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CONGRUENCE NORMAL COVERS OF FINITELY GENERATED LATTICE VARIETIES

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ABSTRACT. We consider certain pseudovarieties K of lattices which are closed under the doubling of convex sets. For each such K, given an arbitrary finite lattice $\mathcal{L}$, we describe the covers of the variety $V(\mathcal{L})$ of the form $V(\mathcal{L}, \mathcal{K})$ with $\mathcal{K}$ a subdirectly irreducible lattice in K.

A general open problem in the theory of lattice varieties is: *Given a finite lattice $\mathcal{L}$, find all varieties* $\mathbf{W}$ *with* $\mathbf{W} \succ \mathbf{V}(\mathcal{L})$. This has several parts:

(1) Is there an algorithm for finding all such $\mathbf{W}$?

(2) Are there only finitely many covering varieties?

(3) Is each one generated by a finite lattice?

Here we give a partial answer which relates to parts (1) and (2), by showing that $\mathbf{V}(\mathcal{L})$ has only finitely many covers of the form $\mathbf{W} = \mathbf{V}(\mathcal{L}, \mathcal{K})$ where $\mathcal{K}$ is a finite *congruence normal* lattice (see Section 4 for the definition). The proof implicitly provides an algorithm for finding these covers, albeit a hopelessly impractical one. Usually $\mathbf{V}(\mathcal{L})$ will also have covers not of this form.

This result generalizes Theorem 5.5 of [5], that $\mathbf{V}(\mathcal{L})$ has only finitely many covers of the form $\mathbf{W} = \mathbf{V}(\mathcal{L}, \mathcal{K})$ where $\mathcal{K}$ is a finite upper or lower bounded homomorphic image of a free lattice.

Much of the effort in this paper will be directed toward developing the fine structure of the pseudovariety of congruence normal lattices (and related pseudovarieties). This enables us to get a better picture of the congruence normal lattices $\mathcal{K}$ such that $\mathbf{V}(\mathcal{L}, \mathcal{K}) \succ \mathbf{V}(\mathcal{L})$. However, we regard these results as equally interesting in their own right.

1. **Size functions.** In this section we prove a very general result, which will be refined later. Recall that a pseudovariety (*alias* prevariety) is a collection $\mathbf{K}$ of finite algebras such that $\mathrm{HSP}_{\mathrm{fin}}(\mathbf{K}) = \mathbf{K}$.

THEOREM 1. *Let* $\mathbf{K}$ *be a pseudovariety of algebras in a congruence distributive variety* $\mathbf{D}$ *of finite type. Assume that there is an increasing function* $s: \omega \longrightarrow \omega$ *such that if* $\mathcal{B}$ *is subdirectly irreducible in* $\mathbf{K}$ *with monolith* μ, *then* $|B| \leq s(|B/\mu|)$. *Let* $\mathcal{A}$ *be any finite algebra in* $\mathbf{D}$. *Then in the lattice of subvarieties of* $\mathbf{D}$,

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(1) *$\mathbf{V}(\mathcal{A})$ has only finitely many covering varieties of the form $\mathbf{W} = \mathbf{V}(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} \in \mathbf{K}$, and*
(2) *for all other varieties $\mathbf{W} \succ \mathbf{V}(\mathcal{A})$, $\mathbf{W} \cap \mathbf{K} \subseteq \mathbf{V}(\mathcal{A})$.*

PROOF. Let $\mathbf{W}$ be any variety with $\mathbf{D} \geq \mathbf{W} \succ \mathbf{V}(\mathcal{A})$ and $\mathbf{W} \cap \mathbf{K} \nsubseteq \mathbf{V}(\mathcal{A})$. Take any algebra $\mathcal{C} \in \mathbf{W} \cap \mathbf{K} - \mathbf{V}(\mathcal{A})$. By Baker's theorem [1], $\mathbf{V}(\mathcal{A})$ has a finite equational basis in say p variables. Then $\mathcal{C} \notin \mathbf{V}(\mathcal{A})$ implies that $\mathcal{C}$ has a p-generated subalgebra $\mathcal{D} \notin \mathbf{V}(\mathcal{A})$, which has a subdirectly irreducible factor $\mathcal{B} \notin \mathbf{V}(\mathcal{A})$. Note $\mathcal{B} \in \mathbf{W} \cap \mathbf{K}$, and $\mathbf{W} = \mathbf{V}(\mathcal{A}, \mathcal{B})$.

