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p-GROUPS WITH CYCLIC OR GENERALISED QUATERNION HUGHES SUBGROUPS: CLASSIFYING TIDY p-GROUPS

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Abstract

Let G be a p-group for some prime p. Recall that the Hughes subgroup of G is the subgroup generated by all of the elements of G with order *not* equal to p. In this paper, we prove that if the Hughes subgroup of G is cyclic, then G has exponent p or is cyclic or is dihedral. We also prove that if the Hughes subgroup of G is generalised quaternion, then G must be generalised quaternion. With these results in hand, we classify the tidy p-groups.

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1. Introduction

In this paper, all groups are finite. Given a group G and a prime p, Hughes considered the subgroup $\mathrm{H}_p(G)$ generated by all elements of G whose order is not p. In [8], Hughes asked if it is always the case that when $\mathrm{H}_p(G)$ is proper and nontrivial, then it has index p in G. Hughes proved that this is true for 2-groups in [7]. Strauss and Szekeres proved it is true for 3-groups in [14], and Hughes and Thompson proved it is true when G is not a p-group in [9]. However, the conjecture is not true in general. Wall published a counterexample for $p = 5$ [15]. See the discussion in [6] for more background regarding the Hughes subgroup problem.

In this paper, our goal is quite modest. We wish to consider p-groups that have Hughes subgroups that are cyclic or generalised quaternion. We begin by considering p-groups with a cyclic Hughes subgroup.

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THEOREM 1.1. *Let G be a p-group. Then $\mathrm{H}_p(G)$ is cyclic if and only if one of the following occurs:*

(1) *G has exponent p and $\mathrm{H}_p(G) = 1$;*
(2) *G is cyclic and $\mathrm{H}_p(G) = G$;*
(3) *$p = 2$, G is a dihedral group and $\mathrm{H}_2(G)$ has index 2 in G.*

Next, we consider a 2-group with a Hughes subgroup that is generalised quaternion. In this case, we prove that G must equal its Hughes subgroup.

THEOREM 1.2. *Let G be a 2-group. Then $\mathrm{H}_2(G)$ is generalised quaternion if and only if $G = \mathrm{H}_2(G)$.*

Our interest in groups of prime power order with a Hughes subgroup that is cyclic or generalised quaternion arises in the context of tidy groups. For each element x in a group G, let $\mathrm{Cyc}_G(x) = \{g \in G \mid \langle x, g\rangle \text{ is cyclic}\}$. It is not difficult to find examples of a group G and an element x where $\mathrm{Cyc}_G(x)$ is *not* a subgroup. In the literature, a group G is said to be *tidy* if $\mathrm{Cyc}_G(x)$ is a subgroup of G for every element $x \in G$. As far as we can determine, tidy groups were introduced in [13] and in a second paper [12]. We note that in [12], the authors define an object they call cycels, so the word 'cycels' in the title of that paper is not a typo. Tidy groups have been studied in [2–5].

In [12, Theorem 14], O'Bryant *et al.* prove that if G is a p-group, then G is tidy if and only if there is a normal subgroup H that is cyclic or generalised quaternion such that every element in $G \setminus H$ has order p. It is not difficult to see that H must be the Hughes subgroup of G. Hence, the task of classifying the tidy p-groups becomes that of determining the p-groups whose Hughes subgroup is either cyclic or generalised quaternion. With that in mind, we obtain the following classification of tidy p-groups.

THEOREM 1.3. *Let G be a p-group for some prime p. Then the following are equivalent.*

(1) *G is a tidy group.*
(2) *The subgroup $\mathrm{H}_p(G)$ is cyclic or generalised quaternion.*
(3) *One of the following occurs:*
   (a) *G has exponent p;*
   (b) *G is cyclic;*
   (c) *$p = 2$ and G is dihedral or generalised quaternion.*

2. Results

To prove our results, we make use of the following classification of p-groups that have a cyclic maximal subgroup (see, for example, [10, Satz I.14.9]).

THEOREM 2.1. *Let G be a nonabelian p-group for some prime p and assume that $H = \langle h\rangle$ is a cyclic maximal subgroup of G with $|\langle h\rangle| = p^e$. If H has a complement $\langle g\rangle$ in G, then one of the following situations occurs:*

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(1) *$p \neq 2$ and $h^g = h^{1+p^{e-1}}$ (for suitably chosen g);*
(2) *$p = 2$ and $h^g = h^{-1}$;*
(3) *$p = 2$, $e \geq 3$ and $h^g = h^{-1+2^{e-1}}$;*
(4) *$p = 2$, $e \geq 3$ and $h^g = h^{1+2^{e-1}}$.*

Theorem 2.1 depends on the structure of $Aut(H)$, which we mention explicitly. If H is a cyclic p-group of order p^e, where p is an odd prime, then $Aut(H)$ is cyclic of order $p^{e-1}(p-1)$. If H is a cyclic 2-group of order 2^e, $e \geq 1$, then $Aut(H)$ is cyclic of order 2^{e-1} for $e \in \{1, 2\}$ and is isomorphic to $C_2 \times C_{2^{e-2}}$ for $e \geq 3$.

