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ON INTEGERS OCCURRING AS THE MAPPING DEGREE BETWEEN QUASITORIC 4-MANIFOLDS

ĐORĐE B. BARALIĆ

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Abstract

We study the set $D(M, N)$ of all possible mapping degrees from M to N when M and N are quasitoric 4-manifolds. In some of the cases, we completely describe this set. Our results rely on Theorems proved by Duan and Wang and the sets of integers obtained are interesting from the number theoretical point of view, for example those representable as the sum of two squares $D(\mathbb{C}P^2\sharp\mathbb{C}P^2, \mathbb{C}P^2)$ or the sum of three squares $D(\mathbb{C}P^2\sharp\mathbb{C}P^2\sharp\mathbb{C}P^2, \mathbb{C}P^2)$. In addition to the general results about the mapping degrees between quasitoric 4-manifolds, we establish connections between Duan and Wang's approach, quadratic forms, number theory and lattices.

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1. Introduction

The *mapping degree* is one of the oldest topological invariants and almost every textbook has a section devoted to the definition and the calculation of this invariant. Given two oriented n-manifolds M and N, every map $f : M \to N$ induces a homomorphism

$$f_* : H_*(M) \to H_*(N).$$

The degree of f is defined as an integer k such that

$$f_*([M]) = k\,[N],$$

where $[M] \in H_n(M)$ and $[N] \in H_n(N)$ are the fundamental classes of M and N, respectively. It is a natural question to find all integers that occur as the mapping degree of some $f : M \to N$.

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DEFINITION 1.1. Given two closed oriented n-manifolds M and N, $D(M, N)$ is the set of integers that could be realized as the degree of a map from M to N

$$D(M, N) = \{\deg f \mid f : M \to N\}.$$

The problem of determining $D(M, N)$ is extensively examined in topology. In dimension two, the problem is completely solved (see [19] and [13]). In dimension three, the problem has been studied by several authors and it is solved for numerous classes of 3-manifolds (see [3] and [17]). Results about 3-manifolds usually assume some additional geometrical or topological structure on the manifolds. The most important results about 3-manifolds can be found in the survey article of Wang [27]. In dimensions higher than three, there are only a few relevant results obtained for some special classes of manifolds (see [2, 10, 26] and [14]). It is clear, even from the results in dimension two and three, that $D(M, N)$ significantly depends on the homotopy types of both M and N. Significant progress in the problem has been achieved by Haibao Duan and Shicheng Wang in [11], who gave algebraic conditions for the existence of certain map degree between two given closed $(n-1)$-connected $2n$-manifolds. Their algebraic conditions are obtained from the topology of this wide class of manifolds. However, even in the simplest case of dimension four, it is not easy to check these conditions.

The goal of this article is to improve the known results in dimension four. We identify the sets $D(M, N)$ when N is $\mathbb{C}P^2$, $\mathbb{C}P^2\sharp\mathbb{C}P^2$, $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ and M is a quasitoric 4-manifold. By Orlik and Raymond [24], four-dimensional quasitoric manifolds are connected sums of some copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$. The sets $D(M, N)$ are also obtained in some other more general cases. Among other results, the following theorems are proved.

THEOREM 1.2. *There is a degree k map $f : (\mathbb{C}P^2)^{\sharp 4n} \to (\mathbb{C}P^2)^{\sharp 4n}$ if and only if k is a nonnegative integer.*

THEOREM 1.3. *Let l, m and n be positive integers such that $l \geq m + n$. Then there is a degree k map*

$$f : (S^2 \times S^2)^{\sharp l} \to (\mathbb{C}P^2)^{\sharp n}\sharp(\overline{\mathbb{C}P^2})^{\sharp n}\sharp(S^2 \times S^2)^{\sharp m}$$

if and only if k is an even number.

THEOREM 1.4. *Let m and n be positive integers such that $m \geq 3n$. Then there is no nonzero degree map*

$$f : (\mathbb{C}P^2)^{\sharp m} \to (\mathbb{C}P^2)^{\sharp n}\sharp(S^2 \times S^2)^{\sharp n}.$$

Our efforts lead us to a compelling result that, for every quasitoric 4-manifold N, each integer can appear as the mapping degree of a map $f : M \to N$ for all 'sufficiently complicated' quasitoric 4-manifolds M. The following theorem states this more precisely.

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THEOREM 1.5. *Let M be a given quasitoric 4-manifold. Then there exist integers a_0, b_0 and c_0 such that, for any integers a, b and c, $a \geq a_0$, $b \geq b_0$ and $c \geq c_0$,*

$$D((\mathbb{C}P^2)^{\sharp a}\sharp(\overline{\mathbb{C}P^2})^{\sharp b}\sharp(S^2 \times S^2)^{\sharp c}, M) = \mathbb{Z}.$$

However, although the intersection forms of the manifolds that we consider are relatively simple, the description of $D(M, N)$ for arbitrary quasitoric 4-manifolds remains an open problem. The main obstacle in solving the problem using our methods is that the algebraic conditions that we are checking are too complicated even for this class of manifolds.

In Section 2, we give a review of the known facts about the mapping degrees and, in particular, of the work of Duan and Wang. Theorem 2 from [12] is reproved and slightly improved. The latter result is crucial for the results of the next sections.

Section 3 focuses on the complete calculation of $D(M, N)$ when N is $\mathbb{C}P^2$, $\mathbb{C}P^2\sharp\mathbb{C}P^2$, $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ and M is a quasitoric 4-manifold. In Section 4, we study the mapping degrees between connected sums of $\mathbb{C}P^2$ and their relationship to the problems about the lattice discriminants. The proof of Theorem 1.2 is given in this section.

Theorems 1.3, 1.4 and 1.5 are proved in Section 5.

2. The mapping degree

From the standard topology course ([16] and [4]) we know several effective methods for calculating the mapping degree. Proposition 2.30 and Exercises 8, page 258 in [16] can be easily generalized and summarized in the following theorem.

THEOREM 2.1. *For a map $f : M \to N$ between connected closed orientable n-manifolds and a point $y \in N$ such that $f^{-1}(y) = \{x_1, \ldots, x_k\}$ and for which there is ball $B \subset N$, $y \in B$ such that $f^{-1}(B)$ is the union of k disjoint balls $B_1, \ldots, B_k$, $x_i \in B_i$ for every i, $1 \leq i \leq k$, and the mapping degree $\deg f$ is the sum*

$$\deg f = \sum_{i=1}^{k} \deg f \mid x_i,$$

where $\deg f \mid x_i$ is the local map degree, that is, the degree of map $f : \partial B_i \to \partial B$.

Theorem 2.1 states that $\deg f$ evaluates the number of times the domain manifold M 'wraps around' the range manifold N under the mapping f. This geometrical principle is the guiding idea in most papers studying the mapping degrees. From Theorem 2.1 it is easy to produce the map of any given degree into the sphere S^n. We take k disjoint balls on M and map their interiors by an orientation-preserving homeomorphism onto $S^n - \{\mathrm{pt}\}$ and the rest of M maps to the point $\{\mathrm{pt}\}$ (see Figure 1). Thus $D(M^n, S) = \mathbb{Z}$.