If μ denotes the monolith of $\mathcal{B}$, then $\mathcal{B}/\mu$ is p-generated and in $\mathbf{V}(\mathcal{A})$ (by Jónsson's Lemma). Hence $|\mathcal{B}/\mu|$ is at most the cardinality of the free algebra $F_{\mathbf{V}(\mathcal{A})}(p)$, and thus $|\mathcal{B}| \leq s\big(|F_{\mathbf{V}(\mathcal{A})}(p)|\big)$. Since the algebras in $\mathbf{D}$ have finite type, there are only finitely many such algebras $\mathcal{B}$. ∎

Theorem 1, combined with Lemma 11, already suffices to prove the result stated in the introduction. However, we will postpone the applications until we have developed a stronger version (Theorem 6).

2. **Rank functions.** Again let $\mathbf{K}$ be a pseudovariety of algebras in a congruence distributive variety $\mathbf{D}$ of finite type. Assume now that we have a rank function $\rho\colon \mathbf{K} \longrightarrow \omega$, and as before an increasing function $s\colon \omega \longrightarrow \omega$, with properties to be described momentarily. Let $\mathbf{K}_n := \{\mathcal{A} \in \mathbf{K} : \rho(\mathcal{A}) \leq n\}$. Of course, $\mathbf{K} = \bigcup_{n\in\omega} \mathbf{K}_n$.

Let us assume that ρ and s satisfy the following properties for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$.

(1) If $\theta \in \mathbf{Con}\ \mathcal{A}$, then $\rho(\mathcal{A}/\theta) \leq \rho(\mathcal{A})$.
(2) If $\rho(\mathcal{A}) = m > n$, then there exists $\theta \in \mathbf{Con}\ \mathcal{A}$ with $\rho(\mathcal{A}/\theta) = n$.
(3) If $\mathcal{A}$ is subdirectly irreducible in $\mathbf{K}$ with monolith μ, and $\rho(\mathcal{A}) > 0$, then (a) $\rho(\mathcal{A}) = \rho(\mathcal{A}/\mu) + 1$, and (b) $|A| \leq s(|A/\mu|)$.
(4) If $\mathcal{A} \leq \mathcal{B}$, then $\rho(\mathcal{A}) \leq \rho(\mathcal{B})$.
(5) $\rho(\mathcal{A} \times \mathcal{B}) = \max\{\rho(\mathcal{A}), \rho(\mathcal{B})\}$.
(6) $\mathbf{V}(\mathbf{K}_0)$ is locally finite.

Such a rank function ρ naturally induces a function (which we will also call ρ) on the subvarieties of $\mathbf{D}$:

$$\rho(\mathbf{V}) = \sup\{\rho(\mathcal{A}) : \mathcal{A} \in \mathbf{V} \cap \mathbf{K}\}.$$

LEMMA 2. *For each $n \in \omega$, $\mathbf{K}_n$ is a pseudovariety.*

PROOF. Use (1), (4) and (5). ∎

LEMMA 3. *For every finite $\mathcal{A} \in \mathbf{D}$, we have $\rho\big(\mathbf{V}(\mathcal{A})\big) < \infty$, and if $\mathcal{A} \in \mathbf{K}$ then $\rho\big(\mathbf{V}(\mathcal{A})\big) = \rho(\mathcal{A})$.*