Let G be a group and let p be a prime. We define the *Hughes subgroup* of G to be the subgroup generated by all of the elements of G whose order does not equal p. The Hughes subgroup of G with respect to the prime p is denoted by $\mathrm{H}_p(G)$. Hence,

$$\mathrm{H}_p(G) = \langle\, g \in G \mid o(g) \neq p \,\rangle.$$

When a p-group G is cyclic, then it will equal its Hughes subgroup. However, a p-group G has exponent p and order at least p^2 if and only if its Hughes subgroup is trivial. The following preliminary lemma about the Hughes subgroup is useful.

LEMMA 2.2. *If G is a p-group for a prime p and $\mathrm{H}_p(G) \neq 1$, then $\mathbf{C}_G(\mathrm{H}_p(G)) \leq \mathrm{H}_p(G)$.*

PROOF. Suppose that $\mathbf{C}_G(\mathrm{H}_p(G)) \nleq \mathrm{H}_p(G)$ and fix $x \in \mathbf{C}_G(\mathrm{H}_p(G)) \setminus \mathrm{H}_p(G)$. Note that $o(x) = p$. The subgroup $\mathrm{H}_p(G)$ has an element of order p^2, say h. However, now, we deduce that the element hx has order p^2 and does not belong to $\mathrm{H}_p(G)$, which is a contradiction. □

We now prove the case when the Hughes subgroup is cyclic.

PROOF OF THEOREM 1.1. Let $H = \mathrm{H}_p(G)$ and assume that H is cyclic. If $H = 1$, then G has exponent p and G satisfies item (1). If $H = G$, then G satisfies item (2). We therefore proceed with the hypothesis that $1 < H < G$. Note that $|H| \geq p^2$ since H is nontrivial.

Since H is cyclic, $H \leq \mathbf{C}_G(H)$. Using Lemma 2.2, we conclude that $H = \mathbf{C}_G(H)$. By the normaliser/centraliser theorem [11, Corollary X.19], G/H is isomorphic to a subgroup of $Aut(H)$.

If p is odd, then $Aut(H)$ is cyclic. Hence, the section G/H is also cyclic. Since every nonidentity element of G/H has order p, we conclude that $|G : H| = p$. In particular, H is a cyclic maximal subgroup of G.

Now, let $H = \langle h \rangle$ and write $|\langle h \rangle| = p^e$. Fix $g \in G \setminus H$. Note that $|\langle g \rangle| = p$ and that $\langle g \rangle$ serves as a complement to H in G. By Theorem 2.1, $h^g = h^{1+p^{e-1}}$ (where g may have to be re-chosen). Observe that

$$(h^p)^g = (h^g)^p = (h^{1+p^{e-1}})^p = h^{p+p^e} = h^p.$$

Hence, $\langle h^p \rangle \leq Z(G)$ and it follows that $|H : Z(G)| = p$. Now, if $|Z(G)| > p$, then there would exist elements of order p^2 outside of H, which is a contradiction. We conclude that $|Z(G)| = p$, $|G| = p^3$ and the exponent of G is p^2. Hence, G is extraspecial.

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Now G is extra-special of order p^3 and has exponent p^2. This implies that G has nilpotence class 2. We claim that G is generated by elements of order p^2. We know that G has an element a whose order is p^2. It suffices to show that $G \setminus \langle a \rangle$ contains an element of order p^2. Consider $b \in G \setminus \langle a \rangle$, and assume b has order p. Then using induction, it is not difficult to compute that $(ab)^n = a^n b^n [b, a]^{(n-1)n/2}$ for every positive integer n. So, if p is odd, then $(ab)^p = a^p b^p [b, a]^{(p-1)p/2} = a^p \neq 1$. Hence, ab has order p^2. Thus, we conclude that $G = H$, which is a contradiction. In particular, if H is a nontrivial, proper cyclic subgroup of G, then $p = 2$.

So, assume that $p = 2$, while still operating under the assumption that $H = \mathrm{H}_2(G)$ is cyclic. Again, fix $g \in G \setminus H$. Lemma 2.2 guarantees that $h^g \neq h$. Consider the subgroup $X = \langle h, g \rangle$. We claim that g inverts h: that is, $h^g = h^{-1}$. By Theorem 2.1 applied to X, the possibilities for h^g are $h^g = h^{-1}$, $h^g = h^{-1+2^{e-1}}$ or $h^g = h^{1+2^{e-1}}$. If $h^g = h^{-1+2^{e-1}}$, then there exist elements of order 4 in $X \setminus \langle h \rangle \subseteq G \setminus H$ (see [11, Problem 3A.1]), which is a contradiction.

