Canad. Math. Bull. Vol. **31** (2), 1988

THE SOLUTION TO A PROBLEM OF GRÜNBAUM

BY

PETER SALAMON AND PAUL ERDÖS

ABSTRACT. The paper characterizes the set of all possible values for the number of lines determined by n points for n sufficiently large. For $\binom{k}{2} \leqq (n - k)$, the lower bound of Kelly and Moser for the number of lines in a configuration with $n - k$ collinear points is shown to be sharp and it is shown that all values between $M_{\min}(k)$ and $M_{\max}(k)$ are assumed with the exception of $M_{\max} - 1$ and $M_{\max} - 3$. Exact expressions are obtained for the lower end of the continuum of values leading down from $\binom{n}{2} - 4$. In particular, the best value of $c = 1$ is obtained in Erdös' previous expression $cn^{3/2}$ for this lower end of the continuum.

In the paper below we characterize for large n the possible values of the number of connecting lines determined by a set P_n of n points in the plane, where a connecting line is any straight line containing at least two points of P_n. This solves a problem posed by B. Grünbaum [5, 6] which asks for the sequence of all integers m with the property that some configuration of n points determine exactly m lines. The approach of the present paper is likely to prove useful also for the related problems discussed in Grünbaum [6] and Cordovil [2].

Besides its significance for combinatorial geometry, the problem is also of interest as an example to help elucidate the connection between statistical physics and the "spectrum" of values for a combinatorial problem. In fact the possible values for $n = 22$ through $n = 28$ shown in figure 1 can be seen to bear a strong resemblance to physical spectra. A similar structure has been observed for the problem of possible values of the permanent for (0, 1) matrices [3]. This connection with statistical physics is expected to prove useful in certain implementations of simulated annealing. In keeping with this analogy, our analysis proceeds in "bands", where the k-th band consists of those configurations in which the largest number of collinear points has $n - k$ elements.

The problem requires a careful analysis only for the case when k is small, i.e. of order $\sqrt{n}$ or less. For such k, the values for the number of connecting lines determined by configurations in the k-th band do not overlap with values

Received by the editors March 19, 1986, and, in final revised form, March 4, 1987.

AMS Subject Classification (1980): 05-A15.

https://doi.org/10.4153/CMB-1988-020-2 Published online by Cambridge University Press

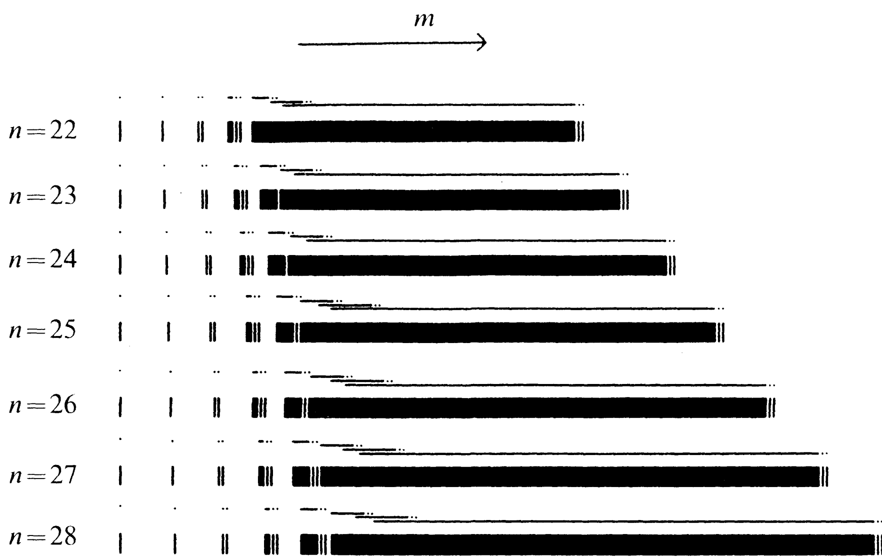

FIGURE 1 — Possible values of m for n between 22 and 28. The values for different k are shown on different lines to display the overlap. Below these the overlapped values are shown with each point fattened to a vertical bar to display the similarity to physical spectra. This figure may be omitting some m values from the lower end of high k bands.