PROOF. Applying Jónsson's Lemma, $\mathbf{V}(\mathcal{A})$ contains only finitely many subdirectly irreducible algebras, say $\mathcal{B}_1, \ldots, \mathcal{B}_r$, and each $\mathcal{B}_i$ is in $\mathrm{HS}(\mathcal{A})$. Let $\mathcal{B}_1, \ldots, \mathcal{B}_t$ be the ones which are also in $\mathbf{K}$. Using (1), (4) and (5), we see that $\rho(\mathcal{C}) \leq \max_{1\leq i\leq t} \rho(\mathcal{B}_i)$ for all $\mathcal{C} \in \mathbf{V}(\mathcal{A}) \cap \mathbf{K}$. Moreover, if $\mathcal{A} \in \mathbf{K}$, then $t = r$ and $\rho(\mathcal{B}_i) \leq \rho(\mathcal{A})$ for each i. ∎

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LEMMA 4. *For every $n \in \omega$, $\mathbf{V}(\mathbf{K}_n)$ is locally finite.*

PROOF. The proof is of course by induction, with assumption (6) providing the case $n = 0$. So assume that $n > 0$ and $\mathbf{V}(\mathbf{K}_{n-1})$ is locally finite. Since $\mathbf{K}_n$ is a pseudovariety of algebras of finite type, the following are equivalent.

(1) $\mathbf{V}(\mathbf{K}_n)$ is locally finite.

(2) For every $m \in \omega$, there is a bound B_n^m on the size of the m-generated algebras in $\mathbf{K}_n$.

(3) For every $m \in \omega$, there is a bound C_n^m on the size of the m-generated subdirectly irreducible algebras in $\mathbf{K}_n$.

Let $\mathcal{A}$ be an m-generated subdirectly irreducible algebra in $\mathbf{K}_n$ with monolith μ. By condition (3a), $\mathcal{A}/\mu \in \mathbf{K}_{n-1}$ and hence $|\mathcal{A}/\mu| \leq B_{n-1}^m$. Condition (3b) then implies $|\mathcal{A}| \leq s(B_{n-1}^m)$. Thus we can take $C_n^m = s(B_{n-1}^m)$, and B_n^m the size of the m-generated free algebra in $\mathbf{K}_n$. Hence $\mathbf{K}_n$ is locally finite. ∎

Now let $\mathcal{A}$ be a finite algebra in $\mathbf{D}$. By Baker's theorem [1], $\mathbf{V}(\mathcal{A})$ has a finite equational basis in say p variables, and by Lemma 3 $\rho\big(\mathbf{V}(\mathcal{A})\big) = k$ for some integer k. We will use these parameters in the next lemma.

LEMMA 5. *If $\mathbf{D} \geq \mathbf{W} \succ \mathbf{V}(\mathcal{A})$ and $\mathbf{W} \cap \mathbf{K} \not\subseteq \mathbf{V}(\mathcal{A})$, then there exists a subdirectly irreducible p-generated algebra $\mathcal{S} \in \mathbf{W} \cap \mathbf{K}_{k+1}$ such that $\mathbf{W} = \mathbf{V}(\mathcal{A}, \mathcal{S})$.*

PROOF. Take any algebra $\mathcal{B} \in \mathbf{W} \cap \mathbf{K} - \mathbf{V}(\mathcal{A})$. Then $\mathcal{B} \notin \mathbf{V}(\mathcal{A})$ implies that $\mathcal{B}$ has a p-generated subalgebra $\mathcal{C} \notin \mathbf{V}(\mathcal{A})$, which has a subdirectly irreducible factor $\mathcal{D} \notin \mathbf{V}(\mathcal{A})$. Note $\mathcal{D} \in \mathbf{W} \cap \mathbf{K}$, so if $\rho(\mathcal{D}) \leq k+1$ we are done. If $\rho(\mathcal{D}) > k+1$, then by condition (2), $\mathcal{D}$ has a homomorphic image $\mathcal{E}$ with $\rho(\mathcal{E}) = k+1$, and $\rho\big(\mathbf{V}(\mathcal{A})\big) = k$ implies $\mathcal{E} \notin \mathbf{V}(\mathcal{A})$. In this case we can take $\mathcal{S}$ to be any subdirectly irreducible factor of $\mathcal{E}$ not in $\mathbf{V}(\mathcal{A})$. ∎

Combining Lemmas 4 and 5, we obtain a version of Theorem 1 with more information about the covering varieties of $\mathbf{V}(\mathcal{A})$.