Assume that $h^g = h^{1+2^{e-1}}$. Under this hypothesis, $Z(X)$ is cyclic of order 2^{e-2} (see [1, Exercise 8.2(1)]). If $|Z(X)| > 2$, then, as before, there exist elements of order 4 in $X \setminus \langle h \rangle \subseteq G \setminus H$, which is a contradiction. So $|Z(X)| = 2$ and $e = 3$. Hence, $h^g = h^5$. Note that $hg \in X \setminus H$ and so $(hg)^2 = 1$. Now, $1 = (hg)(hg) = h(g^{-1}hg) = hh^5 = h^6$, which is a contradiction to the fact that $o(h) = 8$.

The remaining possibility is, of course, that g inverts h. Indeed, g always inverts h, and so X is dihedral. As noted below Theorem 2.1, $Aut(H) \cong C_2 \times C_{2^{e-2}}$. As G/H embeds in $Aut(H)$, we conclude that $G/H \cong C_2$ or $G/H \cong C_2 \times C_2$. Suppose that $G/H \cong C_2 \times C_2$. Then we can choose $x, y \in G \setminus H$ such that $Hx \neq Hy$. Both elements x and y are involutions and invert h. So $h^{xy^{-1}} = (h^{-1})^{y^{-1}} = h$. However, now $xy^{-1} \in \mathbf{C}_G(H) = H$ and so $Hx = Hy$, which is a contradiction. This argument rules out the possibility that $G \cong C_2 \times C_2$. Hence, $|G : H| = 2$ and $G = X$ is dihedral, giving item (3).

Finally, if item (1), (2) or (3) occurs, then it is not difficult to see in each case that H is cyclic. □

Finally, we consider the case when the Hughes subgroup is generalised quaternion.

PROOF OF THEOREM 1.2. Assume that $H = \mathrm{H}_2(G)$ is generalised quaternion. In this case, H is generated by elements x, y such that $o(x) = 2^a$, $o(y) = 4$, $x^y = x^{-1}$, $x^{2^{a-1}} = y^2$. Recall that $x^{2^{a-1}} = y^2$ is the unique involution of H. If $G = H$, then we are done. So, assume that $H < G$ and fix $s \in G \setminus H$. Conjugation by s induces an automorphism of H. If s induces an inner automorphism of H, then, for all $h \in H$, $h^s = h^t$ for some $t \in H$. However, then $st^{-1} \in \mathbf{C}_G(H) \leq H$ (using Lemma 2.2) and so $s \in H$, which is a contradiction. Hence, s induces an outer automorphism of H.

At this point, we recall a result mentioned previously. Reference [12, Theorem 14] says that if G is a p-group, then G is tidy if and only if there is a normal subgroup K that is cyclic or generalised quaternion such that every element in $G \setminus K$ has order p. So, setting $K = H$ in our present situation, we conclude that G is tidy. If $a = 2$, then the semi-direct product resulting from the action of $\langle s \rangle$ on H is necessarily semi-dihedral, which contradicts the fact that G is tidy.

Assume $a \geq 3$. Note that $\langle x \rangle$ is characteristic in H and so s induces an automorphism of $\langle x \rangle$. An analysis of the possibilities, similar to the argument in the proof of Theorem 1.1, shows that s acts as the inversion map on $\langle x \rangle$.

Write $y^s = yx^d$ for $0 \leq d \leq 2^a - 1$. Suppose that d is even and write $d = 2b$ for $b \in \mathbb{Z}$. However, now,

$$(yx^b)^s = yx^{2b}x^{-b} = yx^b.$$

Thus, s and yx^b commute. Since H is generalised quaternion and yx^b does not lie in $\langle x \rangle$, we see that $o(yx^b) = 4$. Now, $o(syx^b) = 4$ and $syx^b \in G \setminus H$, which is a contradiction.

We now suppose that d is odd. Observe that

$$(sy)^2 = sysy = y^s y = yx^d y = y^2 y^{-1} x^d y = y^2 x^{-d} = x^{2^{a-1}-d}.$$

Since d is odd, $o((sy)^2) = 2^a$. Hence, $o(sy) = 2^{a+1}$. Next, note that $\langle x \rangle \leq \langle sy \rangle$ and that $|H\langle s \rangle : \langle sy \rangle| = 2$. Now, as $H\langle s \rangle$ contains subgroups of index 2 that are generalised quaternion and cyclic, it can be deduced that $H\langle s \rangle$ is semi-dihedral (which is not tidy), in contrast to the fact that it is a subgroup of a tidy group. So, if H is generalised quaternion, then $G = H$.

If G is generalised quaternion, it is not difficult to see that it is its own Hughes subgroup. □

Combining all of the results, we see that Theorem 1.3 follows.

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