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ON COATOMS OF THE LATTICE OF MATRIC-EXTENSIBLE RADICALS

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A radical α in the universal class of all associative rings is called matric-extensible if for all natural numbers n and all rings A, $A \in \alpha$ if and only if $M_n(A) \in \alpha$, where $M_n(A)$ denotes the $n \times n$ matrix ring with entries from A. We show that there are no coatoms, that is, maximal elements in the lattice of all matric-extensible radicals of associative rings.

1. INTRODUCTION

We work in the universal class of all associative rings. A radical α is called matric-extensible if for all natural numbers n and all rings A, we have $A \in \alpha$ if and only if $M_n(A) \in \alpha$, where $M_n(A)$ denotes the $n \times n$ matrix ring with entries from A. Many of the most important radicals such as, the prime radical β, Levitzki, Jacobson and Brown-McCoy radicals are well known to be matric-extensible and one of the hardest and still open problems in ring theory raised by Koethe in 1930 is equivalent to the matric-extensibility of the nil radical. Thus there is a motivation for studying matric-extensible radicals. Snider [7] shown that the class $\mathbb{L}_m$ of all matric-extensible radicals has a complete lattice structure with respect to inclusion. If $\{\alpha_i, i \in I\}$ is a class of matric-extensible radicals, then the meet and join are given by $\bigwedge_{i\in I}\alpha_i = \bigcap_{i\in I}\alpha_i$ and $\bigvee_{i\in I}\alpha_i = l\Big(\bigcup_{i\in I}\alpha_i\Big)$, where $l\Big(\bigcup_{i\in I}\alpha_i\Big)$ denotes the smallest radical containing the class $\bigcup_{i\in I}\alpha_i$. In [2] all atoms, that is, minimal elements of the sublattice of $\mathbb{L}_m$ consisting of all hereditary matric-extensible radicals were described and it was shown that this sublattice is atomic. In this paper we shall show that $\mathbb{L}_m$ does not contain coatoms, that is, maximal elements.

In what follows the notation $A \lhd B$ means that A is an ideal of B, $A \lhd_l B$ means that A is a left ideal of B and $A \lhd' B$ means that the factor ring B/A is a prime ring. As usual, the cardinality of a set S will be denoted by $|S|$, $S \subseteq R$ means that S is contained in R and $S \subset R$ means that S is strictly contained in R. A class μ is called hereditary if $I \lhd R \in \mu$ implies $I \in \mu$, left hereditary if $I \lhd_l R \in \mu$ implies $I \in \mu$, and homomorphically closed if $I \lhd R \in \mu$ implies $R/I \in \mu$. For a radical α and a ring A, $\alpha(A)$ denotes the α-radical of A and $\mathcal{S}(\alpha) = \{A : \alpha(A) = 0\}$. A ring A is called α-semisimple

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if $\alpha(A) = 0$. A radical α is left strong if for any ring A we have $L \lhd_l A$ and $L \in \alpha$ imply $L \subseteq \alpha(A)$. For a class μ of rings, $l(\mu)$ denotes the smallest radical containing μ and $l_s(\mu)$ denotes the smallest left strong radical containing μ. All undefined terms and used facts on radicals can be found in [**1**] and [**4**].

2. Main Results

We require the following.

CONSTRUCTION. ([**3**]) For any cardinal $\kappa > 1$, let $S = W(\kappa)$ be the set of all finite words made from a (well-ordered) alphabet of cardinality κ, lexicographically ordered. Then S is a linearly ordered set with no greatest element, with least element and such that every interval $[x, y]$, $x < y$, has cardinality κ. Moreover, S is a semigroup with multiplication defined by $xy = \max\{x, y\}$. For any nonzero semiprime ring A let $A(S)$ denote the semigroup ring of S over A.