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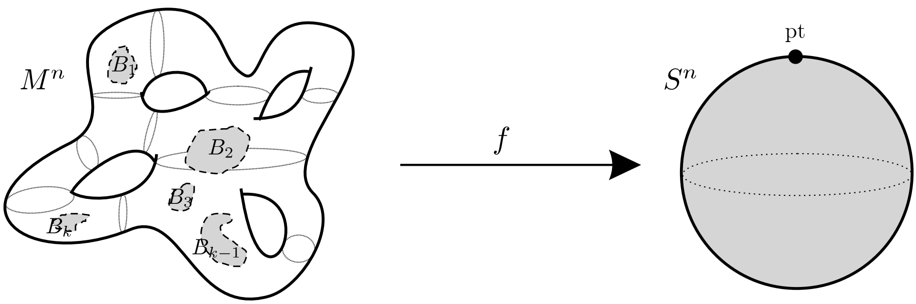

FIGURE 1. The degree k map from M^n to S^n.

Every map $f : M \to N$ induces homomorphisms on homology f_* and cohomology f^*. From the following commutative diagram (see [16, page 241]),

$$\begin{array}{ccc}
H_n(M;\mathbb{Z}) \times H^k(M;\mathbb{Z}) & \xrightarrow{\cap} & H_{n-k}(M;\mathbb{Z}) \\
\downarrow f_* \qquad\qquad f^* \uparrow & & \downarrow f_* \\
H_n(N;\mathbb{Z}) \times H^k(N;\mathbb{Z}) & \xrightarrow{\cap} & H_{n-k}(N;\mathbb{Z})
\end{array}$$

we conclude that for a nonzero degree map f, every $f^* : H^k(N;\mathbb{Z}) \to H^k(M;\mathbb{Z})$ is a monomorphism if $H^k(N;\mathbb{Z})$ is torsion free.

It is easy to produce maps of zero degree, so $0 \in D(M, N)$. The identity map shows that $1 \in D(M, M)$. In general, it is not known if there exists any degree one map from M to N for arbitrary manifolds. The following simple result predicts that M must have 'more complexity' than N.

LEMMA 2.2. *Let $f : M\sharp N \to M$ be a map obtained by pinching N to a point. Then* $\deg f = 1$.

PROOF. Take a point $p \in M$ outside the part which got pinched. Consider its inverse image $f^{-1}(p)$ in $M\sharp N$, which is the unique point. The map f is an orientation-preserving local homeomorphism. By Theorem 2.1 the claim is therefore proved. □

We proceed with an exciting corollary of Lemma 2.2.

COROLLARY 2.3. *If there is a degree k map $f : M \to N$ between two closed oriented n-manifolds, then there is a degree k map from $M\sharp Q$ to N, where Q is a closed oriented n-manifold.*

PROOF. Consider the composition of maps

$$M\sharp Q \xrightarrow{g} M \xrightarrow{f} N,$$

where g is the map pinching Q to a point.

Then $\deg(f \circ g) = \deg f \cdot \deg g = 1 \cdot k = 1$. □

The following simple examples show that, in general, the sets $D(M, N)$ are distinct and depend on both manifolds. Even for Problem 2, it is hard to say when the answer is positive or negative.

Proposition 2.4. *For a simply connected closed orientable manifold M^{2n-1}, the set*

$$D(M, \mathbb{R}P^{2n-1}) = 2\mathbb{Z}.$$

Proof. The sphere S^{2n-1} is the universal covering for $\mathbb{R}P^{2n-1}$. Let p be the covering map. Let $f : M \to \mathbb{R}P^{2n-1}$ be a map. We consider the following diagram.

$$\begin{tikzcd} & S^{2n-1} \arrow[d, "p"] \\ M \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & \mathbb{R}P^{2n-1} \end{tikzcd}$$

Because $\pi_1 M$ is trivial, there is a lifting map $\tilde{f} : M \to S^{2n-1}$ of f. From the functoriality of homology $f_* = p_* \tilde{f}_*$ and thus $\deg f = \deg p \cdot \deg \tilde{f}$. But $\deg p = 2$ when n is odd and so $\deg f$ is even.

However, it is not hard to produce a degree $2k$ map. Take any degree k map from M to S^{2n-1} and compose it with p. □

Proposition 2.5. *Let P be the Poincaré homology sphere. Then*

$$D(S^3, P) = 120\mathbb{Z}.$$

Proof. Here S^3 is the universal cover of P and the degree of p is 120. The argument is the same as in the previous proof. □

In articles [12] and [11], theorems are given which significantly contribute to our knowledge about mapping degrees between closed orientable $2n$-manifolds. In this section, we prove Theorem 2 from [12], and extend Corollary 3 of Wang and Duan's result.

Let M be a $2n$-dimensional closed, connected and orientable manifold where $n > 1$ and let $\bar{H}^n(M; \mathbb{Z})$ be the free part of $H^n(M; \mathbb{Z})$. Then the cup product operator

$$\bar{H}^n(M; \mathbb{Z}) \otimes \bar{H}^n(M; \mathbb{Z}) \to H^{2n}(M; \mathbb{Z})$$

defines the intersection form X_M over $\bar{H}^n(M; \mathbb{Z})$, which is bilinear and unimodular by Poincaré duality (see [16, Proposition 3.38]). This form is n-symmetric in the sense that

$$X_M(x \otimes y) = (-1)^n X_M(y \otimes x).$$

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a basis for $\bar{H}^n(M; \mathbb{Z})$. Then X_M determines an $m \times m$ matrix $A = (a_{ij})$, where a_{ij} is given by

$$a_{ij} = \alpha_i \cup \alpha_j [M]$$

and $[M]$ is the fundamental class of $H_{2n}(M)$.

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Let $f : M \to L$ be a map between two connected, closed and orientable $2n$-manifolds M and L and let f^* and f_* be the induced homomorphisms on the cohomology rings and homology groups. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_l)$ be the basis of $\bar{H}^n(M;\mathbb{Z})$ and $\bar{H}^n(L;\mathbb{Z})$, respectively. The induced homomorphism f^* determines $m \times l$ matrix $P = (p_{ij})$ such that

$$f^*(\beta_i) = \sum_{j=1}^{m} p_{ij}\alpha_j$$

for every i, $1 \le i \le l$.

Theorem 2.6 (Duan and Wang). *Suppose that M and L are closed oriented $2n$-manifolds with intersection matrices A and B under some given bases α for $\bar{H}^n(M;\mathbb{Z})$ and β for $\bar{H}^n(L;\mathbb{Z})$. If there is a map $f : M \to L$ of degree k such that $f^*(\beta) = \alpha P$, then*

$$P^t A P = kB.$$

Moreover, if $k = 1$, then X_L is isomorphic to a direct summand of X_M.

