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THE QUOTIENT PROBLEM FOR NOETHERIAN RINGS

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1. Introduction. Our work was motivated by attempts to find a criterion for the existence of a classical quotient ring, for a noetherian ring, in analogy with the various known criteria for the existence of an artinian classical quotient ring ([**9**], [**10**], [**13**], [**2**]).

We have restricted our attention to Krull symmetric noetherian rings R, and we make heavy use of the fact that all their Krull composition factors are non-singular (Proposition 7). The collection Kprime R of the associated primes of the Krull composition factors of R plays a central role, taking the place of the collection of the associated primes of R.

Our main result (Theorem 19) allows the lifting of the property "classical" from factor rings modulo the Krull radicals. As an immediate corollary, we obtain our contribution to the classical quotient ring problem, again in terms of conditions on these factor rings (Corollary 20). A known criterion for the existence of an artinian classical quotient ring [**2**] is an immediate consequence (Corollary 21). Further applications concern the lifting of clans from these factor rings, under the assumption that R has a classical quotient ring.

We make two remarks relating to our basic concepts: we say that a noetherian ring R has a classical quotient ring, if $\mathscr{C}(0)$ is an Ore set, and if the Jacobson radical of the resulting semi-local ([**12**]) quotient ring has the AR-property. The second property holds automatically if the quotient ring is artinian, if the ring R is fully bounded, and in all known examples. It allows the use of the machinery of clans, in particular of Lemma 18.

No example of a noetherian ring R which is not Krull symmetric is known, either. This condition holds if R is fully bounded, or of Krull dimension one. We hope that it will, eventually, be derived from a weaker assumption, namely the equality of the left- and right-Krull dimensions of the factor rings R/P, which is true in many more cases. In the presence of this weaker assumption, Krull symmetry is equivalent to ideal invariance on the left and on the right [**2**].

2. Preliminaries. The rings under consideration are at least right-noetherian, and usually noetherian. $|M|$ denotes the Krull dimension, and

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$E(M)$ the injective hull, of the module M. For a prime ideal P, E_P is the indecomposable injective right-module a finite power of which is isomorphic to $E(R/P)$. We use $\operatorname{ann}_M I$ for the annihilator of the ideal I in M, and rt-ann M and rt-ass M, for the annihilator in R, and the collection of associated prime ideals, of the right-module M. For a bi-module B, we write

$$\text{ann } B = \text{lt-ann } B \cap \text{rt-ann } B$$

and

$$\text{ass } B = \text{lt-ass } B \cup \text{rt-ass } B.$$

For a subset $\mathfrak{A}$ of spec R, the collection of all prime ideals of R, we find the following notations convenient:

$$\max \mathfrak{A} = \{P \in \mathfrak{A}: P \text{ is maximal in } \mathfrak{A}\};$$
$$\text{her } \mathfrak{A} = \{Q \in \text{spec } R: \text{there exists } P \in \mathfrak{A} \text{ with } Q \subset P\};$$
$$\alpha\text{-}\mathfrak{A} = \{P \in \mathfrak{A}: |R/P| = \alpha\}$$
$$\mathfrak{A}/I = \{P \in \mathfrak{A}: P \supset I\} \text{ for an ideal } I; \quad \text{and}$$
$$\mathscr{C}(\mathfrak{A}) = \cap\{\mathscr{C}(P): P \in \mathfrak{A}\}.$$

We say that $\mathfrak{A}$ is *incomparable* if its members are pairwise incomparable, and that it is *α-homogeneous* if $|R/P| = \alpha$ holds for all $P \in \mathfrak{A}$. We call $\mathfrak{A}$ (*right-*)*Ore*, or (*right-*)*classical*, if the multiplicative set $\mathscr{C}(\mathfrak{A})$ has the respective property, and we denote the corresponding quotient ring by $R_{\mathfrak{A}}$. We regard $\mathfrak{A}/I$ as a set of prime ideals of R/I, as well as of R, and we use the phrase that it possesses a certain property "in R/I" or "in R".

The following lemmas collect auxiliary information concerning the localization of a right-noetherian ring R at a finite set $\mathfrak{A}$ of prime ideals. Similar observations were made independently by Stafford and Warfield.

LEMMA 1. *Let R be right-noetherian, let $\mathfrak{A}$ be a finite set of prime ideals, and let I be a right-ideal such that $I \cap \mathscr{C}(P) \neq \emptyset$ for all $P \in \mathfrak{A}$. Then $I \cap \mathscr{C}(\mathfrak{A}) \neq \emptyset$.*

Proof. The special case where $\mathfrak{A}$ is incomparable, is well known [**4**]. We index the members of $\mathfrak{A}$ as $P_1, \ldots, P_n$ in such a manner that the sequence of Krull dimensions $|R/P_i|$ is non-decreasing; and we show $I \cap \bigcap_{i=1}^{t} \mathscr{C}(P_i) \neq \emptyset$, by induction over t.

For $t = 1$, this is true by assumption. We consider $2 \leqq t \leqq n$, and we pick the smallest $s \leqq t$ such that $|R/P_s| = |R/P_t|$. We put $V = P_s \cap \ldots \cap P_t$, a semi-prime ideal, and $I' = I \cap P_1 \cap \ldots \cap P_{s-1}$, a right-ideal. By assumption of induction, there is an element

$$a \in I \cap \bigcap_{i=1}^{t-1} \mathscr{C}(P_i).$$

We have $|R/(I + V)| < |R/V|$, by the well known special case mentioned above, since $P_s, \ldots, P_t$ are pairwise incomparable and consequently $I \cap \mathscr{C}(V) \neq \emptyset$. Moreover, we obtain

$$\begin{aligned} |(I + V)/(I' + V)| &= |I/(I \cap I' + V)| \leqq |I/I'| \\ &= |I + (P_1 \cap \ldots \cap P_{s-1})/I| \leqq |R/(P_1 \cap \ldots \cap P_{s-1})| < |R/V|, \end{aligned}$$

and consequently $|R/(I' + V)| < |R/V|$.