from other bands. The largest number of lines m in the k-th band, i.e. with $n - k$ points known to lie on a line, can easily be seen to be $M_{\max}(k) = k(n - k) + \binom{k}{2} + 1$, which results when the remaining k points are in general position. The smallest m in the k-th band is known to be bounded below by $M_{\min}(k) = k(n - k) - \binom{k}{2} + 1$ [7]. Another purpose of the presentation below will be to show that for $\binom{k}{2} \leqq (n - k)$ this lower bound is sharp and that m assumes all values between $M_{\min}(k)$ and $M_{\max}(k)$ with the exception of $M_{\max} - 1$ and $M_{\max} - 3$.

For larger k, the bands overlap and all values are assumed up to $\binom{n}{2}$ except $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$. The fact that $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$ do not occur was observed by Grünbaum and follows by noting that if three of the points are collinear while the other points are in general position we get $\binom{n}{2} - 2$ lines while if two sets of three points are collinear we get $\binom{n}{2} - 4$ and for four collinear points we get $\binom{n}{2} - 5$. P. Erdös has shown that except for these two values, $\binom{n}{2} - 1$ and $\binom{n}{2} - 3$, all values occur between $cn^{3/2}$ and $\binom{n}{2}$ [5]. The best value of c is one of the results presented below.

The structure of a band for large k still remains elusive. The upper bound $M_{\max}(k)$ holds sharply for all k. Our methods show that the upper portions of the large k bands except for $M_{\max}(k) - 1$ and $M_{\max}(k) - 3$ are again "continua", i.e. include all integers in an interval. The missing, and apparently difficult, information concerns the minimum value in the bands with large

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values of k. While the bound $M_{\min}(k)$ of Kelly and Moser remains valid, it loses sharpness and eventually becomes negative. For our purposes it will be sufficient to show that the large k bands overlap and that they do not stretch down into the discrete region.

We remark that our approach of focusing on the values of m assumed in the k-th band is far easier than the related question of asking for the minimum value of m on all bands from some k on. In this direction, Erdös established that if all the points are not collinear, i.e. $k \geqq 1$, then they determine at least n lines. He further conjectured that if no $n - 1$ of the n points are collinear, then the resulting configurations define at least $2n - 4$ lines. Elliot [4] proved this for $n \geqq 10$ while Kelly and Moser [7] proved a more general result to the effect that if *at most* $n - k$ points are collinear with $n \geqq (3(3k - 2)^2 + 3k - 1)/2$, then the n points determine at least $k(n - k) - \binom{k}{2} + 1$ lines. Unfortunately their restriction on k is too strong to be useful for us. Instead we will make use of another, more recently proved conjecture of Erdös due to Beck [1, 8] which says that in any configuration with $k \geqq x$, the number of connecting lines is greater than $cx(n - x)$, where c is an absolute constant.

We now proceed to demonstrate our results through a sequence of lemmas. The key technique employed in the proofs will be to slide points on the "large line" with $n - k$ points into coincidence with a connecting line determined by the k points off the line. (See figure 2.) For convenience in these arguments, we let P_{n-k} denote the set of points on the large line and P_k denote $P_n - P_{n-k}$. We also drop the adjective "connecting" when referring to the lines of P_n.