Proof. For a map $f : M \to L$ it holds that $f_*([M]) = k[L]$. From the functoriality of the cup and the cap product functor,

$$\begin{aligned} X_M(f^*(x) \otimes f^*(y)) &= f^*(x) \cup f^*(y)[M] \\ &= f^*(x \cup y)[M] = (x \cup y)f_*([M]) = (x \cup y)k[L] = kX_L(x \otimes y) \end{aligned}$$

for every $x, y \in \bar{H}^n(L;\mathbb{Z})$. Thus the following diagram commutes.

$$\begin{tikzcd} \bar{H}^n(L;\mathbb{Z}) \times \bar{H}^n(L;\mathbb{Z}) \arrow[r, "X_L"] \arrow[d, "f^*\otimes f^*"] & \mathbb{Z} \arrow[d, "\times k"] \\ \bar{H}^n(M;\mathbb{Z}) \times \bar{H}^n(M;\mathbb{Z}) \arrow[r, "X_M"] & \mathbb{Z} \end{tikzcd}$$

Consequently, for the bases α for $\bar{H}^n(M;\mathbb{Z})$ and β for $\bar{H}^n(L;\mathbb{Z})$, this fact is written in the form $P^tAP = kB$, where $f^*(\beta) = \alpha P$.

In particular, when $k = 1$, the restriction of X_M on the subgroup

$$f^*(\bar{H}^n(L;\mathbb{Z})) \subset \bar{H}^n(M;\mathbb{Z})$$

is isomorphic to X_L and unimodular. By the orthogonal decomposition lemma [22, page 5],

$$X_M = X_{f^*(\bar{H}^n(L;\mathbb{Z}))} \oplus X_{H^\perp} = X_L \oplus X_{H^\perp},$$

where $H^\perp$ is the orthogonal complement of $f^*(\bar{H}^n(L;\mathbb{Z}))$ and $X_{H^\perp}$ is the restriction of X_M on $H^\perp$. □

In the same paper, Duan and Wang proved the following theorem that gives the complete criteria for the existence of a degree k map from a 4-manifold M to a simply connected 4-manifold L.

Theorem 2.7. *Suppose that M and L are closed oriented 4-manifolds with intersection matrices A and B under given bases α for $\bar{H}^2(M;\mathbb{Z})$ and β for $\bar{H}^2(L;\mathbb{Z})$. If L is simply connected, then there is a map $f : M \to L$ of degree k such that $f^*(\beta) = \alpha P$ if and only if*

$$P^t A P = kB.$$

Moreover, there is a map $f : M \to L$ of degree one if and only if X_L isomorphic to a direct summand of X_M.

Duan and Wang proved Theorem 2.6 in [12, Corollary 3]. Implicitly, it is clear from their paper that this corollary could be generalized.

Corollary 2.8. *Suppose that M and L are closed oriented $2n$-manifolds such that* $\operatorname{rank} \bar{H}^n(M;\mathbb{Z}) = \operatorname{rank} \bar{H}^n(L;\mathbb{Z}) = 2r + 1$. *Then, for any map $f : M \to L$, the absolute value of the degree of f is a square of an integer.*

Proof. Let P be the matrix realized by f. By Theorem 2.6,

$$P^t A P = kB,$$

where P, A and B are square matrices of order $2r + 1$. By taking the determinant, $|P|^2 |A| = k^{2r+1} |B|$. Since A and B are unimodular, $|P|^2 = |k|^{2r+1}$. Thus $|k|$ is a perfect square. □

Theorem 2.9. *Let M and N be closed, connected and oriented $2n$-manifolds such that* $1 \le \operatorname{rank} \bar{H}^n(M;\mathbb{Z}) < \operatorname{rank} \bar{H}^n(N;\mathbb{Z})$. *Then there is no nonzero degree map $f : M \to N$.*

Proof. This is a corollary of the Cauchy–Binet formula applied to matrices P^t and $A \cdot P$. According to the formula,

$$0 = k^{\operatorname{rank} \bar{H}^n(N;\mathbb{Z})} \det B$$

and, from the unimodularity of the intersection form, $k = 0$. □

In the same fashion, we give an algebraic proof of the subsequent enthralling result.

Theorem 2.10. *If there are degree k maps $f : M \to N$ and $g : M' \to N'$ between closed oriented 4-manifolds and if N and N' are simply connected, then there is a degree k map from $M \sharp M'$ to $N \sharp N'$.*

Proof. Let A and A' be the intersection matrices for M and M' and B and B' for N and N', and let P and P' be matrices such that

$$P^t A P = kB,$$
$$P'^t A' P' = kB'.$$

By Theorem 2.7, it is sufficient to check that

$$\begin{bmatrix} P^t & 0 \\ 0 & P'^t \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} \cdot \begin{bmatrix} P & 0 \\ 0 & P' \end{bmatrix} = k \begin{bmatrix} B & 0 \\ 0 & B' \end{bmatrix}.$$

This concludes the proof. □

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In papers [12] and [11], Duan and Wang developed a technique for studying nonzero degree maps between $(n-1)$-connected closed and oriented $2n$-manifolds. They demonstrated applications on various concrete examples of manifolds.

3. Mapping degrees between quasitoric 4-manifolds

Quasitoric manifolds appeared as a topological generalization of nonsingular projective toric varieties in [9]. A nice introduction to the subject is given in the monograph [5].

We are particularly interested in the case of 4-dimensional quasitoric manifolds. In this special case, the classification problem for 4-quasitoric manifolds is completely solved by Orlik and Raymond in [24]. Their result states that a quasitoric manifold of dimension four is diffeomorphic to the connected sum of several copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$. Many important 4-manifolds are quasitoric, such as *the Hirzebruch surfaces* introduced by Hirzebruch in [18]. A Hirzebruch surface H_k is a quasitoric manifold whose orbit space is a combinatorial square. As complex manifolds they are pairwise distinct while, as smooth manifolds, there are only two diffeomorphism types (see also [20]).

Quasitoric manifolds are simply connected, so they are perfect test examples for the application of Theorem 2.7. The mapping degree between some quasitoric 4-manifolds is studied in [12] where some simple examples illustrate the application of Theorems 2.6 and 2.7. However, the results they obtain in the simplest cases are not trivial. Our goal is to study the topic further and to obtain even more exciting sets of integers realizing the mapping degree.

We specify Corollary 2.3 for application in the rest of the paper.