This allows us to apply Lemma 2 of [**6**] to the semi-prime ring $\bar{R} = R/V$, the right-ideal $\overline{I'} = (I' + V)/V$ and the element $\bar{a}$. We obtain an element $b \in I'$ such that $|\bar{R}/(\bar{a} + \bar{b})\bar{R}| < |\bar{R}|$. Then, $a + b \in I + I' = I$, and $a + b \in \mathscr{C}(V) \subset \mathscr{C}(P_i)$ for $s \leqq i \leqq t$ since $|\bar{R}/(\bar{a} + \bar{b})\bar{R}| < |\bar{R}|$, and $a + b \in \mathscr{C}(P_i)$ for $i < s$ since $a \in \mathscr{C}(P_i)$ and $b \in I' \subset P_i$. We conclude

$$a + b \in I \cap \bigcap_{i=1}^{t} \mathscr{C}(P_i),$$

as desired.

(We mention that, although the proof of Lemma 1 in [**6**], on which Lemma 2 there is based, contains an error, the proof of the same lemma in [**11**], viz. (1.2), is correct.)

COROLLARY 2. *Let R be noetherian, $P_1, \ldots, P_n$ prime ideals and $c_i \in \mathscr{C}(P_i)$. Then, there exist $r_i \in R$ such that*

$$\sum_{i=1}^{n} c_i r_i \in \bigcap_{i=1}^{n} \mathscr{C}(P_i).$$

Proof. Apply Lemma 1 to the right-ideal $I = \sum_{i=1}^{n} r_i R$.

LEMMA 3. *Let R be right-noetherian, and let $\mathfrak{A}$ be a finite collection of prime ideals. Then, $\mathfrak{A}$ is right-Ore if and only if* $\max \mathfrak{A}$ *is right-Ore. Under this assumption, we have* $\mathscr{C}(\mathfrak{A}) = \mathscr{C}(\max \mathfrak{A})$, *and the quotient ring is semi-local with Jacobson radical* $\bigcap(\max \mathfrak{A})_{\mathfrak{A}}$.

Proof. If $\max \mathfrak{A}$ is right-Ore, then the quotient ring $R_{\max \mathfrak{A}}$ is semi-local, with prime ideals corresponding to the members of her $\max \mathfrak{A} =$ her $\mathfrak{A}$. Therefore, $\mathscr{C}(P) \supset \mathscr{C}(\max \mathfrak{A})$ holds for all $P \in \mathfrak{A}$ (in fact, for all $P \in$ her $\mathfrak{A}$), and we have $\mathscr{C}(\mathfrak{A}) = \mathscr{C}(\max \mathfrak{A})$. In particular, $\mathscr{C}(\mathfrak{A})$ is right-Ore.

Now we assume, conversely, that $\mathscr{C}(\mathfrak{A})$ is right-Ore. We consider any proper right-ideal M which is maximal among $\mathscr{C}(\mathfrak{A})$-closed right-ideals, and any $P \in \mathfrak{A}$. There arise three cases: either $P \not\subset M$ whence $P + M$ is $\mathscr{C}(\mathfrak{A})$-dense and therefore $M \cap \mathscr{C}(P) \neq \emptyset$, or $P \subset M$ and M/P is essential in R/P whence again $M \cap \mathscr{C}(P) \neq \emptyset$, or $P \subset M$ and M/P is not essential in R/P whence $R_{\mathfrak{A}}/P_{\mathfrak{A}}$ is simple artinian and $P_{\mathfrak{A}}$ is the bound of $M_{\mathfrak{A}}$.

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If, for fixed M and variable $P \in \mathfrak{A}$, only the first two cases occur, then Lemma 1 implies the contradiction $M \cap \mathscr{C}(\mathfrak{A}) \neq \emptyset$. Therefore, there exists $P \in \mathfrak{A}$ which produces the third case; and we conclude that every primitive right-ideal of $R_{\mathfrak{A}}$ is of the form $P_{\mathfrak{A}}$ for some $P \in \mathfrak{A}$, with $R_{\mathfrak{A}}/P_{\mathfrak{A}}$ simple artinian. It follows readily that $R_{\mathfrak{A}}$ is semi-local, with Jacobson radical $\bigcap(\max \mathfrak{A})_{\mathfrak{A}}$.

If $c \in \mathscr{C}(\max \mathfrak{A})$, then $\bar{c}$ is regular in the radical factor ring of $R_{\mathfrak{A}}$, hence is invertible in this ring, and in $R_{\mathfrak{A}}$ itself. This fact implies $c \in \mathscr{C}(P)$ for all $P \in$ her $\max \mathfrak{A}$, in particular for all $P \in \mathfrak{A}$, and we conclude $\mathscr{C}(\max \mathfrak{A}) = \mathscr{C}(\mathfrak{A})$. Consequently, $\mathscr{C}(\max \mathfrak{A})$ is right-Ore; and the proof is complete.

We recall from [7] that a finite incomparable right-Ore set of prime ideals, of a right-noetherian ring, is called *right-classical* if the Jacobson radical of the quotient ring has the right-AR-property. In view of Lemma 3, a finite right-Ore set $\mathfrak{A}$ can be replaced by the incomparable set $\max \mathfrak{A}$; and then the right-AR-property in question is equivalent to the condition

$$E_P = \bigcup_{n=1}^{\infty} \operatorname{ann}_{E_P}(A^n),$$

for $A = \bigcap \max \mathfrak{A}$ and all $P \in \max \mathfrak{A}$.