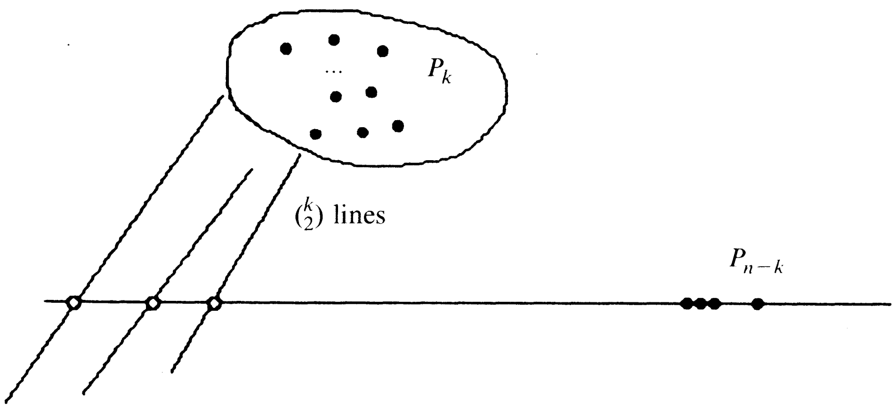

FIGURE 2 – The configuration which achieves $M_{\max}(k)$ with $n - k$ collinear points and k points in general position. These k points determine $\binom{k}{2}$ lines of intersection with the line of P_{n-k} and thus create holes for the points of P_{n-k} to fill.

LEMMA 1. *For all* $0 \leqq k \leqq n - 2$, *the maximum number of lines in the* k-th *band is* $M_{\max}(k) = k(n - k) + \binom{k}{2} + 1$. *The minimum number of lines is bounded below by* $M_{\min}(k) = k(n - k) - \binom{k}{2} + 1$.

PROOF. As stated above, the upper bound is trivial. The lower bound is the result of Kelly and Moser [7]. Their result follows by counting tie lines of the configuration, i.e. lines which connect points of P_{n-k} and points of P_k. There are $k(n-k)$ (possibly coincident) tie lines. We can get a bound on the amount of coincidence between such lines by noting that if $a, b \in P_k$, then they share at most one tie line. Thus there are at least $k(n-k) - \binom{k}{2}$ distinct tie lines and thus at least $k(n-k) - \binom{k}{2} + 1$ connecting lines, counting the large line of the points in P_{n-k}. Note that, with the usual interpretation that $\binom{0}{2}$ and $\binom{1}{2}$ are zero, the lemma remains valid for $k = 0$, and $k = 1$.

LEMMA 2. *All values in the k-th band, for $n \geqq k(k+1)/2$, between the largest m value $M_{\max}(k)$ and the smallest m value $M_{\min}(k)$ are realized except for* $M_{\max} - 1$ and $M_{\max} - 3$.

PROOF. We argue from the configuration depicted in figure 2 which achieves $M_{\max}$, i.e. one in which the points of P_k are in general position, determining $\binom{k}{2}$ lines, the points of P_{n-k} are in general position on their line ensuring $k(n-k)$ tie lines plus the large line of the P_{n-k} giving a total of $k(n-k) + \binom{k}{2} + 1$ lines. We consider the configurations which can be made by moving the points in P_{n-k} into special position so as to become coincident with one of the $\binom{k}{2}$ lines determined by the P_k. Each such move decreases the number of lines by two. Assuming there are enough points among the P_{n-k} to move one on to each of the $\binom{k}{2}$ lines determined by the points in P_k, then the number of lines decreases in increments of two to a value $M_{\max}(k) - 2\binom{k}{2} = M_{\min}(k)$.

To get the remaining values we start from a configuration with $m = M_{\max} - 2$ obtained by making three points of P_k collinear. Our moves will still consist only of moving points of P_{n-k} into coincidence with lines determined by the P_k. When a point is brought into coincidence with the line containing three points of P_k, the number of lines drops by three. When a point is brought into coincidence with one of the other lines determined by exactly two of the P_k, the number of lines again drops by two. Thus all values of the form $(M_{\max} - 2) - 2j$ and $(M_{\max} - 2) - 2j - 3$ for $j = 1, \ldots \binom{k}{2} - 3$ are generated. Except for $M_{\max} - 3$, all integers between $M_{\min} + 1$ and $M_{\max} - 2$ are of this form.

By comparing $M_{\max}(k)$ and $M_{\min}(k+1)$ for small k, we have $M_{\max}(k) < M_{\min}(k+1)$. Eventually, the reverse inequality holds. The transition from the discrete bands in the low k values to the "continuum" of values up to $\binom{n}{2} - 3$ occurs with this first overlap. We note here only that such overlap takes place in the $k = [\sqrt{n+2}]$ band and postpone a careful discussion of this first overlap until after Lemma 3 which gives some limited information concerning the structure of the bands for k values beyond $n \geqq k(k+1)/2$.