Corollary 3.1. *If there is a degree k map $f : M \to N$ between two quasitoric 4-manifolds M and N, then there is a degree k map from $M \sharp Q$ to N for every quasitoric 4-manifold Q.*

Quasitoric 4-manifolds are topologically classified as the connected sums of several copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ and we could easily determine their intersection forms. The matrices representing the intersection form for $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ are

$$\begin{bmatrix}1\end{bmatrix}, \quad \begin{bmatrix}-1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

respectively. Thus the intersection form for $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ has the matrix representation

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Consequently, the intersection form for a quasitoric 4-manifold diffeomorphic to

$$\begin{aligned}&(\mathbb{C}P^2)^{\sharp a} \sharp (\overline{\mathbb{C}P^2})^{\sharp b} \sharp (S^2 \times S^2)^{\sharp c}\\ &\quad = \underbrace{\mathbb{C}P^2 \sharp \cdots \sharp \mathbb{C}P^2}_{a\,\text{times}} \sharp \underbrace{\overline{\mathbb{C}P^2} \sharp \cdots \sharp \overline{\mathbb{C}P^2}}_{b\,\text{times}} \sharp \underbrace{S^2 \times S^2 \sharp \cdots \sharp S^2 \times S^2}_{c\,\text{times}}\end{aligned}$$

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is represented by the $(a+b+2c)$-square matrix

$$\begin{bmatrix} I_{a\times a} & 0 & 0 \\ 0 & -I_{b\times b} & 0 \\ 0 & 0 & A_{c\times c} \end{bmatrix},$$

where $A_{c\times c}$ is a $2c$-square matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Let

$$M = (\mathbb{C}P^2)^{\sharp a}\sharp(\overline{\mathbb{C}P^2})^{\sharp b}\sharp(S^2\times S^2)^{\sharp c} \quad \text{and}$$
$$N = (\mathbb{C}P^2)^{\sharp d}\sharp(\overline{\mathbb{C}P^2})^{\sharp e}\sharp(S^2\times S^2)^{\sharp f}$$

be two quasitoric manifolds and let A and B be the matrices of their intersection forms, respectively. From Theorem 2.7 it follows that there is a degree k map between M and N if and only if there is a $(a+b+2c)\times(d+e+2f)$ matrix P such that $P^tAP = kB$. Direct calculation gives that elements of $(d+e+2f)$-square matrix $C = P^tAP$ are

$$c_{ij} = \sum_{r=1}^{a} p_{ri}p_{rj} - \sum_{r=1}^{b} p_{a+r\,i}p_{a+r\,j} + \sum_{r=1}^{c}(p_{a+b+2r-1\,i}p_{a+b+2r\,j} + p_{a+b+2r\,i}p_{a+b+2r-1\,j}).$$

Solving the equation $P^tAP = kB$ means to solve the corresponding system of Diophantine equations. There is no algorithm for solving a Diophantine equation, so there is no standard way to approach this problem. Note that only $d+e+f$ out of these $((d+e+f)(d+e+f+1))/2$ expressions is equal to $\pm k$ and the others are zero.

3.1. Maps to $\mathbb{C}P^2$. We study the maps from quasitoric manifolds to $\mathbb{C}P^2$ (and $\overline{\mathbb{C}P^2}$). Let M be a quasitoric manifold diffeomorphic to

$$(\mathbb{C}P^2)^{\sharp a}\sharp(\overline{\mathbb{C}P^2})^{\sharp b}\sharp(S^2\times S^2)^{\sharp c}.$$

Theorem 2.7 reduces problem to the existence of a nontrivial solution of Diophantine equation

$$\sum_{i=1}^{a} p_{i\,1}^2 - \sum_{i=1}^{b} p_{a+i\,1}^2 + 2\sum_{i=1}^{c} p_{a+b+2i-1\,1}p_{a+b+2i\,1} = k. \tag{3.1}$$

Theorem 3.2.

(i) *If $a \geq 1$ (or $b \geq 1$) and $c \geq 1$, then Equation (3.1) has a solution for every $k \in \mathbb{Z}$.*

(ii) *If $a \geq 4$ and $b = c = 0$, then Equation (3.1) has a solution for every nonnegative integer k and no solution for negative k.*

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(iii) *If* $b \geq 4$ *and* $a = c = 0$*, then Equation (3.1) has a solution for every integer* $k \leq 0$ *and no solution for positive* k*.*

(iv) *If* $a = 3$ *and* $b = c = 0$*, then Equation (3.1) has a solution for every nonnegative integer* $k \neq 4^p(8q + 7)$ *and no solution for positive integers* $k = 4^p(8q + 7)$ *and negative integers.*

(v) *If* $b = 3$ *and* $a = c = 0$*, then Equation (3.1) has a solution for every integer* $k \neq -4^p(8q + 7)$ *and no solution for negative integers* $k = -4^p(8q + 7)$ *and positive integers.*

(vi) *If* $a = 2$ *and* $b = c = 0$*, then Equation (3.1) has a solution for every nonnegative integer* k *such that every prime number* $4p - 1$ *that divides* k *occurs an even number of times in the prime factorization of* k *and no solutions in other cases.*

(vii) *If* $b = 2$ *and* $a = c = 0$*, then Equation (3.1) has a solution for every integer* $k \leq 0$ *such that every prime number* $4p - 1$ *that divides* $|k|$ *occurs an even number of times in the prime factorization of* $|k|$ *and no solutions in other cases.*

(viii) *If* $a = b = 1$ *and* $c = 0$*, then Equation (3.1) has a solution for every integer* $k \neq 4p + 2$ *and no solution for* $k = 4p + 2$*.*

(ix) *If* $a = 1$ *and* $b = c = 0$*, then Equation (3.1) has a solution for every integer that is a square of an integer and no solution in other cases.*

(x) *If* $b = 1$ *and* $a = c = 0$*, then Equation (3.1) has a solution for every integer that is a square of an integer multiplied by* -1 *and no solution in other cases.*

(xi) *If* $a = b = 0$ *and* $c \geq 1$*, then Equation (3.1) has a solution for every even integer* k*.*

PROOF. We observe that every integer can be represented in the form $u^2 + 2vw$ for some integers u, v and w. This guarantees the existence of a k-degree map $f : \mathbb{C}P^2\sharp(S^2 \times S^2) \to \mathbb{C}P^2$. Corollary 3.1 extends the result for cases $a \geq 1$ and $c \geq 1$. Since every nonnegative integer has a decomposition into the sum of four perfect squares, there is a k-degree map for every $k \geq 0$ when $a \geq 4$ and $b = c = 0$.

The cases $a = 3$ and $b = c = 0$ are curios. It follows from the nontrivial result of Legendre [21] and Gauss [15] that a nonnegative integer has presentation as the sum of three squares of integers if and only if it is not of the type $4^p(8q + 7)$.

If $a = 2$ and $b = c = 0$, then, for any map $f : \mathbb{C}P^2\sharp\mathbb{C}P^2 \to \mathbb{C}P^2$, $\deg f$ must have a representation as the sum of two perfect squares. It is a well known fact that a nonnegative integer k can be written as the sum of two perfect squares if and only if every prime number $4p - 1$ that divides k occurs an even number of times in the prime factorization of k (see [7] and [23]). For $k = u^2 + v^2$, we take the matrix

$$P = \begin{bmatrix} u & v \\ v & -u \end{bmatrix}$$

and get the map of degree k.

In the case $a = b = 1$ and $c = 0$, the map degree must be the difference of two squares and this is possible if and only if the integer is not equal to two modulo four. Again, it

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is straightforward to check that

$$P = \begin{bmatrix} u & v \\ v & -u \end{bmatrix}$$

gives map of degree $u^2 - v^2$.