We mention that no semi-local noetherian ring is known whose Jacobson radical does not possess the AR-property. Therefore, no example of a finite Ore set is known which is not classical.

LEMMA 4. *Let R be right-noetherian, $\mathfrak{A}$ a finite set of prime ideals, and I an ideal. If $\mathfrak{A}$ is right-Ore (right-classical) in R, then $\mathfrak{A}/I$ is right-Ore (right-classical) in R/I.*

Proof. Consider elements $x \in R$ and $c \in \mathscr{C}(\mathfrak{A}/I)$. For all $P \in \mathfrak{A}$ with $I \not\subset P$, there exist elements $c_P \in I \cap \mathscr{C}(P)$; and therefore, by Corollary 2, there are $r, r_P \in R$ such that $c^* = cr + \sum c_P r_P \in \mathscr{C}(\mathfrak{A})$. From the right-Ore condition on $\mathscr{C}(\mathfrak{A})$, we obtain $x' \in R$ and $c' \in \mathscr{C}(\mathfrak{A})$ with $xc' = c^*x'$, and consequently $\bar{x}\bar{c}' = \bar{c}^*\bar{x}' = \bar{c}\bar{r}\bar{x}'$, since $c_P \in I$. This verifies the right-Ore condition for $\mathfrak{A}/I$ in R/I.

For $P \in \max \mathfrak{A}$ with $I \not\subset P$, we have $\bar{P}_{\mathfrak{A}/I} = \bar{R}_{\mathfrak{A}/I}$, and consequently

$$\overline{\bigcap(\max \mathfrak{A})_{\mathfrak{A}/I}} = \overline{\bigcap \max (\mathfrak{A}/I)_{\mathfrak{A}/I}}.$$

On the other hand, for any $P \in (\max \mathfrak{A})/I = \max(\mathfrak{A}/I)$, the annihilator $\operatorname{ann}_{E_P} I$ is the $\bar{R}$-module analogous to E_P; it is $\bar{R}$-injective and therefore an $\bar{R}_{\mathfrak{A}/I}$-module. For each $e \in E_P$, we have $e(\bigcap \max \mathfrak{A})^n = 0$ for suitable n, provided that $\mathfrak{A}$ is right-classical. If $e \in \operatorname{ann}_{E_P} I$, then we deduce

$$0 = e\overline{(\bigcap \max \mathfrak{A})^n}_{\mathfrak{A}/I} = e\overline{(\bigcap \max (\mathfrak{A}/I))^n}_{\mathfrak{A}/I} \supset e(\bigcap \max (\mathfrak{A}/I))^n,$$

verifying that $\mathfrak{A}/I$ is indeed right-classical in $\bar{R} = R/I$.

3. Krull primes. We develop here the fundamental properties of our main technical tool, the collection of Krull primes of a noetherian ring, elaborating on ideas in [**12**], and to some extend in [**2**].

The *Krull dimension sequence* $-1 = \kappa_0 < \kappa_1 < \ldots < \kappa_n = M$ of a finitely generated right-module M over a right-noetherian ring R, is the sequence of ordinals occuring as the Krull dimensions of non-zero submodules of M. The *Krull radicals* are the largest submodules K_i of the respective Krull dimensions κ_i. Each factor K_i/K_{i-1} is κ_i-homogeneous, which is to say that all its non-zero submodules have Krull dimension κ_i.

A *Krull series* of M is a series

$$0 = M_0 \subsetneqq M_1 \subsetneqq \ldots \subsetneqq M_n = M$$

of submodules such that the *Krull factors* M_k/M_{k-1} are critical, for a non-decreasing sequence of ordinals. The ordinals occurring here are just the κ_i of the Krull dimension sequence (possibly with repetitions). Every Krull series passes through the Krull radicals.

There is a Jordan Hölder Theorem for Krull series: the injective hulls of the Krull factors of two Krull series of M are pairwise isomorphic, after a suitable permutation. Therefore, the collection of the associated prime ideals P_k of the Krull factors M_k/M_{k-1} is independent of the Krull series. We denote this (finite) collection by rt-Kprime M, and we call it the set of (*right-*) *Krull primes* of M.

For a bi-module B, we write

$$K\text{prime } B = \text{lt-}K\text{prime } B \cup \text{rt-}K\text{prime } B$$

for the set of all its (left- or right-) Krull primes.

Krull primes were studied in [**3**], under the name of composition series primes. It is mentioned there (p. 500) that ass $M = K$prime M holds if R is commutative noetherian, and that rt-ass $M \subset$ rt-Kprime M is true in general. But the full force of this concept, as the appropriate generalization of associated primes, reveals itself only in its application to bimodules over Krull symmetric noetherian rings.

We shall reserve the term *bi-module* for a bi-module over a noetherian ring R which is finitely generated as left- and as right-module. A fundamental fact about such bi-modules is the equality of the Krull dimensions $|B| = |R/\text{rt-ann } B|$. A *subfactor of* R is a bi-module of the form U/V, where $V \subset U$ are ideals of R. The ring R is called *Krull symmetric* if the left- and right-Krull dimensions of each of its subfactors coincide [**2**].

From here on, we assume that all rings R are noetherian and Krull symmetric.

We call a critical (right-) module X, with associated prime ideal P, *non-singular* if it is a non-singular R/P-module, that is if $XP = 0$ and

$|X| = |R/P|$. With these concepts, we have the following fundamental observation:

LEMMA 5. *Every critical right-module X with $|X| = |R/\text{rt-ann } X|$ is non-singular.*

Proof. This is an immediate consequence of Theorem 14 in [**6**], in conjunction with Lemma 7 in [**8**], applied to the ring $R/\text{rt-ann } X$. (We mention that, though we cannot understand the proof of Theorem 14 in [**6**] as it stands, we have convinced ourselves that its statement is true.)