LEMMA 3. *For $n \leqq k(k+1)/2$ and $k \leqq n - 3$ all values in the interval between $M_{\max}(k)$ and $M_{\max}(k) - 2(n-k)$ are taken on except $M_{\max} - 1$ and $M_{\max} - 3$.*

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PROOF. Same as above except that $\binom{k}{2} \geqq n - k$ and thus we run out of points on P_{n-k} which can be move into coincidence with the $\binom{k}{2}$ lines of the P_k. Thus these moves can only move us down $n - k$ steps of 2. Although in this case such moves do not move us to the bottom of the band, they are sufficient to demonstrate the overlap between the bands by noting that $M_{\max}(k) - 3 \geqq M_{\max}(k + 1) - 2(n - k)$ for all $k < n - 2$.

For the bands with $k = n - 2$ and $k = n - 3$, these arguments do not give information about the bands' structure since moving the points of P_{n-k} into coincidence with the lines of P_k moves us to a configuration belonging to a different band. The arguments do however suffice to deduce that all values with the exception of $M_{\max}(k) - 1$ and $M_{\max}(k) - 3$ between $M_{\max}(k)$ and $M_{\max}(k) - 2(n - k)$ occur in some band. In fact for $k = n - 2$ only the single value $M_{\max}(n - 2) = \binom{n}{2}$ is possible, while for $k = n - 3$ only values of the form $M_{\max}(n - 3) - 2j$ are present.

The problem of the bottoms of these bands need be considered only to show that they remain sufficiently large to stay out of the discrete region.

LEMMA 4. *For n sufficiently large, any configuration in a band with* $k > [\sqrt{n + 2}]$ *has more than* $M_{\max}([\sqrt{n + 2}] - 1)$ *lines.*

PROOF. On reexamining the result of Kelly and Moser which gives a lower bound $M_{\min}(k) = k(n - k) - \binom{k}{2} + 1$ to the values in a band, we note that these lower bounds are increasing until $k = [(n + 0.5)/3]$. Furthermore, for $k > [\sqrt{n + 2}]$, they exceed $M_{\max}([\sqrt{n + 2}] - 1)$. Thus we need only worry about bands with $k > n/3$. But by the result of Beck [1] such configurations must give rise to at least $c(n/3)(2n/3)$ lines which for sufficiently large n exceeds $M_{\max}([\sqrt{n + 2}] - 1)$ which grows as $n^{3/2}$.

The structure is thus a sequence of non-overlapping bands until $k = [\sqrt{n + 2}]$ at which time a transition occurs to a continuum of values which persist until $\binom{n}{2} - 4$. The remaining two values at the top are $\binom{n}{2} - 2$ and $\binom{n}{2}$. A careful examination of the first overlap requires us to break the analysis into five cases according to the extent of the overlap between the $k = [\sqrt{n + 2}] - 1$ band and the $k = [\sqrt{n + 2}]$ band. This is conveniently done by means of the function $f(n) = [\sqrt{n + 2}]^2 - n \leqq 2$. As illustrated in figure 3, the values of f lead to the cases

CASE 1	$f(n) = 2$	$M_{\max}([\sqrt{n + 2}] - 1) - 2 = M_{\min}([\sqrt{n + 2}])$
CASE 2	$f(n) = 1$	$M_{\max}([\sqrt{n + 2}] - 1) - 1 = M_{\min}([\sqrt{n + 2}])$
CASE 3	$f(n) = 0$	$M_{\max}([\sqrt{n + 2}] - 1) = M_{\min}([\sqrt{n + 2}])$
CASE 4	$f(n) = -1$	$M_{\max}([\sqrt{n + 2}] - 1) + 1 = M_{\min}([\sqrt{n + 2}])$
CASE 5	$f(n) < -1$	$M_{\max}([\sqrt{n + 2}] - 1) + 1 < M_{\min}([\sqrt{n + 2}])$.