THEOREM 6. *Let $\mathbf{D}$ be a congruence distributive variety, $\mathbf{K}$ a pseudovariety contained in $\mathbf{D}$ satisfying properties (1)–(6), and $\mathcal{A}$ a finite algebra in $\mathbf{D}$.*

(1) In the lattice of subvarieties of $\mathbf{D}$, $\mathbf{V}(\mathcal{A})$ has only finitely many covering varieties of the form $\mathbf{W} = \mathbf{V}(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} \in \mathbf{K}$. For each such $\mathcal{B}$, $\rho(\mathcal{B}) \leq \rho\big(\mathbf{V}(\mathcal{A})\big) + 1$.

(2) For all other varieties $\mathbf{W} \succ \mathbf{V}(\mathcal{A})$, $\mathbf{W} \cap \mathbf{K} \subseteq \mathbf{V}(\mathcal{A})$.

3. **Rank functions using depth.** In a finite lattice $\mathcal{L}$, let $J(\mathcal{L})$ denote the set of (non-zero) join irreducible elements, and let $M(\mathcal{L})$ be the set of (non-unit) meet irreducible elements. For any finite ordered set $\mathcal{P}$, define the *length* $l(\mathcal{P})$ to be the length of the longest chain in $\mathcal{P}$.

Now let $\mathcal{C}$ be a finite distributive lattice. (In application, $\mathcal{C}$ will be $\mathbf{Con}\ \mathcal{A}$ for some $\mathcal{A} \in \mathbf{K}$.) Define a depth function on $\mathcal{C}$ by

$$\delta(x) = l\big(\uparrow x \cap M(\mathcal{C})\big).$$

As usual, we extend this map by defining $\delta(C) := \delta(0_C) = l\big(M(C)\big)$.

The depth function δ will be used to induce the rank function ρ on congruence normal lattices. For other applications we need a relativized depth function. Given $z \in C$, let $M_z(C) := M(C) - \uparrow z = \{q \in M(C) : q \not\geq z\}$. Then define

$$\delta_z(x) = \begin{cases} 0 & \text{if } x \geq z, \\ l\big(\uparrow x \cap M_z(C)\big) + 1 & \text{otherwise.} \end{cases}$$

It is not hard to see that our original unadorned δ is given by $\delta = \delta_b$, where b is the meet of all the coatoms (maximal meet irreducible elements) of C.

As before, define $\delta_z(C) := \delta_z(0_C) = l\big(M_z(C)\big) + 1$.

LEMMA 7. *Let C be a finite distributive lattice and $z \in C$. Then the function $\gamma = \delta_z$ satisfies*

(1) $\gamma(1) = 0$,

(2) $\gamma(x \wedge y) = \max\{\gamma(x), \gamma(y)\}$,

(3) if $q \in M(C)$ and $\gamma(q) > 0$, then $\gamma(q) = \gamma(q^*) + 1$.

Conversely, if $\gamma \colon C \longrightarrow \omega$ satisfies (1)–(3), then $\gamma = \delta_z$ for $z = \bigwedge\{x \in C : \gamma(x) = 0\}$.

PROOF. It is clear that δ_z satisfies (1) and (3). For (2), note that because meet irreducible elements in a distributive lattice are meet prime,

$$\uparrow (x \wedge y) \cap M_z(C) = \big(\uparrow x \cap M_z(C)\big) \cup \big(\uparrow y \cap M_z(C)\big),$$

whence $\delta_z(x \wedge y) = \max\{\delta_z(x), \delta_z(y)\}$.