LEMMA 6. *Let B be a right-β-homogeneous bi-module. Then, all right-Krull factors of B are nonsingular, and* rt-Kprime B *is β-homogeneous and consists precisely of the minimal primes over* rt-ann B.

Proof. For a right-Krull factor X of B, with associated prime ideal P, we have $P \supset \text{rt-ann } X \supset \text{rt-ann } B$, and therefore

$$\beta = |X| \leqq |R/P| \leqq |R/\text{rt-ann } X| \leqq |R/\text{rt-ann } B| = |B| = \beta.$$

Consequently, X is non-singular by Lemma 5, and P is minimal over rt-ann B and satisfies $|R/P| = \beta$.

Moreover, we obtain $BP_s \dots P_1 = 0$, for the associated primes P_k of a right-Krull series of B. Therefore, for an arbitrary minimal prime Q over rt-ann B, we have $P_s \dots P_1 \subset \text{rt-ann } B \subset Q$, hence rt-ann $B \subset P_k \subset Q$ for some k, hence $Q = P_k \in$ rt-Kprime B.

PROPOSITION 7. *Let R have Krull dimension sequence κ_i and Krull radicals K_i $(i = 0, \dots, n)$. Then, all the Krull factors of R are non-singular,*

$$K \text{ prime } R = \bigcup_{i=1}^{n} \kappa_i\text{-}K \text{ prime } R,$$

and κ_i-Kprime R *consists precisely of the minimal primes over* $\text{ann}(K_i/K_{i-1})$.

Proof. Note that, by Krull symmetry, the Krull dimension sequences and Krull radicals, of R as left- and as right-module, coincide. We can apply Lemma 6 to the bi-modules K_i/K_{i-1}.

COROLLARY 8. *Every minimal prime of Rbe longs to* lt-Kprime R $\cap$ rt-Kprime R.

Proof. Due to the non-singularity of the Krull factors of R, we have $RP_s \dots P_1 = 0$, for the associated primes P_k of a right-Krull series of R. We obtain $P_s \dots P_1 \subset Q$, for each minimal prime Q, and therefore $P_k = Q$ for some k.

PROPOSITION 9. $\mathscr{C}(0) = \mathscr{C}(K\text{prime } R)$.

Proof. Consider $c \in \mathscr{C}(K\text{prime } R)$, and $x \in R$ with $xc = 0$. Let $0 = X_0 \subsetneqq \ldots \subsetneqq X_s = R$ be a right-Krull series, with right-Krull primes $P_1, \ldots, P_s$. If $x \in X_k$, then $\bar{x}c = 0$ holds in X_k/X_{k-1}, a non-singular R/P_k-module. Since $c \in \mathscr{C}(K\text{prime } R) \subset \mathscr{C}(P_k)$, we conclude $\bar{x} = 0$, or $x \in X_{k-1}$. Eventually, we obtain $x = 0$, proving $\mathscr{C}(K\text{prime } R) \subset \mathscr{C}(0)$.

For the converse, we consider any Krull prime P of R, for instance $P \in \kappa_i$-rt-Kprime R. We obtain the inclusions

$$\mathscr{C}(0) \subset \mathscr{C}(\text{rt-ann } K_i) \subset \mathscr{C}(\sqrt{\text{rt-ann } K_i}) \subset \mathscr{C}(P_k).$$

The first one holds since $c \in \mathscr{C}(0)$ and $K_ixc = 0$ implies $K_ix = 0$; the second one is well known ([**1**], (2.5)); and the last one is true since P is not only minimal over rt-ann(K_i/K_{i-1}), but also over rt-ann K_i, due to

$$|R/\text{rt-ann } K_i| = |K_i| = \kappa_i.$$

We conclude $\mathscr{C}(0) \subset \mathscr{C}(K\text{prime } R)$.

Our next two corollaries are related to [**12**]. The first one makes the same assertion as the main result of that paper, but we need the (tacit) extra assumption of Krull symmetry of R, and we do not need that the classical right-quotient ring is left-noetherian. With the help of Proposition 7, (2) in the second corollary is seen to be the same as the condition

$$\text{lt-ann } A(R) \cap \text{rt-ann } A(R) \subset J$$

in [**12**].

COROLLARY 10. *If R has a classical right-quotient ring, then this quotient ring is semi-local.*

Proof. Combine Proposition 9 with Lemma 3.

COROLLARY 11. *The following statements are equivalent*:
(1) *R is its own classical quotient ring;*
(2) rt-Primitive $R \subset 0$-Kprime R;
(3) *R is semi-local, and* max spec $R \subset K$prime R.

Proof. If we assume (1), then the equality of max spec R and max Kprime R follows from Proposition 9 and Lemma 3, and

$$\text{rt-Primitive } R = \text{max spec } R = 0\text{-spec } R$$

holds since R is semi-local. We conclude (2).

(2) implies (3), trivially. If (3) is given, then the Jacobson radical of R is $J = \cap$ max spec R, and the set U of invertible elements satisfies

$$U = \mathscr{C}(J) = \mathscr{C}(\text{max spec } R) \supset \mathscr{C}(K\text{prime } R) = \mathscr{C}(0).$$

We deduce (1).

4. Bimodule links. We recall from [**8**] our definition of a *bi-module link* $Q \sim P$ between prime ideals Q and P: such a link is provided by a subfactor X of R with lt-ann $X = Q$ and rt-ann $X = P$. Jategaonkar and Warfield have used variants of this definition; but all these concepts coincide, at least for Krull symmetric noetherian rings R, according to the next lemma.