LEMMA 1. *The semigroup ring $A(S)$ enjoys the following properties:*

(a) *$A(S)$ is a subdirect sum of copies of A.*

(b) *If $Q \lhd' A(S)$, then $A(S)/Q \simeq A/P$, for some $P \lhd' A$.*

(c) *If $0 \neq L \lhd_l A(S)$, then $|L/\beta(L)| \geqslant \kappa$.*

PROOF: Parts (a) and (b) are proved in [**3**]. Our proof of part (c) is an adaptation of ([**3**], proof of Lemma 1(d)). Since A is a semiprime ring, it follows from part (a) that so is $A(S)$. Let $0 \neq L \lhd_l A(S)$. Then $L\beta(L) \lhd_l \beta(L)$ and, since $\beta(L) \in \beta$ and β is left hereditary ([**4**, Example 3.2.12]), it follows that $L\beta(L) \in \beta$. But then, since β is left strong ([**4**, Example 3.17.2 (i)]), it follows that $L\beta(L) \subseteq \beta(A(S)) = 0$. Moreover, $\beta(L) \neq L$ since otherwise we would have $L^2 = L/\beta(L) = 0$ which is impossible because $A(S)$ is a semiprime ring. Let $x = a_1u_1 + \cdots + a_ku_k \in L\backslash\beta(L)$, $0 \neq a_i \in A$ and $u_i \in S$, where $u_1 < \cdots < u_k$. If $a = \sum_{i=1}^{k} a_i \neq 0$ then, since A is a semiprime ring, it follows that $aba \neq 0$ for some $b \in A$. Then for all $v \in S$ such that $v > u_k$ we have $0 \neq bau_k + bav = (bu_k + bv)x \in L$. Moreover, $bau_k + bav \notin \beta(L)$ since otherwise we would have $0 \neq abau_k + abav = x(bau_k + bav) \in L\beta(L) = 0$, a contradiction. Thus $bau_k + bav \in L\backslash\beta(L)$. Let $c \in A$ be such that $(ba)c(ba) \neq 0$. Then for each $u \in S$ such that $u_k < u < v$ we have $0 \neq cbau + cbav = cu(bau_k + bav) \in L\backslash\beta(L)$ since otherwise we would have

$$0 \neq bacbau + bacbav + bacbav + bacbav = (bau_k + bav)(cbau + cbav) \in L\beta(L) = 0,$$

a contradiction. If $a = 0$, then $c = \sum_{i=1}^{k-1} a_i \neq 0$ and, since A is a semiprime ring, we have $cdc \neq 0$ for some $d \in A$. Hence $0 \neq dcu_{k-1} + da_ku_k = du_{k-1}x \in L\backslash\beta(L)$ since otherwise we would have

$$0 \neq cdcu_{k-1} + a_kdcu_k = x(dcu_{k-1} + da_ku_k) \in L\backslash\beta(L) = 0,$$

a contradiction. Let $f \in A$ be such that $dcfdc \neq 0$. Then for every $w \in S$ such that $u_{k-1} < w < u_k$ we have $0 \neq fdcw + fda_k u_k = fw(dcu_{k-1} + da_k u_k) \in L \backslash \beta(L)$ since otherwise we would have

$$0 \neq dcfdcw + dcfda_k u_k + da_k fdcu_k + da_k fda_k u_k$$
$$= (dcu_{k-1} + da_k u_k)(fdcw + fda_k u_k) \in L\beta(L) = 0,$$

a contradiction. In any case there are $s, t \in S$, $s < t$ and nonzero $g, h \in A$ such that $gs + ht \in L \backslash \beta(L)$ and such that for every z in the interval $[s, t]$ and some $l \in A$ such that

$$(gl)g \neq 0, \ l(gz) + lht = (lz)(gs + ht) \in L \backslash \beta(L).$$

Let z_1 and z_2 be in the interval $[s, t]$ and suppose

$$l(gz_1) + lht + \beta(L) = l(gz_2) + lht + \beta(L).$$

Then

$$l(gz_1) - l(gz_2) \in \beta(L).$$

Hence

$$(gs + ht)\big(l(gz_1) - l(gz_2)\big) \in L\beta(L) = 0.$$

Thus

$$gl(gz_1) - gl(gz_2) + (hl)gt - (hl)gt = 0$$

which implies that $z_1 = z_2$. This shows that the mapping $z \longmapsto l(gz) + lht + \beta(L)$ is an injection of the interval $[s, t]$ into $L/\beta(L)$ which implies that $\left|L/\beta(L)\right| \geqslant \left|[s, t]\right| = \kappa$ and ends the proof. ∎