Conversely, let γ satisfy (1)–(3) on C. Fixing $x \in C$, we may assume that $\gamma(y) = \delta_z(y)$ for all $y > x$. W.l.o.g. $x \not\geq z$. Since $\gamma(\bigwedge y_i) = \max \gamma(y_i)$, and the same holds for δ_z, it suffices to consider the case $x \in M(C)$. In that case $\gamma(x) = \gamma(x^*) + 1 = \delta_z(x^*) + 1 = \delta_z(x)$. ∎

At one point we will need the following observation.

LEMMA 8. *Let C be a finite distributive lattice and $z \in C$. If $x \prec y$, then $\delta(x) \leq \delta(y) + 1$.*

PROOF. Let $x \prec y$ in C, and w.l.o.g. $x \not\geq z$. Then there is a unique $q \in M(C)$ such that $q \geq x$, $q \not\geq y$, and

$$\uparrow x \cap M_z(C) \subseteq \big(\uparrow y \cap M_z(C)\big) \cup \{q\}.$$

Thus $\delta_z(x) \leq \delta_z(y) + 1$. ∎

Now, as in Section 2, let **K** be a pseudovariety of algebras in a congruence distributive variety **D** of finite type, and for $\mathcal{A} \in \mathbf{K}$ define $\rho(\mathcal{A}) = \delta(\mathbf{Con}\mathcal{A})$. Then we immediately have the following, using only the definitions and the fact that $\mathbf{Con}\mathcal{A} \times \mathcal{B} \cong \mathbf{Con}\mathcal{A} \times \mathbf{Con}\mathcal{B}$, and hence $M(\mathbf{Con}\mathcal{A} \times \mathcal{B}) \cong M(\mathbf{Con}\mathcal{A}) \dot\cup M(\mathbf{Con}\mathcal{B})$, in a congruence distributive variety.

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LEMMA 9A. *The rank function $\rho(\mathcal{A}) = \delta(\mathbf{Con}\mathcal{A})$ on* $\mathbf{K}$ *satisfies (1), (2), (3a) and (5).*

So we can apply Theorem 6 with a specific pseudovariety $\mathbf{K}$ whenever conditions (3b), (4) and (6) hold for $\mathbf{K}$, ρ and a suitable function s.

More generally, with $\mathbf{K}$ and $\mathbf{D}$ as above, let $\mathbf{U}$ be a locally finite subvariety of $\mathbf{D}$. For any $\mathcal{A} \in \mathbf{K}$, let $\zeta(\mathbf{U}) = \bigwedge\{\theta \in \mathbf{Con}\mathcal{A} : \mathcal{A}/\theta \in \mathbf{U}\}$, and define $\rho_{\mathbf{U}}(\mathcal{A}) = \delta_{\zeta(\mathbf{U})}(\mathbf{Con}\mathcal{A})$. Since $\mathbf{U}$ is locally finite, we also get (6), leaving only (3b) and (4).

LEMMA 9B. *If* $\mathbf{U}$ *is a locally finite variety, then the rank function $\rho_{\mathbf{U}}(\mathcal{A}) = \delta_{\zeta(\mathbf{U})}(\mathbf{Con}\mathcal{A})$ on* $\mathbf{K}$ *satisfies (1), (2), (3a), (5) and (6).*

4. **Congruence normal lattices.** Recall that if $\mathcal{C}$ is a finite distributive lattice, then $J(\mathcal{C})$ and $M(\mathcal{C})$ are isomorphic ordered sets; in fact, the map $\kappa: J(\mathcal{C}) \longrightarrow M(\mathcal{C})$ with $\kappa(p) = \bigvee\{x \in \mathcal{C} : x \not\geq p\}$ does the trick.