LEMMA 12. *Let X provide a bi-module link $Q \sim P$. Then, there exists a bi-module factor $\bar{X}$, which also provides such a link, with the following additional properties: every proper bi-module factor of $\bar{X}$ has strictly smaller Krull dimension and strictly larger left- and right-annihilators; every non-zero bi-submodule of $\bar{X}$ is essential as left- and as right-submodule; and $\bar{X}$ is non-singular as R/Q-left-module and as R/P-right-module.*

Proof. Let Y be a maximal bi-submodule of X, with respect to $|X/Y| = |X|$. Obviously, every proper bi-module factor of $\bar{X} = X/Y$ has strictly smaller Krull dimension. From

$$|R/\text{rt-ann } \bar{X}| = |\bar{X}| = |X| = |R/P|$$

we obtain immediately rt-ann $\bar{X} = P$; and similarly we deduce rt-ann $\bar{\bar{X}} \supsetneq P$ for every proper bi-module factor $\bar{\bar{X}}$.

If the non-zero bi-submodule B of $\bar{X}$ is not right-essential, then we have a non-zero right-submodule U of $\bar{X}$ with $U \cap B = 0$. U embeds into $\bar{X}/B$, hence has strictly smaller right-Krull dimension. Then, the largest submodule of $\bar{X}$ of strictly smaller right-Krull dimension, a bi-submodule K, is non-zero and satisfies $|\bar{X}/K| = |X|$, a contradiction.

The singular submodule Z of $\bar{X}$, as right-R/P-module, is a bi-submodule with $|Z| < |R/P| = |\bar{X}|$. Therefore, we have $|\bar{X}/Z| = |\bar{X}|$, we conclude $Z = 0$, and the proof is complete.

If X provides a bi-module link between Q and P, and if $|X| = \alpha$, then $\{Q\} = \alpha$-lt-Kprime X and $\{P\} = \alpha$-rt-Kprime X. The next lemma is a useful converse of this observation.

LEMMA 13. *Let B be a subfactor of R, and let $P \in$ rt-Kprime B. Then, there exists a bi-module subfactor of B which provides a bi-module link $Q \sim P$ with some $Q \in$ lt-Kprime B.*

Proof. Replacing B by a subfactor, a quotient of Krull radicals, we may assume that B is α-homogeneous, for $\alpha = |R/P|$. Then, by Lemma 6, all Krull factors of B are non-singular, and all Krull primes lie in α-spec R.

Let N be a right-submodule of B, maximal with respect to $P \notin$ rt-Kprime N. N is automatically a bi-submodule: indeed we have $RNP_s \ldots P_1 = 0$, for the associated primes P_k of a right-Krull series of N, and therefore $P_s \ldots P_1 \subset P'$ for every $P' \in$ rt-Kprime RN. We conclude $P_k \subset P'$ for some k, hence $P_k = P'$ since both lie in α-spec R.

We claim rt-ass $B/N = \{P\}$. If, to the contrary, there exists $P \neq P' \in$ rt-ass B/N, then we can pick a critical right-submodule U/N with associated prime P'. We consider a right-Krull series $0 = U_0 \subsetneqq U_1 \subsetneqq \ldots \subsetneqq U_s = U$, with associated primes P_k. The modules $U_k \cap N$ form a right-Krull series for N, except that the inclusions between them need not all be strict. We conclude that the P_k corresponding to the strict inclusions are right-Krull primes of N, and therefore different from P. Thus, by the maximality of N, there must be at least one k with $U_k \cap N = U_{k-1} \cap N$, and we deduce

$$U/N \hookleftarrow U_k + N/N \cong U_k/U_k \cap N = U_k/U_{k-1} \cap N \twoheadrightarrow U_k/U_{k-1}$$

and

$$|U/N| \geqq |U_k/U_{k-1}| = \alpha.$$

Therefore, we can splice a right-Krull series of N with U/N, obtain a right-Krull series of U, and conclude

$$\text{rt-}K\text{prime } U = \text{rt-}K\text{prime } N \cup \{P'\} \not\ni P,$$

in contradiction to the maximality of N.

It follows now readily that B/N is α-homogeneous, and that we can find a bi-submodule whose right-annihilator is P, and whose left-annihilator is some left-Krull prime Q of B/N, hence of B.

LEMMA 14. *Let X be a finitely generated submodule of E_P with* $|R/\text{rt-ann } X| = |R/P|$. *Then, there exist bi-module links $P_k \sim P$, for all* $P_k \in$ rt-Kprime X.

Proof. Let $|R/P| = \alpha$. Then, X is α-homogeneous, its right-Krull factors are non-singular by Lemma 5, and its right-Krull primes lie in the α-stratum of spec R. With these observations, the second half of the proof of Theorem 15 in [**8**] goes through, verbatim.

LEMMA 15. *Let $\mathfrak{A}$ be a finite set of prime ideals which is left-Ore, and assume* her $\mathfrak{A} \ni Q \sim P$. *Then $P \in$* her $\mathfrak{A}$.

Proof. By assumption, there is a subfactor U/V of R with left-annihilator Q and right-annihilator P. If we suppose $P \notin$ her $\mathfrak{A}$, then there exists $c \in \mathscr{C}(\mathfrak{A}) \cap P$, by Lemma 1, and we have $Uc \subset V$. Since R is noetherian, there is n with $Vc^{-n} = Vc^{-(n+1)}$. We write $U = \sum_{j=1}^{m} u_jR$, and we obtain from the Ore condition an element $c' \in \mathscr{C}(\mathfrak{A})$ with $c'u_j = u_j'c^n$ for all j. Then, we have $u_j'c^{n+1} = c'u_jc \in V$, and therefore

$$u_j' \in Vc^{-(n+1)} = Vc^{-n}$$

hence $u_j'c^n \in V$ and

$$c'U = \sum_{j=1}^{m} u_j'c^nR \subset V.$$

We conclude $c' \in Q$, in contradiction to $Q \in$ her $\mathfrak{A}$.