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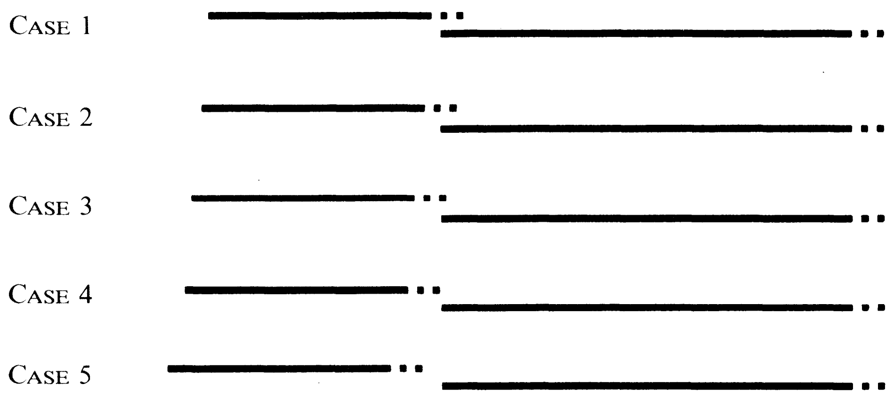

FIGURE 3 – Overlap between the $k = [\sqrt{n+2}] - 1$ and $k = [\sqrt{n+2}]$ bands illustrating the five cases.

The last gap before the continuum can therefore be seen to be

CASES 1 & 2 continuum $= \{m; M_{\max}([\sqrt{n+2}] - 1) - 3 < m < \binom{n}{2} - 3\}$

CASES 3 & 4 continuum $= \{m; M_{\max}([\sqrt{n+2}] - 1) - 1 < m < \binom{n}{2} - 3\}$

CASE 5 continuum $= \{m; M_{\min}([\sqrt{n+2}]) - 1 < m < \binom{n}{2} - 3\}$.

From these expressions we obtain the best value of $c = 1$ in the $cn^{3/2}$ bound to the bottom of the continuum proved by Erdös [5]. We can also get the $m_i^{(n)}$ of Grünbaum's problem. We first note that for $k \geqq 3$, the number of values in a band is $2\binom{k}{2} - 1$. Summing these from 3 to j and adding 4 for the first three bands we find that there are $h(j) = 4 + j(j+2)(j-2)/3$ values in the first $j + 1$ bands for $j \geqq 2$.

$$m_1^{(n)} = 1$$
$$m_2^{(n)} = n$$
$$m_3^{(n)} = 2n - 4$$
$$m_4^{(n)} = 2n - 2.$$

For $i > 4$, we determine a j such that i is between $h(j)$ and $h(j+1)$. Provided the resulting j is less than $[\sqrt{n+2}] - 1$, we use

(1a) $m_i^{(n)} = M_{\max}(j) + i - h(j) - 2 \qquad h(j-1) < i \leqq h(j) - 2$

(1b) $m_i^{(n)} = M_{\max}(j) - 2 \qquad i = h(j) - 1$

(1c) $m_i^{(n)} = M_{\max}(j) \qquad i = h(j)$.

For larger j, we again have to distinguish between the five cases.

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CASES 1 & 2. $f(n) = [\sqrt{n+2}]^2 - n = 1, 2.$

For

$$h([\sqrt{n+2}] - 2) < i \leqq h([\sqrt{n+2}] - 1) - 2$$
$$m_i^{(n)} = M_{\max}([\sqrt{n+2}] - 1) + i - h([\sqrt{n+2}] - 1) - 2$$

for

$$h([\sqrt{n+2}] - 1) - 2 < i \leqq h([\sqrt{n+2}] - 1) - 3 + \binom{n}{2} - M_{\max}([\sqrt{n+2}] - 1)$$
$$m_i^{(n)} = M_{\max}([\sqrt{n+2}] - 1) + i - h([\sqrt{n+2}] - 1) - 1,$$

and finally for

$$i = h([\sqrt{n+2}] - 1) - 2 + \binom{n}{2} - M_{\max}([\sqrt{n+2}] - 1)$$
$$m_i^{(n)} = \binom{n}{2} - 2$$

and for

$$i = h([\sqrt{n+2}] - 1) - 1 + \binom{n}{2} - M_{\max}([\sqrt{n+2}] - 1)$$
$$m_i^{(n)} = \binom{n}{2}.$$

CASES 3 & 4. $f(n) = [\sqrt{n+2}]^2 - n = -1, 0.$

Formulas (1) apply for $j = [\sqrt{n+2}] - 1$.