Let us introduce some notation. If $\mathcal{L}$ is a finite lattice and $p \in J(\mathcal{L})$, $m \in M(\mathcal{L})$, let $\Phi_p := \mathrm{con}(p_*, p)$ and $\Phi^m := \mathrm{con}(m, m^*)$. Then $\mathbf{Con}\mathcal{L}$ is a finite distributive lattice, and it is well known that

$$J(\mathbf{Con}\mathcal{L}) = \{\Phi_p : p \in J(\mathcal{L})\} = \{\Phi^m : m \in M(\mathcal{L})\}.$$

Of course, the map $\Phi: J(\mathcal{L}) \twoheadrightarrow J(\mathbf{Con}\mathcal{L})$ is usually not one-to-one. If we let $\Psi_p := \kappa(\Phi_p)$ and $\Psi^m = \kappa(\Phi^m)$, then the corresponding meet irreducibles are given by

$$M(\mathbf{Con}\mathcal{L}) = \{\Psi_p : p \in J(\mathcal{L})\} = \{\Psi^m : m \in M(\mathcal{L})\}.$$

We say that a finite lattice $\mathcal{L}$ is *congruence normal* if for all $p \in J(\mathcal{L})$, $m \in M(\mathcal{L})$,

(CN) $$\Phi_p = \Phi^m \text{ implies } p \not\leq m.$$

Let $\mathbf{CN}$ denote the class of all (finite) congruence normal lattices. The importance of this concept is in a result due to Winfried Geyer [4].

THEOREM 10. $\mathbf{CN}$ *is the smallest class of lattices containing the one-element lattice and closed under the doubling of convex subsets.*

For comparison, lower bounded lattices are the smallest class of lattices containing $\mathbf{1}$ and closed under the doubling of convex sets with a unique minimal element, and they are precisely the finite lattices satisfying the congruence condition

(LB) $$\Phi_p = \Phi_q \text{ implies } p = q.$$

Upper bounded lattices are closed under the dual type of doublings, and satisfy

(UB) $$\Phi^m = \Phi^n \text{ implies } m = n.$$

Bounded lattices are the smallest class containing $\mathbf{1}$ and closed under the doubling of intervals, and are characterized by the conjunction of (LB) and (UB). These results are all due to Day [2].

We want to show that Theorem 6 applies to varieties of lattices with $\mathbf{K} = \mathbf{CN}$. To begin, we need a pair of results found in both Geyer [4] and Day [3].

LEMMA 11. *(a)* **CN** *is a pseudovariety. (b) If* $\mathcal{L} \in$ **CN** *and* $\theta \succ 0$ *in* **Con**$\mathcal{L}$, *then* $\mathcal{L} \cong \mathcal{L}/\theta[\mathrm{C}]$ *for some convex subset* $\mathrm{C} \subseteq \mathcal{L}/\theta$.

Now we introduce the rank function $\rho(\mathcal{L}) = \delta(\mathbf{Con}\mathcal{L})$ on **CN**. According to Lemma 9A, we now have to check conditions (3b), (4) and (6). By Lemma 11(b), if $\mathcal{L} \in$ **CN** is subdirectly irreducible, then $\mathcal{L}$ is obtained by doubling a convex set in $\mathcal{L}/\mu$. Hence condition (3b) holds with $s(n) = 2n$.

Condition (6) is also easy, being a consequence of the following lemma.

LEMMA 12. $\mathbf{CN}_0$ *is the class of all finite distributive lattices.*

PROOF. By the definition of ρ, every subdirectly irreducible lattice $\mathcal{L} \in \mathbf{CN}_0$ is simple. Then (CN) yields $p \not\leq m$ for all $p \in J(\mathcal{L})$, $m \in M(\mathcal{L})$, implying $|\mathcal{L}| = 2$. ∎

So it remains to show that condition (4) holds, which requires three lemmas and a little work.