In the same manner, we can verify the other cases, so we omit the rest of the proof. □

An explicit realization of the maps from Theorem 3.2 in most of its cases is fairly easy. The best approach to the construction of such maps is via geometric topology. The reader can easily convince themselves that the map $f_k : \mathbb{C}P^2 \to \mathbb{C}P^2$ given by $[x : y : z] \mapsto [x^k : y^k : z^k]$ has the mapping degree equal to k^2. For example, a map $f : \mathbb{C}P^2 \sharp \mathbb{C}P^2 \to \mathbb{C}P^2$ of degree $n = a^2 + b^2$ can be constructed using the maps f_a and f_b and a homeomorphism $h : \mathbb{C}P^2 \setminus D^4 \to \mathbb{C}P^2 \setminus [1 : 0 : 0]$. At first, the S^3 belonging to both of the $\mathbb{C}P^2$ in the connected sum is pinched to $[1 : 0 : 0]$. The rest of the first $\mathbb{C}P^2$ is mapped to $\mathbb{C}P^2$ by $f_a \circ h$, and the same is done for the second copy using $f_b \circ h$. From Theorem 2.1, it follows that a map constructed in this way has degree equal to $a^2 + b^2$.

3.2. Maps to $S^2 \times S^2$. Maps from $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ to $S^2 \times S^2$ are studied in [12]. We are interested in the mapping degrees from an arbitrary quasitoric manifold to $S^2 \times S^2$.

Proposition 3.3. *There is no nonzero degree map from $(\mathbb{C}P^2)^{\sharp k}$ $((\overline{\mathbb{C}P^2})^{\sharp k})$ to $S^2 \times S^2$.*

Proof. For every matrix P induced by a map $f : (\mathbb{C}P^2)^{\sharp k} \to S^2 \times S^2$,

$$P^t P = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}.$$

Thus P must be the zero matrix and hence $k = 0$. □

Proposition 3.4. *For every integer k, there is a k-degree map from $(S^2 \times S^2)^{\sharp n}$ to $S^2 \times S^2$.*

Proof. According to Corollary 3.1, it is enough to show that there is a k-degree map $f : S^2 \times S^2 \to S^2 \times S^2$. However, the matrix P given by

$$P = \begin{bmatrix} 0 & k \\ 1 & 0 \end{bmatrix}$$

guarantees the existence of such a map by Theorem 2.7. □

Proposition 3.5. *For every integer k, there is a k-degree map from $(\mathbb{C}P^2)^{\sharp 2} \sharp \overline{\mathbb{C}P^2}$ $(\mathbb{C}P^2 \sharp (\overline{\mathbb{C}P^2})^{\sharp 2})$ to $S^2 \times S^2$.*

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Proof. We directly check that the matrix

$$P = \begin{bmatrix} 0 & 0 \\ 1 & k \\ 1 & k \end{bmatrix}$$

satisfies the condition of Theorem 2.7. □

Theorem 3.6. *Let M be a quasitoric manifold such that $c \geq 1$. Then there is a k-degree map from M to $S^2 \times S^2$ for every integer k.*

Proof. We easily check that matrix P, such that

$$P = \begin{bmatrix} & 0 & \\ 0 & & 1 \\ k & & 0 \end{bmatrix},$$

satisfies Theorem 2.7. □

Theorem 3.7. *Let M be a quasitoric manifold such that $a \geq 2$ and $b \geq 1$ (or $a \geq 1$ and $b \geq 2$). Then there is a k-degree map from M to $S^2 \times S^2$ for every integer k.*

Proof. We easily check that matrix P, such that

$$P = \begin{bmatrix} 0_{(a-2)\times 2} & \\ k & 0 \\ 0 & 1 \\ -k & 1 \\ 0_{(b+2c-1)\times 2} & \end{bmatrix},$$

satisfies Theorem 2.7. □

3.3. Maps to $\mathbb{C}P^2 \sharp \mathbb{C}P^2$. There is no nonzero degree map from $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ to $\mathbb{C}P^2 \sharp \mathbb{C}P^2$, according to the examples presented in [12]. For other quasitoric manifolds, the sets $D(M, \mathbb{C}P^2 \sharp \mathbb{C}P^2)$ are much more attractive because they are described by some intriguing number theoretical conditions. The appearance of quadratic forms is not unusual as they play an important role in modern mathematics (see [8] and [22]).

Proposition 3.8. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 2} \to (\mathbb{C}P^2)^{\sharp 2}$ if and only if $k \geq 0$ and every prime number $4p - 1$ that divides k occurs an even number of times in the prime factorization of k.*

Proof. Let $P = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be a matrix induced by f. The corresponding system of equations is

$$a^2 + c^2 = b^2 + d^2 = k,$$
$$ab + cd = 0.$$

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It is obvious that k should have a representation in the form $k = u^2 + v^2$ for some integers u and v.

However, for every such k, the matrix $P = \left[\begin{smallmatrix} u & v \\ v & -u \end{smallmatrix}\right]$ implies the existence of a k-degree map. □

Proposition 3.9. *There is a k-degree map $f : (S^2 \times S^2)^{\sharp n} \to (\mathbb{C}P^2)^{\sharp 2}$, $n \geq 2$ if and only if k is an even integer.*

Proof. Let

$$P = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_{2n} & b_{sn} \end{bmatrix}$$

be a matrix induced by f.

$$2a_1a_2 + \cdots + 2a_{2n-1}a_{2n} = 2b_1b_2 + \cdots + 2b_{2n-1}b_{2n} = k,$$
$$a_1b_1 + \cdots + a_{2n}b_{2n} = 0.$$

Hence k is even.

It is sufficient to prove that there is a map $f : (S^2 \times S^2)^{\sharp 2} \to (\mathbb{C}P^2)^{\sharp 2}$ having the mapping degree equal to $k = 2t$. But it exists due to the matrix

$$P = \begin{bmatrix} t & 0 \\ 1 & 0 \\ 0 & t \\ 0 & 1 \end{bmatrix}.$$

This concludes the proof. □

Proposition 3.10. *For every integer k, there is a k-degree map $f : \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \sharp (S^2 \times S^2) \to (\mathbb{C}P^2)^{\sharp 2}$.*

Proof. Suppose that $k \neq 4t + 2$. Then k has a representation in the form $u^2 - v^2$ for some integers u and v. But the matrix

$$P = \begin{bmatrix} u & u \\ v & v \\ u^2 - v^2 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfies the condition of Theorem 2.7.

If $k = 4t + 2$, we take

$$P = \begin{bmatrix} t+2 & t \\ t & t \\ 1 & 2t+1 \\ -1 & -1 \end{bmatrix}.$$

This concludes the proof. □

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Theorem 3.11. *There is a k-degree map $f : (\mathbb{C}P^2)\sharp(S^2 \times S^2) \to \mathbb{C}P^2\sharp\mathbb{C}P^2$ if and only if $k \geq 0$ and every prime number $4p-1$ that divides k occurs an even number of times in the prime factorization of k.*

Proof. Let P be a matrix induced by f, given by

$$P = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

We need to find out all integers k such that the system

$$a^2 + 2ce = b^2 + 2df = k, \tag{3.2}$$
$$ab + de + cf = 0 \tag{3.3}$$

has solutions in integers. Multiplying (3.3) by $2cd$,

$$0 = 2abcd + d^2(2ce) + c^2(2df) = 2abcd + d^2(k - a^2) + c^2(k - b^2).$$

It follows that

$$k(c^2 + d^2) = (ad - bc)^2.$$

Thus $k \geq 0$ and k has the form $u^2 + v^2$.

