$$\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} = 0.$$

This relation is, of course, normally derived by solving the pair of simultaneous equations requiring for example $y = ax + b$ to go through (x_2, y_2) and (x_3, y_3).

Yours sincerely,

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Editor's Note

Robert Pargeter mentioned in a letter that he remembers teaching the formula to sixth-formers a few years ago. Overseas candidates for 'Additional Maths' continue to use essentially the same method in papers I have recently marked.

DEAR EDITOR,

While not being a school teacher, I get the impression that there has been a steady decline in the teaching of geometry in schools over the past decade, despite moves towards graphic communication in, for example, user interfaces for computers. With the lack of geometric education, I feel there could be a new literacy problem arising. It is therefore laudable that there have been a number of articles in recent editions of the Mathematical Gazette related to art and geometry. While the article "The Portrait of Fra Luca Pacioli" by Nick MacKinnon was an interesting piece of research, it contains errors which illustrate the misunderstanding that can arise when mathematicians do not understand the geometry of perspective.

Pacioli's ellipse

On page 139 of the Pacioli article there are two sentences (just above the diagram) which are in conflict with elementary geometry: "In fact a circle drawn in linear perspective does not give an ellipse. The difference here is immaterial."

Linear perspective is concerned with projection and section. An artist takes lines from the eye which are the reverse of the path of light rays followed from an object. These rays intersect the picture plane to get the image. Normally, the picture plane is a plane perpendicular to the main line of sight, but need not necessarily be so. If you are creating the image of a circle, the circle together with the eye-point form a cone. The picture plane intersects this cone in a conic as has been known from Greek times – indeed the modern term conic is a shortening of "conic section". In normal cases the conic is an ellipse, though it could equally well be a hyperbola or parabola, or even a circle if the circle lies in a plane which is parallel to the picture plane.

The article then goes on to the method used to produce ellipses for the diagrams. "I stretched the diagram until the circle was an ellipse ..." However, in linear perspective this does not happen. If a circle is stretched, the centre of the circle is transformed into the centre of the ellipse. In perspective, because of foreshortening, this does not occur. The centre is behind the ellipse centre, because the major axis of the ellipse is formed by the chord joining the tangents to the circle from the eye.

I think Nick MacKinnon meant to say that the perspective view used by Barbari was such that it did not matter that such an approximation was made.

Durer's solid

Page 203 of the article says "when the solid is described it is always said to be a truncated rhombihedron". There is more evidence than this since Durer drew the solid independently of the engraving. It occurs in the so called Dresden Notebook. Moreover, the drawing shows the complete solid, without "hidden line removal" showing that this is indeed so. By comparing sizes, it is a good supposition that he actually transferred the diagram to the plate. This has relevance to your eyepoint which I will return to in a moment.

In the paragraph on page 205 there is confusion about "the vanishing point" which is alluded to as if there were only one. A vanishing point is where any pair of parallel lines in space meet in the picture plane. **Vanishing points do not even have to be on the horizon line.** What is referred to is the *central* vanishing point. This is the point where all the lines which are perpendicular to the picture plane (*and* parallel to the perpendicular from the eye) meet on the horizon line. The significance of this point is that the eye-point is somewhere on the line which is the perpendicular in space to the picture through this point (assuming the picture to have been painted with a perpendicular picture plane). Moreover, unless the eye is placed exactly at this point (in space) the picture cannot be placed in its correct mathematical correspondence. Just looking at it, as we normally do, means that we often see distortions, and artists, once they got the hang of perspective, made compensations for these problems.

It is normally easy to find the correct distance for the eye-point, that is how far along the line from the central vanishing point you need to go to view the picture from the artist's viewpoint. That Durer knew this to be true is evident from his many engravings of perspective aparati which gave methods to fix the viewpoint. In this case, however, as was pointed out, there are few clues to perspective construction lines. It is possible to find the eye-point empirically by moving the eye back and forth along the line through the central vanishing point you have constructed. If you put your eye roughly a distance equal to the width of the engraving then I believe the scales look right. What Nick MacKinnon sees as a "joke" is an artifact of the way it is being viewed from a different place than the artist's viewpoint.

This leads me to disprove the hypothesis about the truncated cube. To do so, let me take the simpler case of a quadrilateral in perspective. There is no way, if you have a perspective view showing a quadrilateral, that you know whether it was originally a square, a rectangle or any other quadrilateral, even a skew one in space, because all give the same image when transformed from a three dimensional position in space to a two dimensional one. This is easily proved by considering a pyramid coming from the eye which goes to a square. If you were to make a number of slices of the pyramid, then you would end up with many differently shaped quadrilaterals. This is analogous to the sections of a circular cone. If the pyramid were glass and you drew each one of these section lines on its surface and then looked from the apex, you would still only see one "image", the square, since all the other ones you have drawn are in line with one another. In order to make a perspective reconstruction, you need to make an assumption about an object, usually that there is a square tiled floor.

Now taking this a step further to solids which spatially have the same properties as a cube, that is they have 8 vertices and 6 faces made of quadrilaterals. By looking at intersections of planes and lines and using the fact that the lines meet in three points (or appear to, as in the case of a cube in three point perspective) they must conform to Reye's configuration (see figure 148 in Hilbert and Cohn Vossen's *Geometry and the Imagination*). This is so for the rhombihedron we are considering which means that all such solids when looked at from the correct point will appear to be a cube. This is why Nick MacKinnon was able to see a cube because he moved his eye until he did so. In addition to this, if you see the original engraving where the solid rests on a square plinth, you will see the central vanishing point for the perspective marked with an eye which agrees with the central vanishing point for the Melencolia.

Yours sincerely,

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DEAR EDITOR,

I have just picked up a copy of the *Gazette* Volume 78, March 1994 and I see the short note (78.7) by R. E. Scraton (pp. 60-63). The problem he is considering is an old one and is now called Shapiro's problem. Shapiro made the conjecture that $S \geqslant n/2$. This is of course false in general as is shown. A history of this problem and the current knowledge is given in the book *Classical and new inequalities in analysis* by Mitrinovic, Pecaric and Fink (Kluwer Academic Publishers, 1993). Your readers might like to know this.

Yours sincerely,

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