \geq 0$, $[S^n, G]$ is nilpotent.*

PROOF OF LEMMA (2.6). If $\mathrm{nil}[C, G] \leq a$, the choices $s := a$ and $t := 0$ clearly satisfy (i), (ii).

Conversely, suppose there exist $s, t \in \mathbb{N}$ such that (i), (ii) are satisfied. It is known (and can be shown by using the idea of [7] Theorem X 3.10) that the sequence (2.5) is a central extension. By (i), $\Gamma^s_{\psi(C)}[C, G_0]$ is contained in the center of $[C, G_0]$. Hence, for $a \in [C, G_0]$, $b \in \Gamma^s_{\psi(C)}[C, G_0]$, $p \in \Pi$

$$\Upsilon_{\psi(C)}(a, p, b) = a\psi(C)_p(b)a^{-1}b^{-1} \overset{(*)}{=} \psi(C)_p(b)b^{-1} = \Upsilon_{\psi(C)}(e, p, b).$$

Thus, for $k \geq 0$

$$\Gamma^{k+s}_{\psi(C)}[C, G_0] = \Gamma^k_{\psi(C)}\Gamma^s_{\psi(C)}[C, G_0]$$

so that

$$\Gamma^{t+s}_{\psi(C)}[C, G_0] = \Gamma^t_{\psi(C)}(\Gamma^s_{\psi(C)}[C, G_0]) \overset{(**)}{\subset} \nu\Gamma^t_{\psi(SA)}(\nu^{-1}\Gamma^s_{\psi(C)}[C, G_0]) \subset \nu(\ker \nu) = \{1\}.$$

Equation (*) is true because $\Gamma^s_{\psi(C)}[C, G_0]$ is invariant under the action of Π. Inclusion (**) is true because ν is an operator homomorphism.

Consequently, $[C, G]$ is nilpotent because Π is nilpotent and $[C, G_0]$ is $\psi(C)$-nilpotent. □

PROOF OF COROLLARY (2.7). Suppose $\mathrm{nil}_{\psi(Y)}[Y, G_0] \leq s$. Then (because ϵ is an operator homomorphism)

$$\epsilon\Gamma^s_{\psi(C)}[C, G_0] \subset \Gamma^s_{\psi(Y)}[Y, G_0] = \{1\}.$$

Thus $\Gamma^s_{\psi(C)}[C, G_0] \subset \ker \epsilon = \mathrm{im}\,\nu$ and conditions (i), (ii) of Lemma (2.6) are satisfied. □

PROOF OF THEOREM (2.9). Using Corollary (2.8), we show by induction on the stages of the stratification of K that $[K, G_0]$ is $\psi(K)$-nilpotent.

$[S_0, G_0]$ is $\psi(S_0)$-nilpotent by hypothesis. So suppose $0 \leq i < k$ and $[S_i, G_0]$ is $\psi(S_i)$-nilpotent. Let Λ_{i+1} be an indexing set for the $(i + 1)$-st wedge corresponding to the $\mathcal{A}$-stratification of K. By distributivity of wedge and suspension, we get

$$P := \left[S\Big(\bigvee_{\lambda \in \Lambda_{i+1}} A_\lambda\Big), G_0\right] = \left[\bigvee_{\lambda \in \Lambda_{i+1}} (SA_\lambda), G_0\right] \cong \prod_{\lambda \in \Lambda_{i+1}} [SA_\lambda, G_0] =: Q.$$

The isomorphism above is an operator isomorphism. Hence

$$\mathrm{nil}_\psi P = \mathrm{nil}_\psi Q \leq N.$$

By (2.8), $\mathrm{nil}_{\psi(S_{i+1})}[S_{i+1}, G_0] \leq \mathrm{nil}_{\psi(Y)}[Y, G_0] + (i + 1)N$, which completes the induction. Summing up, we obtain

$$\mathrm{nil}_{\psi(K)}[K, G_0] \leq \mathrm{nil}_{\psi(Y)}[Y, G_0] + kN. \quad \square$$

PROOF OF COROLLARY (2.11). For connected K, the statement follows from (2.10). If K is not connected, it is the topological sum of its connected components K_i. Therefore

$$[K, G] \cong \Pi[K_i, G] \cong \Pi[K_i, G_0] \rtimes \Pi)$$

and the bounded nilpotency of the factors $[K_i, G_0] \rtimes \Pi$ ensures the nilpotency of the product. □

Analogously to [7] p. 465, in the situation of Theorem (2.9), we may derive the following explicit $\psi(K)$-central series for $[K, G_0]$.

(2.12) PROPOSITION. *Let F_i be the set of homotopy classes of maps $K \to G_0$ whose restriction to S_i is nullhomotopic. Let* $\mathrm{nil}_{\psi(S_0)}[S_0, G_0] \leq c$. *Let $e_i : S_i \to K$ denote the inclusion map and let $W_i = \bigvee_{\lambda \in \Lambda_i} SA_\lambda$. Then*

$$[K, G_0] \supset (e_0^*)^{-1}\Gamma^1_{\psi(S_0)}[S_0, G_0] \supset \ldots \supset (e_0^*)^{-1}\Gamma^c_{\psi(S_0)}[S_0, G_0] = F_0 \supset$$

$$\supset (e_1^*)^{-1}(\nu_1\Gamma^1_{\psi(W_1)}[W_1, G_0]) \supset \ldots \supset (e_1^*)^{-1}(\nu_1\Gamma^N_{\psi(W_1)}[W_1, G_0]) = F_1 \supset$$

$$\vdots$$

$$\supset (e_k^*)^{-1}(\nu_k\Gamma^1_{\psi(W_k)}[W_k, G_0]) \supset \ldots \supset (e_k^*)^{-1}(\nu_k\Gamma^N_{\psi(W_k)}[W_k, G_0]) = F_k = \{1\}$$

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is a $\psi(K)$*-central series for* $[K, G_0]$.

PROOF. This follows from the proof of Lemma (2.6) and the fact that the e_i's and ν_i's are operator homomorphisms. □

Finally, we remark that the investigation of the nilpotency of $[X, G]$ would benefit from knowledge about the sets $\mathrm{Hom}(\Pi, H\mathscr{E}(G_0))$ and, hence, from knowledge about the groups $H\mathscr{E}(G_0) \subset \mathscr{E}(G_0)$. Some information in this direction can be found in [8], [9], [10]. The author is grateful to the referee for bringing these to his attention.

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