It is left to show that there is a map whose mapping degree is equal to $u^2 + v^2$ for every two integers u and v. It is sufficient to take the matrix

$$P = \begin{bmatrix} u+v & u-v \\ -u & v \\ v & u \end{bmatrix}.$$

This concludes the proof. □

Theorem 3.12. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 2}\sharp(S^2 \times S^2) \to \mathbb{C}P^2\sharp\mathbb{C}P^2$ if and only if $k \geq 0$ and every prime number $4p-1$ that divides k occurs an even number of times in the prime factorization of k.*

Proof. Let

$$P = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}.$$

We are solving the system

$$a^2 + c^2 + 2ge = b^2 + d^2 + 2fh = k,$$
$$ab + cd + ef + gh = 0.$$

It is clear that when $k = m^2 + n^2$ there is a solution $a = d = m$, $b = n$, $c = -n$ and $g = e = h = k = 0$.

We are going to prove that k has the form $m^2 + n^2$.

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Multiply the equation $ab + cd + ef + 1gh = 0$ by $2fg$ and use the other two to get

$$2fg(ab + cd) + f^2(k - a^2 - c^2) + g^2(k - b^2 - d^2) = 0.$$

It follows that

$$k(f^2 + g^2) = (af - bg)^2 + (cf - dg)^2.$$

Thus it is clear that k must be of the form $u^2 + v^2$. □

Using the same approach, we obtain the following theorems.

THEOREM 3.13. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 3} \to \mathbb{C}P^2\sharp\mathbb{C}P^2$ if and only if $k \geq 0$ and every prime number $4p - 1$ that divides k occurs an even number of times in the prime factorization of k.*

PROOF. According to Proposition 3.8, there is a map $f : \mathbb{C}P^2\sharp\mathbb{C}P^2 \to \mathbb{C}P^2\sharp\mathbb{C}P^2$ with the mapping degree equal to $u^2 + v^2$. So it is only left to prove that k has a representation as the sum of two perfect squares.

Let the induced matrix be

$$P = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}.$$

We are solving the system

$$a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2 = k, \quad a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

We can suppose that $\mathrm{GCD}(a_1, a_2, a_3, b_1, b_2, b_3) = 1$, since, if $\mathrm{GCD}(a_1, a_2, a_3, b_1, b_2, b_3) = d$, we can switch to the same system of Diophantine equations with k/d^2 instead of k.

Expressing $a_3^2b_3^2$ in two different ways,

$$a_3^2b_3^2 = (k - a_1^2 - a_2^2)(k - b_1^2 - b_2^2),$$
$$a_3^2b_3^2 = (a_1b_1 + a_2b_2)^2,$$

yields

$$k^2 - k(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (a_1b_2 + a_2b_1)^2 = 0.$$

We shall prove that there is no prime $q = 4r - 1$ such that $q^{2s+1} \mid k$ and $q^{2s+2} \nmid k$, which clearly implies our claim. If such q exists, then $q^{2s+1} \mid (a_1b_2 + a_2b_1)^2$, and hence $q^{s+1} \mid (a_1b_2 + a_2b_1)$. Thus $q^{2s+2} \mid k(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ and $q \mid a_1^2 + a_2^2 + b_1^2 + b_2^2$. But

$$2k = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2$$

yields $q \mid a_3^2 + b_3^2$, and $q \mid a_3$ and $q \mid b_3$. Analogously, we prove that $q \mid a_1$, $q \mid b_1$, $q \mid a_2$ and $q \mid b_2$. This contradicts the assumption that $\mathrm{GCD}(a_1, a_2, a_3, b_1, b_2, b_3) = 1$! □

THEOREM 3.14. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp n} \to \mathbb{C}P^2\sharp\mathbb{C}P^2$, $n \geq 4$, if and only if k is a nonnegative integer.*

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Proof. From the condition of Theorem 2.7, k has a representation as the sum of n perfect squares. Thus $k \geq 0$.

According to Lagrange's theorem, every nonnegative integer has a representation as the sum of four perfect squares. Let $k = a^2 + b^2 + c^2 + d^2$ for some integers a, b, c and d. So the matrix

$$P = \begin{bmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{bmatrix}$$

guarantees the existence of a degree k map. □

Theorem 3.15. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 3}\sharp(S^2 \times S^2) \to \mathbb{C}P^2\sharp\mathbb{C}P^2$ for every nonnegative integer k.*

Proof. Let the induced matrix be

$$P = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ a_5 & b_5 \end{bmatrix}.$$

Then the corresponding system is

$$a_1^2 + a_2^2 + a_3^2 + 2a_4a_5 = b_1^2 + b_2^2 + b_3^2 + 2b_4b_5 = k, \tag{3.4}$$

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 + a_5b_5 = 0. \tag{3.5}$$

Multiplying (3.5) by $2a_4b_5$ and using (3.4) in a similar way as in the proof of Theorem 3.12 yields

$$k(a_4^2 + b_5^2) = (a_4b_1 - b_5a_1)^2 + (a_4b_2 - b_5a_2)^2 + (a_4b_3 - b_5a_3)^2.$$

It is clear that $k \geq 0$.

Since $k \geq 0$, k has the form $k = u^2 + v^2 + w^2 + z^2$. Then the matrix

$$P = \begin{bmatrix} u & -w \\ v & -z \\ w - z & u - v \\ w & v \\ z & u \end{bmatrix}$$

implies the existence of a k-degree map. □

Theorem 3.16. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 2}\sharp\overline{\mathbb{C}P^2} \to \mathbb{C}P^2\sharp\mathbb{C}P^2$ if and only if $k \geq 0$ and every prime number $4p - 1$ that divides k occurs an even number of times in the prime factorization of k.*

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PROOF. Let

$$P = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

The corresponding system of equations is

$$a^2 + c^2 - e^2 = b^2 + d^2 - f^2 = k, \quad ab + cd - ef = 0. \tag{3.6}$$

From (3.6), we deduce that

$$k^2 - (a^2 + b^2 + c^2 + d^2)k + (bc - ad)^2 = 0.$$

By Vièta's formulas, we obtain $k \geq 0$.

Without loss of generality, we suppose that $\mathrm{GCD}(a, b, c, d, e, f) = 1$. We shall prove that there is no prime $q = 4p - 1$ such that $q^{2r+1} \mid k$ and $q^{2r+2} \nmid k$. If such q exists, from the quadratic equation above, $q^{r+1} \mid bc - ad$ and $q \mid a^2 + b^2 + c^2 + d^2$. It is not hard to deduce that $q \mid (a + d)^2 + (b - c)^2$ and $q \mid (a - d)^2 + (b + c)^2$. Thus $a + d \equiv b - c \equiv 0 \bmod q$ and $a - d \equiv b + c \equiv 0 \bmod q$. Finally, $a \equiv b \equiv c \equiv d \equiv 0 \bmod q$ yields $q \mid e$ and $q \mid f$. This contradicts $\mathrm{GCD}(a, b, c, d, e, f) = 1$!