COROLLARY 16. *Let R have Krull dimension sequence κ_i and Krull radicals K_i $(i = 0, \ldots, n)$, and let $\mathfrak{A}$ be a finite set of prime ideals such that $\mathfrak{A}/K_j$ is right-Ore in R/K_j. Assume $Q \sim P \in$ her $\mathfrak{A}$ and $|R/P| \geqq \kappa_j$. Then, $Q \in$ her $\mathfrak{A}$ or $P, Q \in \kappa_j$-Kprime R.*

Proof. We write $R/K_j = \bar{R}$ and $|R/P| = \alpha$. If the $\bar{R}$-subfactor $U + K_j/V + K_j$ has Krull dimension α, then its annihilators are precisely $\bar{Q}$ and $\bar{P}$, and it provides a bi-module link $\bar{Q} \sim \bar{P}$ in $\bar{R}$. Therefore, the right-left-analogue of Lemma 15, applied to $\bar{R}$, shows $Q \in$ her $\mathfrak{A}/K_j \subset$ her $\mathfrak{A}$.

If, on the other hand, the bi-module above has Krull dimension less than α, then $U \cap (V + K_j)/V$ has Krull dimension α and has P among its right-Krull primes. This bi-module is isomorphic to the subfactor $U \cap K_j/V \cap K_j$ of K_j. We conclude $\alpha = \kappa_j$, and we can splice right-Krull series of $V \cap K_j$ and of $U \cap K_j/V \cap K_j$, to obtain

$$P \in \text{rt-}K\text{prime } (U \cap K_j) \subset \text{rt-}K\text{prime } R.$$

Similarly, we show $Q \in \text{lt-}K\text{prime } R$.

5. Quotient rings.

We start by listing two lemmas, which were already proved, under somewhat stronger assumptions, in [**8**].

LEMMA 17. *If $\mathfrak{S}$ is a finite Krull homogeneous set of prime ideals, and if $E_P = \bigcup_{n=1}^{\infty} \text{ann}_{E_P}(S^n)$ holds for all $P \in \mathfrak{S}$, with $S = \bigcap \mathfrak{S}$, then $\mathfrak{S}$ is right-classical.*

LEMMA 18. *Every clan is Krull homogeneous.*

Proofs. The appropriate portions of the proofs of the Theorems 9 and 13 in [**8**] go through, once we observe that all Krull factors of E_P are non-singular, for $P \in \mathfrak{S}$, respectively for

$$P \in \mathfrak{C}^* = \{Q \in \mathfrak{C}\colon |R/Q| \text{ is maximal}\},$$

for a clan $\mathfrak{C}$.

But, if $\bar{X}$ is such a Krull factor, for a finitely generated submodule X of E_P, then $XS^n = 0$ by assumption, respectively $XC^n = 0$ since $C = \bigcap \mathfrak{C}$ is classical. We deduce

$$|R/P| \leqq |\bar{X}| \leqq |R/\text{rt-ann } \bar{X}| \leqq |R/S^n|,$$

respectively $\leqq |R/C^n|$, which equals $|R/P|$ in both cases. Therefore, we can apply Lemma 5.

THEOREM 19. *Let R have Krull dimension sequence κ_t and Krull radicals K_i $(i = 0, \ldots, n)$. Let $\mathfrak{A}$ be a finite incomparable set of prime ideals, and put*

$$\mathfrak{A}_t = \{P \in \kappa\colon \kappa_t \leqq |R/P| < \kappa_{t+1}\}.$$

Then, $\mathfrak{A}$ is classical in R, if and only if

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(1) $\mathfrak{A}_i/K_i$ *is classical in* R/K_i $(i = 0, \ldots, n-1)$, *and*

(2) *for bi-module linked Krull primes* Q *and* P, $P \in \mathfrak{A}$ *implies* $Q \in$ her $\mathfrak{A}$.

Proof. One direction is easy: if $\mathfrak{A}$ is classical in R, then $\mathfrak{A}_i$ has the same property by Lemma 18, and therefore $\mathfrak{A}_i/K_i$ is classical in R/K_i by Lemma 4. Moreover, her $\mathfrak{A}$ is bi-module link closed by Lemma 15, and consequently $Q \sim P \in \mathfrak{A}$ implies $Q \in$ her $\mathfrak{A}$.

We demonstrate the converse, by showing inductively that α-$\mathfrak{A}$ is classical in R. To this purpose, we fix $P \in \alpha$-$\mathfrak{A}$ and $e \in E_P$. We determine i by $\kappa_i \leqq \alpha < \kappa_{i+1}$, and we write $K = K_i$ and $V =$ rt-ann $(eR) \cap K$. Then, the fundamental observation is the following claim:

$$\text{lt-}K\text{prime } K/V \subset \alpha\text{-}\mathfrak{A}.$$

We may assume $K/V \neq 0$, since the claim is otherwise trivial. We have $|K/V| \leqq |K| = \kappa_i$. If U/V is a non-zero right-submodule, then $eRU \neq 0$ and we obtain a natural epimorphism $U/V \rightarrow erU \neq 0$. This implies

$$|U/V| \geqq |erU| \geqq |R/P| = \alpha \geqq \kappa_i,$$

and proves that α equals κ_i and that K/V is κ_i-homogeneous.