Let ε denote the class of all rings. A subring S of a ring A is called a left accessible subring of A if there exist subrings $S_0, \ldots, S_n$ of A such that $S = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n = A$ and $S_i \lhd_l S_{i+1}$ for every $i \in \{0, 1, \ldots, n-1\}$.

Theorem 2. *For every radical $\alpha \neq \varepsilon$ there exists a matric-extensible radical α' such that $\alpha \subset l(\alpha \cup \alpha') \subset \varepsilon$.*

Proof: We shall use similar arguments to those of ([**1**], proof of Theorem 5, pp. 252-253). Let α be a radical such that $\alpha \neq \varepsilon$. Let A be a nonzero semiprime and α-semisimple ring. The existence of such a ring A was shown in [**3**]. Let η_A be the class of all rings isomorphic to the homomorphic images of the left accessible subrings of A. Let $\mu = \eta_A \cup \beta$ and let $\alpha' = l_s(\mu)$. We shall show that α' is the required matric-extensible radical. Clearly, η_A is a homomorphically closed and left hereditary class of rings and so is β. Thus μ is a homomorphically closed and left hereditary class of rings. Moreover, $\mu \subseteq \{A : A^\circ \in \mu\}$, where A° denotes the ring with zero multiplication on the additive group of A. Therefore, it follows from ([**5**, Corollary 3]) that α' is left hereditary. Since

α' is also left strong, it follows from ([**4**, Theorem 4.9.6]) that α' is matric-extensible. Moreover, since $\alpha' \cup \alpha \subseteq l(\alpha \cup \alpha')$ and $A \in \alpha' \backslash \alpha$, it follows that $\alpha \subset l(\alpha \cup \alpha')$. To show that $l(\alpha \cup \alpha') \subset \varepsilon$, it is sufficient to build a nonzero $l(\alpha \cup \alpha')$-semisimple ring. Take a cardinal $\kappa > |A|$ and construct the semigroup ring $A(S)$. We shall show that $A(S)$ is $l(\alpha \cup \alpha')$-semisimple. Since A is semiprime and α-semisimple, Lemma 1 (a) implies that so is $A(S)$. Suppose $A(S)$ is not α'-semisimple. Then, since $A(S)$ is semiprime and the class μ is homomorphically closed and hereditary, it follows from ([**6**, Lemma 2.1 (ii)]) that $A(S)$ contains a left ideal L such that $0 \neq L/\beta(L) \in \mu$. Then $0 \neq L/\beta(L) \in \eta_A$ because $L/\beta(L) \notin \beta$. Then applying Lemma 1 (c) one gets $\left|L \backslash \beta(L)\right| \geqslant \kappa$. But this is impossible since clearly, for any $R \in \eta_A$ we have $|R| \leqslant |A| < \kappa$. Thus $A(S)$ is α'-semisimple and so $A(S)$ is $l(\alpha \cup \alpha')$-semisimple because $\mathcal{S}\big(l(\alpha \cup \alpha')\big) = \mathcal{S}(\alpha) \cap \mathcal{S}(\alpha')$. ∎

COROLLARY 3. *The lattice $\mathbb{L}_m$ of all matric-extensible radicals does not contain coatoms.*

PROOF: If $\alpha \neq \varepsilon$ is a matric-extensible radical, then so is the radical $\alpha \vee \alpha' = l(\alpha \cup \alpha')$, where α' is the matric-extensible radical built in the proof of Theorem 2. Moreover, it follows from Theorem 2 that $\alpha \subset \alpha \vee \alpha' \subset \varepsilon$. Thus the result follows. ∎

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