For the existence of a degree $k = u^2 + v^2$ map, we can take the matrix

$$P = \begin{bmatrix} u & v \\ -v & u \\ 0 & 0 \end{bmatrix}.$$

This concludes the proof. □

THEOREM 3.17. *There is a degree k map $f : (\mathbb{C}P^2)^{\sharp 3} \sharp \overline{\mathbb{C}P^2} \to \mathbb{C}P^2 \sharp \mathbb{C}P^2$ if and only if $k \geq 0$ and every prime number $4p - 1$ that divides k occurs an even number of times in the prime factorization of k.*

PROOF. Due to the previous theorem, it is enough to prove that k has the form $u^2 + v^2$.

Let

$$P = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}.$$

Then the corresponding system is

$$a^2 + c^2 + e^2 - g^2 = b^2 + d^2 + f^2 - h^2 = k, \tag{3.7}$$

$$ab + cd + ef - gh = 0. \tag{3.8}$$

Substitute $e = g + m$ and $f = h + n$ in (3.7) and (3.8) and consider the equivalent system

$$a^2 + c^2 + m^2 + 2gm = b^2 + d^2 + n^2 + 2hn = k,$$

$$ab + cd + mn + gn + fm = 0.$$

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Using the same approach as in previous proofs we get

$$2mnab + 2mncd + 2m^2n^2 + n^2(k - a^2 - c^2 - m^2) + m^2(k - b^2 - d^2 - n^2) = 0.$$

So

$$(m^2 + n^2)k = (mb - an)^2 + (md - cn)^2,$$

and the claim clearly follows. □

Theorem 3.18. *There is a degree k map $f : (\mathbb{C}P^2)^{\sharp n}\sharp\overline{\mathbb{C}P^2} \to \mathbb{C}P^2\sharp\mathbb{C}P^2$, $n \geq 4$ if and only if $k \geq 0$.*

Proof. We shall prove that k is nonnegative.

Let the induced matrix be

$$P = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix}$$

and the corresponding system be

$$a_1^2 + a_2^2 + \cdots + a_{n-1}^2 - a_n^2 = b_1^2 + b_2^2 + \cdots + b_{n-1}^2 - b_n^2 = k,$$
$$a_1b_1 + \cdots + a_{n-1}b_{n-1} - a_nb_n = 0.$$

Similarly to the previous proofs,

$$k^2 - k(a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + b_1^2 + b_2^2 + \cdots + b_{n-1}^2) + \Big(\sum_{1\leq i<j\leq n-1} (a_ib_j - a_jb_i)^2\Big) = 0.$$

By Viète's formulas, it must hold that $k \geq 0$.

According to Theorems 3.14 and Corollary 3.1, there is a k-degree map for every nonnegative k. □

Theorem 3.19. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 2}\sharp(\overline{\mathbb{C}P^2})^{\sharp 2} \to \mathbb{C}P^2\sharp\mathbb{C}P^2$ for every integer k.*

Proof. If $k = 2t + 1$ is an odd number, we can take

$$P = \begin{bmatrix} t+1 & 0 \\ 0 & t+1 \\ 0 & t \\ t & 0 \end{bmatrix}.$$

If $k = 2t$ is even, we can take

$$P = \begin{bmatrix} t+1 & 0 \\ 0 & t+1 \\ 1 & t \\ t & -1 \end{bmatrix}.$$

This concludes the proof. □

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The above Theorem, together with Corollary 3.1, implies the following corollary.

COROLLARY 3.20. *Let M be a quasitoric 4-manifold such that* $\operatorname{rank} \bar{H}^2(M;\mathbb{Z}) \geq 5$ *and* $b + 2c \geq 2$. *Then, for every integer k, there is a k-degree map* $f : M \to \mathbb{C}P^2 \sharp \mathbb{C}P^2$.

3.4. Maps to $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$.

PROPOSITION 3.21. *There is no nonzero degree map* $f : (\mathbb{C}P^2)^{\sharp n} \to \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$.

PROOF. Let

$$P = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix}.$$

The corresponding system is

$$\begin{aligned} a_1^2 + a_2^2 + \cdots + a_{n-1}^2 + a_n^2 &= k, \\ b_1^2 + b_2^2 + \cdots + b_{n-1}^2 - b_n^2 &= -k, \\ a_1 b_1 + \cdots + a_{n-1} b_{n-1} + a_n b_n &= 0. \end{aligned}$$

From the first equation, we deduce that $k \geq 0$ but, from the second, we deduce that $k \leq 0$. So $k = 0$. □

PROPOSITION 3.22. *There exists a k-degree map* $f : \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2} \to \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ *if and only if* $k \not\equiv 2 \bmod 4$.

PROOF. Let the induced matrix be

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then the corresponding system is

$$\begin{aligned} a^2 - c^2 = b^2 - d^2 &= k, \\ ac - bd &= 0. \end{aligned}$$

Obviously, $k \not\equiv 2 \bmod 4$.

Every integer $k \not\equiv 2 \bmod 4$ has a representation in the form $u^2 - v^2$ for some integers u, v, so it is sufficient to take the matrix $P = \left[\begin{smallmatrix} u & u \\ v & v \end{smallmatrix}\right]$. □

THEOREM 3.23. *For every integer k, there is a k-degree map* $f : (\mathbb{C}P^2)^{\sharp 2} \sharp \overline{\mathbb{C}P^2} \to \mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$.

PROOF. Let

$$P = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

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We are looking for the solutions of

$$a^2 + c^2 - e^2 = f^2 - b^2 - d^2 = k,$$
$$ab + cd - ef = 0.$$

For $k \neq 4t + 2$, it is known that there are integers m and n such that $k = m^2 - n^2$. In this case, $a = b = 0$, $c = f = m$ and $d = e = n$ finish the proof. For $k = 4t + 2$, we could take $a = 1$, $b = 2$, $c = 2t + 1$, $d = 2t + 2$, $e = 2t$ and $f = 2t + 3$. □

THEOREM 3.24. *For every integer k, there is a k-degree map $f : \mathbb{C}P^2\sharp(S^2 \times S^2) \to \mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$.*

PROOF. Let

$$P = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

We are looking for the solutions of the system

$$a^2 + 2ce = k,$$
$$b^2 + 2df = -k,$$
$$ab + cf + de = 0.$$

For $k = 2t$, we could take $a = b = 0$, $c = d = t$, $e = 1$ and $f = -1$. For $k = 2t + 1$, we could take $a = b = c = d = 2t + 1$, $e = -t$ and $f = -t - 1$. □

As a corollary of the above theorems, we summarize our findings in the next corollary.