For

$$h([\sqrt{n+2}] - 1) < i \leqq h([\sqrt{n+2}] - 1) - 4 + \binom{n}{2} - M_{\max}([\sqrt{n+2}] - 1)$$
$$m_i^{(n)} = M_{\max}([\sqrt{n+2}] - 1) + i - h([\sqrt{n+2}] - 1),$$

and finally for

$$i = h([\sqrt{n+2}] - 1) - 3 + \binom{n}{2} - M_{\max}([\sqrt{n+2}] - 1)$$
$$m_i^{(n)} = \binom{n}{2} - 2$$

and for

$$i = h([\sqrt{n+2}] - 1) - 2 + \binom{n}{2} - M_{\max}([\sqrt{n+2}] - 1)$$
$$m_i^{(n)} = \binom{n}{2}.$$

CASE 5. $f(n) = [\sqrt{n+2}]^2 - n < -1.$

Formulas (1) apply for $j = [\sqrt{n+2}] - 1$.

For

$$h([\sqrt{n+2}]-1) < i \leqq h([\sqrt{n+2}]-1) - 3 + \binom{n}{2} - M_{\min}([\sqrt{n+2}])$$

$$m_i^{(n)} = M_{\min}([\sqrt{n+2}]) + i - h([\sqrt{n+2}]-1) - 1,$$

and finally for

$$i = h([\sqrt{n+2}]-1) - 2 + \binom{n}{2} - M_{\max}([\sqrt{n+2}]-1)$$

$$m_i^{(n)} = \binom{n}{2} - 2$$

and for

$$i = h([\sqrt{n+2}]-1) - 1 + \binom{n}{2} - M_{\max}([\sqrt{n+2}]-1)$$

$$m_i^{(n)} = \binom{n}{2}.$$

These formulas allow us to construct tables of m values for sufficiently large values of n. A graphic form of such a table is shown in figure 4 where for purposes of illustrating the trends embodied in these formulas we have ignored the requirement that n be large. We note that as n varies, the bands show up as lines of fixed width $2\binom{k}{2} - 1$ with slope k. In particular, this implies that they move apart by 1 for a unit increase in n. Thus if n_0 is case 1, then $n_0 + 1$ is case 2,

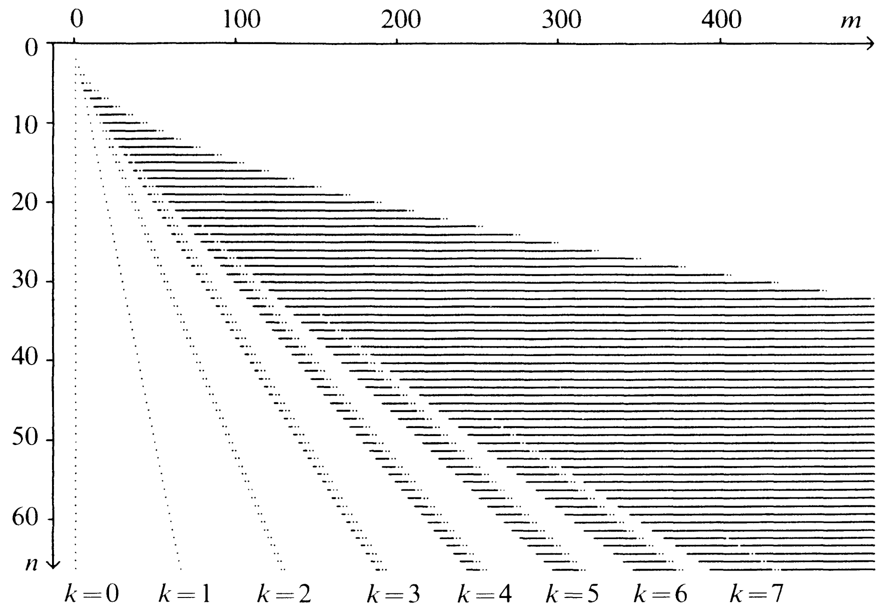

FIGURE 4 – Values of m versus n showing the structure of the solutions to the large n formulas. Actual solutions for small n include additional m values.