Consider any $Q \in$ lt-Kprime K/V. By Lemma 6, we have $|R/Q| = \kappa_i$. By Lemma 13, we obtain a bi-module link $Q \sim P' \in$ rt-Kprime K/V. For a right-Krull series $0 = X_0 \subsetneqq \ldots \subsetneqq X_t = eK$ with associated primes P_k, we notice

$$P_k \supset \text{rt-ann } (X_k/X_{k-1}) \supset \text{rt-ann } (eK) \supset \text{rt-ann } K,$$

and therefore

$$\begin{aligned} \kappa_i &= |K| = |R/\text{rt-ann } K| \geqq |R/\text{rt-ann } (eK)| \\ &\geqq |R/\text{rt-ann } (X_k/X_{k-1})| \geqq |R/P_k| \geqq |X_k/X_{k-1}| \geqq |R/P| = \alpha = \kappa_i. \end{aligned}$$

From Lemma 5, we deduce that all X_k/X_{k-1} are non-singular. We obtain $eKP_t \ldots P_1 = 0$, hence

$$KP_t \ldots P_1 \subset V, P_t \ldots P_1 \subset \text{rt-ann } K/V \subset P', \quad \text{and}$$
$$P_k \subset P' \quad \text{for some } k.$$

But we know $|R/P_k| = \kappa_i = |R/P'|$, and we conclude $P_k = P'$. Lemma 14 applies to the submodule eK of E_P, and provides a bi-module link $P_k \sim P$. Altogether, we have produced two bi-module links $Q \sim P' = P_k \sim P$.

By assumption and Lemma 18, α-$\mathfrak{A}/K$ is classical in R/K. Therefore, $P' \sim P \in \alpha$-$\mathfrak{A}$ and Corollary 16 imply $P' \in$ her $(\alpha$-$\mathfrak{A})$, or $P', P \in K$prime R. In the second case, the assumptions in Theorem 19 allow us to conclude $P' \in \text{her}\,\mathfrak{A}$, too. Now suppose $P' \notin \mathfrak{A}$; then $P' \subsetneq P'' \in \mathfrak{A}$; and therefore $P'' \in \mathfrak{A}_j$ for some $j < i$, since $|R/P''| < |R/P'| = \kappa_i$. But $\mathfrak{A}_j$ is classical in R by assumption of induction, and Lemma 15 yields

$$P \in \text{her}\,\mathfrak{A}_j \cap \alpha\text{-}\mathfrak{A}.$$

This is a contradiction, since $\mathfrak{A}$ is incomparable. The conclusion is $P' \in \alpha\text{-}\mathfrak{A}$.

The same argument can be repeated, for the link $Q \sim P'$, and we arrive at $Q \in \alpha\text{-}\mathfrak{A}$, proving the claim.

With $A = \bigcap \alpha\text{-}\mathfrak{A}$, the claim shows $A \subset Q$ for every $Q \in$ lt-Kprime K/V. Since, by Lemma 6, K/V is annihilated from the left by an appropriate product of its left-Krull primes, we obtain $A^sK \subset (\Pi Q)K \subset V$ and $eA^sK = 0$. This implies that eA^s is contained in $\text{ann}_{E_P}(K \cap P)$, the $R/(K \cap P)$-injective module analogous to E_P.

We define $\mathfrak{A}_P = \alpha\text{-}\mathfrak{A}/K$ if $K \subset P$, and $\mathfrak{A}_P = \{P\}$ if $K \not\subset P$. We claim that $\mathfrak{A}_P$ is classical in $R/(K \cap P)$. In the first case, this is true by Lemma 18 and the assumptions of Theorem 19 (note that for $i = n$, we have $K = R$ and therefore the first case does not arise). In the second case, we argue as follows: we can pick an element $c^* \in K \cap \mathscr{C}(P)$. Then, if $c \in \mathscr{C}(P)$ and $x \in R$ are given, we obtain $cc^* \in \mathscr{C}(P)$ and $xc^* \in R$, and we find elements $c' \in \mathscr{C}(P)$, $x' \in R$ and $p \in P$ with $(xc^*)c' = (cc^*)x' + p$, by the Ore condition on $\mathscr{C}(P)$ in R/P. From $c^* \in K$ we get $p \in K \cap P$, and we obtain $x(c^*c') \equiv c(c^*x')$ in $R/(K \cap P)$. Thus, $\mathscr{C}(P)$ is Ore in $R/(K \cap P)$. Since P is minimal over $K \cap P$, it gives rise to the only prime ideal of the localization. This prime ideal is therefore the prime radical, hence is nilpotent and possesses, trivially, the AR-property. We conclude that $\{P\}$ is classical in $R/(K \cap P)$, as claimed.

In any case, we can find t such that $(\bigcap \mathfrak{A}_P)^t$ annihilates the finitely generated submodule eA^s of $\text{ann}_{E_P}(K \cap P)$. Since $\mathfrak{A}_P \subset \alpha\text{-}\mathfrak{A}$, we deduce $eA^{s+t} = 0$. With Lemma 17, this demonstrates that $\alpha\text{-}\mathfrak{A}$ is classical, and completes the proof.

We are going to apply Theorem 19 in the two instances where the second condition is trivially satisfied: if Kprime $R \subset \text{her}\,\mathfrak{A}$, and if Kprime $R \cap \mathfrak{A} = \emptyset$. If $\mathfrak{A}$ is left- or right-Ore, then the first instance occurs if and only if $\mathscr{C}(\mathfrak{A})$ consists of regular elements, as is easily seen with Proposition 9 and Lemma 3. The simplest case of this first instance arises if we take $\mathfrak{A} = K$prime R. With Propositions 7 and 9, we obtain the main result of this paper:

COROLLARY 20. *R has a classical quotient ring, if and only if κ_i-Kprime $(R)/K_i$ is classical in R/K_i $(i = 1, \ldots, n - 1)$.*

The most obvious situation where this corollary applies happens if κ_i-Kprime $(R)/K_i = \emptyset$ for all i. The next result contains a new proof of Theorem 18 of [**2**], and a formal improvement upon Theorem 2 of [**5**] (since it follows from Proposition 7 that every maximal Krull prime is contained in, but not necessarily equal to, a maximal middle annihilator).