COROLLARY 3.25. *For every quasitoric 4-manifold M such that $a \cdot b \geq 2$ or $ac \geq 1$ and for every integer k, there is a degree k map $f : M \to \mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$.*

4. Orthogonal lattices and maps between connected sums of $\mathbb{C}P^2$

In this section, we focus on the maps between connected sums of $\mathbb{C}P^2$. Our main interest is in the mapping degrees of all maps

$$f : (\mathbb{C}P^2)^{\sharp n} \to (\mathbb{C}P^2)^{\sharp n}.$$

PROPOSITION 4.1. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 2n-1} \to (\mathbb{C}P^2)^{\sharp 2n-1}$, $n \geq 1$ if and only if k is a square of an integer.*

PROOF. From Corollary 2.8, we deduce that k is a perfect square. For the matrix P, we take $(2n - 1) \times (2n - 1)$ square matrix kI, where I is the identity matrix. □

THEOREM 4.2. *There is a k-degree map $f : (\mathbb{C}P^2)^{\sharp 4} \to (\mathbb{C}P^2)^{\sharp 4}$ if and only if k is a nonnegative integer.*

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PROOF. We use the fact that every nonnegative integer can be written as the sum of four perfect squares

$$k = a^2 + b^2 + c^2 + d^2.$$

Then the matrix

$$P = \begin{bmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{bmatrix}$$

guarantees the existence of a k-degree map. □

Theorem 4.2, together with Theorem 2.10, implies Theorem 1.2.

The remaining case to determine the mapping degrees

$$f : (\mathbb{C}P^2)^{\sharp\, 4n+2} \to (\mathbb{C}P^2)^{\sharp\, 4n+2}, \quad n \geq 1$$

is still an open problem. Proposition 3.8 implies that the set of all integers that can be written as the sum of two squares belongs to $D((\mathbb{C}P^2)^{\sharp\, 4n+2}, (\mathbb{C}P^2)^{\sharp\, 4n+2})$. We cannot give the answer even in the case $f : (\mathbb{C}P^2)^{\sharp\, 6} \to (\mathbb{C}P^2)^{\sharp\, 6}$, but we checked directly that there is no degree 3, 7, 11, 15, 19 and various other cases that justify the conjecture that $D((\mathbb{C}P^2)^{\sharp\, 6}, (\mathbb{C}P^2)^{\sharp\, 6})$ is the set of integers that can be written as a sum of two perfect squares. Generally, we make the following conjecture.

CONJECTURE 4.3. The set $D((\mathbb{C}P^2)^{\sharp\, 4n+2}, (\mathbb{C}P^2)^{\sharp\, 4n+2})$ is the set of nonnegative integers such that every prime number $4p - 1$ that divides k occurs an even number of times in the prime factorization of k.

Conjecture 4.3 could be reformulated in the following way. *Is there an integer matrix* $P = [p_{ij}]$ $1 \leq i, j \leq 4n + 2$ *such that*

$$\sum_{j=1}^{4n+2} p_{ij}^2 = k$$

for every $i = 1, \ldots, 4n + 2$ *and*

$$\sum_{t=1}^{4n+2} p_{it} p_{jt} = 0$$

for every $i \neq j$ *if and only if* k *can be written as the sum of two squares?*

We can think about the columns of P as the vectors in $\mathbb{R}^{4n+2}$. Observe that the matrix P satisfies the equality case in the famous Hadamard's inequality (see [6, page 108]). This means that if we look at the columns of P as generators of the lattice (which is the sublattice of $\mathbb{Z}^{4n+2}$), then the integer k is its discriminant. Our question is: *What are the values of discriminants of orthogonal integer lattices in* $\mathbb{R}^{4n+2}$ *with equal lengths of generators?* The matrices that satisfy the equality case of Hadamard's inequality are frequently seen in mathematics. Those with entries -1 and 1 are called Hadamard's matrices (see [1]). There are no $(4n + 2) \times (4n + 2)$ Hadamard's matrices by the result of Paley [25]. We believe that Conjecture 4.3 is very much connected to studying the orthogonal lattices and their discriminants.

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5. Some observations about maps between quasitoric 4-manifolds

In previous sections, we saw several examples of the sets $D(M,N)$ when M and N are quasitoric 4-manifolds. We are not able to determine this set in general for quasitoric 4-manifolds but, due to Theorem 2.10 and Corollary 3.1 and special cases from Section 4, we can detect it for various manifolds and, in most cases, find an infinite subset of $D(M,N)$.

We approach the general problem by starting with decomposing M and N into the connected sums of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $S^2\times S^2$. From the system of Diophantine equations, we look for some general restriction on the mapping degree k if it exists. As has already been seen, the conditions that might be imposed on k are usually of the type that it should be a positive or a negative integer, or have a representation as a sum of a certain number of perfect squares. Then we are working backwards. We completely described the mapping degrees from arbitrary quasitoric manifolds in $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, $S^2\times S^2$, $\mathbb{C}P^2\sharp\overline{\mathbb{C}P^2}$, $\mathbb{C}P^2\sharp\mathbb{C}P^2$. These results, by repeatedly applying Theorem 2.10 and Corollary 3.1, might produce a k-degree map $f: M\to N$. It is not evident that this strategy can work successfully in an arbitrary case but, certainly, it is an algorithm for generating new nontrivial examples.

Now we give the proofs of the main theorems that we formulated in the introductory section.

Proof of Theorem 1.3. It is obvious that k should be even. From the results in [11], there are degree k maps $f: S^2\times S^2\to\mathbb{C}P^2\sharp(\overline{\mathbb{C}P^2})$ and $g: S^2\times S^2\to S^2\times S^2$ for every even k. The claim follows by Theorem 2.10. □

Proof of Theorem 1.4. If there exists a map f of nonzero degree k, then the composition of this map with a degree one map $g:(\mathbb{C}P^2)^{\sharp n}\sharp(S^2\times S^2)^{\sharp n}\to S^2\times S^2$ would have mapping degree equal to k, which would contradict the claim of Proposition 3.3! □

Proof of Theorem 1.5. Let M be diffeomorphic to

$$(\mathbb{C}P^2)^{\sharp m}\sharp(\overline{\mathbb{C}P^2})^{\sharp n}\sharp(S^2\times S^2)^{\sharp p}.$$

By Theorems 3.2 and 2.10, there exist mappings $f_1:(\mathbb{C}P^2\sharp(S^2\times S^2))^{\sharp m}\to(\mathbb{C}P^2)^{\sharp m}$, $f_2:(\overline{\mathbb{C}P^2}\sharp(S^2\times S^2))^{\sharp m}\to(\overline{\mathbb{C}P^2})^{\sharp m}$ and $f_3:(S^2\times S^2))^{\sharp m}\to(S^2\times S^2))^{\sharp m}$ of any given degree k. Take $a_0=m$, $b_0=n$ and $c_0=m+n+p$. According to Theorem 2.10, there exists a map

$$f:(\mathbb{C}P^2)^{\sharp a_0}\sharp(\overline{\mathbb{C}P^2})^{\sharp b_0}\sharp(S^2\times S^2)^{\sharp c_0}\to M.$$

The claim now follows from Corollary 3.1. □

This theorem states that there are infinitely many manifolds that could be mapped to M of any degree.

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ĐORĐE B. BARALIĆ, Mathematical Institute SASA,
Kneza Mihaila 36, p.p. 367, 11001 Belgrade, Serbia
e-mail: djbaralic@mi.sanu.ac.rs