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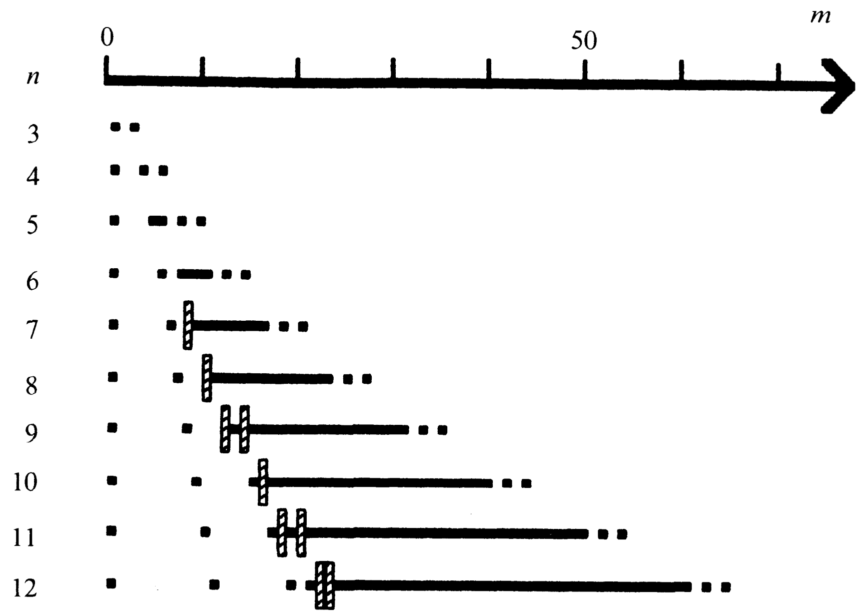

FIGURE 5 – Possible values of m for $n \leqq 12$. Values predicted from formulas for large n are shown as before; "extra" values from low end of high k bands are shown crosshatched and elongated.

$n_0 + 2$ is case 3, $n_0 + 3$ is case 4, and $n_0 + j$ is case 5 for $3 < j \leqq 2[\sqrt{n_0 + 2}] + 1$. Another feature which shows up in the figure is that the lines formed by the top gaps $M_{\max}^{(n)}(k) - 1$ of the bands, i.e. $m = kn - k^2 + \binom{k}{2}$, nearly coincide with the tangents $m = kn - k^2 + \binom{k}{2} - 1/8$ to the parabola $m = \binom{n}{2}$.

Although the above gives a complete answer to Grünbaum's problem for $n \geqq n^*$, it leaves the problem for $n < n^*$ open. This case requires a detailed analysis of the lower end of the high k bands and appears to be difficult. Figure 5 shows an enlargement of the upper corner of figure 4 showing the values of m for $n \leqq 12$. The values predicted from the above formulas for large n are shown as before, the "extra" values resulting from the low end of high k bands are shaded and elongated for emphasis. Note that these values are all at the lower end of the continuum leading down from $\binom{n}{2}$. The size of n^* is unknown but it is likely to be small.

We close by mentioning a related and possibly more fundamental question, both from the point of view of combinatorial geometry and from the point of view of analogies to statistical physics. The question is the characterization of the density of states for the problem, i.e. to give each of the possible values with appropriate multiplicities. This could be characterized in a fashion similar to Grünbaum's problem by asking for all possible values of m for all configurations as a sequence $m_i^{(n)} \leqq m_{i+1}^{(n)}$.

We are grateful to the referee for numerous remarks and corrections and to M. Golomb and J. Nulton for helpful conversations. P. S. would also like to thank the Telluride Summer Research Center for providing an excellent environment for the writing of the manuscript.

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DEPARTMENT OF MATHEMATICAL SCIENCES
SAN DIEGO STATE UNIVERSITY
SAN DIEGO, CA 92182

HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST, HUNGARY