COROLLARY 21. *The following statements are equivalent:*
(1) κ_i-(max) Kprime $(R)/K_i = \emptyset$ $(i = 1, \ldots, n-1)$;
(2) *R has an artinian classical quotient ring;*
(3) *the Krull primes of R are (precisely the) minimal primes.*

Proof. (1) implies (2): By Corollary 20, a classical quotient ring exists. By Proposition 9 and Lemma 3, it is artinian once we know that all Krull primes are minimal. But if $P \in \kappa_i$-Kprime R is not minimal, we obtain $P \subsetneqq P'$ hence $\kappa_i = |R/P| < |R/P'|$. If $K_i \not\subset P'$, then the non-zero module $K_i/(K_i \cap P')$ embeds into the Krull homogeneous module R/P', and we deduce the contradiction

$$\kappa_i = |K_i| \geqq |K_i/(K_i \cap P')| = |R/P'|.$$

Therefore, we have $K_i \subset P' \subset P$, contrary to (1).

That (2) implies (3) is obvious, from Proposition 9.

(3) implies (1): If $P \in \kappa_i$-Kprime $(R)/K_i$, then P is minimal in R by (3). Therefore, $\bar{P}$ is minimal in $\bar{R} = R/K_i$, hence is a Krull prime of $\bar{R}$ by Corollary 8. On the other hand, Proposition 7 says

$$K \text{ prime } \bar{R} = \bigcup_{j=i+1}^{n} \kappa_j\text{-}K \text{ prime } R,$$

and we have a contradiction.

The arguments used in the preceding proof show that a prime ideal P with $|R/P| \geqq \kappa_j$ contains K_j if and only if it is not a minimal prime of R with $|R/P| = \kappa_j$. For such P, the localization behavior in R can often be determined in R/K_j.

COROLLARY 22. *Let P be a non-minimal prime, with $\kappa_j \leqq |R/P| < \kappa_{j+1}$. If P belongs to a clan in R/K_j, and if this clan does not contain any Krull primes of R, then P belongs to a clan in R, and these two clans are identical.*

COROLLARY 23. *Let P be as before, and assume in addition that all the members of κ_j-Kprime R belong to clans in R. If P belongs to a clan in R/K_j, then it belongs to a clan in R.*

Proof. The first corollary follows from Theorem 19, taking for $\mathfrak{A}$ the clan of P in R/K_j. The second one is proved in the same manner, using for $\mathfrak{A}$ the union of the clan of P in R/K_j and the clans of the members of κ_j-Kprime R in R.

Remarks. 1. If R has an artinian classical quotient ring, then all the minimal primes belong to clans in R, and a non-minimal prime belongs to a clan in R if and only if it belongs to a clan in R/K_j, with these two clans then being identical. (This follows immediately from the Corollaries 21 and 22.)

2. The additional assumption in Corollary 23 can also be checked in the factor ring R/K_j: the members of κ_j-Kprime R belong to clans in R, if and only if the members of κ_j-Kprime $(R)/K_j$ belong to clans in R/K_j. (This is shown by applying Theorem 19 to the union $\mathfrak{A}$ of the set of minimal primes of R in the κ_j-stratum and the clans of the members of κ_j-Kprime $(R)/K_j$ in R/K_j.)

Example. Let A be the localization of the commutative ring $k[x, y, z]/\langle xy, xz\rangle$ at the maximal ideal generated by $\bar{x}$, $\bar{y}$ and $\bar{z}$; and let σ, φ and ψ be the automorphisms of A, $A/\langle\bar{y}\rangle$ and $A/\langle\bar{z}\rangle$ which are induced by $x^\sigma = x$, $y^\sigma = y + z$, $z^\sigma = z$ and $x^\varphi = z$, $z^\varphi = x$ and $x^\psi = y$, $y^\psi = x$, respectively.

Our example is the trivial split-extension $R = A \rtimes B$, of the bi-module

$$B = {}_\sigma A \oplus {}_\varphi A/\langle\bar{y}\rangle \oplus {}_\psi A/\langle\bar{z}\rangle \oplus A/\langle\bar{x}, \bar{y}, \bar{z}\rangle$$

with the left-multiplications modified by the indicated automorphisms, by A.

R is a noetherian PI-ring of Krull dimension two, and it is its own classical quotient ring. Since $B^2 = 0$, we may identify the spectra of R and A. With the abbreviations $P_n = \langle\bar{x}, \bar{y} + n\bar{z}\rangle$, $P = \langle\bar{x}, \bar{z}\rangle$ and $Q = \langle\bar{y}, \bar{z}\rangle$, the Krull primes of R are $\langle\bar{x}, \bar{y}, \bar{z}\rangle$, P_0, P, Q and $\langle\bar{x}\rangle$. A certain link component in the 1-stratum looks as follows,

$$\begin{array}{c} \ldots\ldots \sim P_{-1} \sim P_0 \sim P_1 \sim P_2 \sim \ldots\ldots \\ \wr \\ Q \\ \wr \\ P \end{array}$$

where the top row is infinite in characteristic zero, and is a finite cycle in positive characteristic. Since Q does not contain the corresponding Krull radical

$$K = \langle\bar{x}\rangle \rtimes (\langle\bar{x}\rangle \oplus A/\langle\bar{y}\rangle \oplus A/\langle\bar{z}\rangle \oplus A/\langle\bar{x}, \bar{y}, \bar{z}\rangle),$$

this link component decomposes in R/K into two parts, viz. the top row and the singleton P.

In positive characteristic, all primes of R belong to clans, hence Corollary 23 applies, but the clan of P in R is larger than the one in R/K. In characteristic zero, the Krull primes P_0, P and Q do not belong to clans in R, hence Corollary 23 does not apply; and indeed P belongs to a clan in R/K but not in R.

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