LEMMA 13. *For each* $j \in \omega$, $\mathbf{CN}_j$ *is closed under subdirect products.*

PROOF. This is a consequence of Lemma 7(2). ∎

Let $\mathcal{L}[\mathrm{C}]$ denote the lattice obtained by doubling the convex subset C of $\mathcal{L}$. Let $\alpha\mathrm{C} := \{(c,0) : c \in \mathrm{C}\}$ and $\beta\mathrm{C} := \{(c,1) : c \in \mathrm{C}\}$. Let $\gamma\colon \mathcal{L}[\mathrm{C}] \twoheadrightarrow \mathcal{L}$ denote the canonical map.

LEMMA 14. *If* C *is a convex subset of a finite lattice* $\mathcal{L}$, *and* $\mathcal{S}$ *is a sublattice of* $\mathcal{L}[\mathrm{C}]$, *then there exists a convex subset* $\mathrm{D} \subseteq \gamma(\mathcal{S})$ *such that* $\mathcal{S} \cong \gamma(\mathcal{S})[\mathrm{D}]$.

PROOF. Let $\{(m,0) : m \in X\}$ be the maximal members of $\alpha\mathrm{C} \cap \mathcal{S}$, and let $\{(p,1) : p \in Y\}$ be the minimal members of $\beta\mathrm{C} \cap \mathcal{S}$, Note that $(x,0) \in \mathcal{S}$ iff $x \in \gamma(\mathcal{S}) \cap \mathrm{C}$ and $x \leq m$ for some $m \in X$, and dually $(y,1) \in \mathcal{S}$ iff $y \in \gamma(\mathcal{S}) \cap \mathrm{C}$ and $y \geq p$ for some $p \in Y$. Thus for $x \in \gamma(\mathcal{S}) \cap \mathrm{C}$, both $(x,0)$ and $(x,1)$ are in $\mathcal{S}$ iff there exist $p \in Y$ and $m \in X$ with $p \leq x \leq m$. So let $\mathcal{T} = \gamma(\mathcal{S})$ and $\mathrm{D} = \{x \in \mathcal{T} : p \leq x \leq m$ for some $p \in Y, m \in X\}$. Then $\mathcal{T}$ is clearly a sublattice of $\mathcal{L}$, $\mathrm{D} \subseteq \mathrm{C}$ by convexity, and in fact $\mathrm{D} = \{x \in \mathrm{C} : (x,0) \in \mathcal{S}$ and $(x,1) \in \mathcal{S}\}$.

We need to know that D is convex. Assume $x, z \in \mathrm{D}$ and $y \in \mathcal{T}$ with $x \leq y \leq z$. Then for some $p, q \in Y$ and $m, n \in X$ we have $p \leq x \leq m$ and $q \leq z \leq n$. Hence $p \leq x \leq y \leq z \leq n$, so $y \in \mathrm{D}$.

Now define $g\colon \mathcal{S} \longrightarrow \mathcal{T}[\mathrm{D}]$ by

$$g(s) = \begin{cases} s & \text{if } \gamma(s) \in \mathrm{D} \text{ or } \gamma(s) \in \mathcal{T} - \mathrm{C}; \\ \gamma(s) & \text{if } \gamma(s) \in \mathrm{C} - \mathrm{D}. \end{cases}$$

Our earlier observations show that g is one-to-one and onto. Moreover, using the definition of $\leq$ on $\mathcal{T}[\mathrm{D}]$, it is clear that $s \leq t$ implies $g(s) \leq g(t)$, and it remains to prove the reverse implication.

Assume $g(s) \leq g(t)$. The first interesting case is when $g(s) = s = (x,1)$ with $x \in \mathrm{D}$, and $g(t) = \gamma(t) = y$ with $y \in \mathrm{C} - \mathrm{D}$ and $x \leq y$ in $\mathcal{L}$. Then $x \in \mathrm{D}$ implies $y \geq x \geq p$ for some $p \in Y$; since $y \notin \mathrm{D}$, we must have $t = (y, 1) \geq s$. Similarly, suppose $g(s) = \gamma(s) = x$ with $x \in \mathrm{C} - \mathrm{D}$, and $g(t) = t = (y, 0)$ with $y \in \mathrm{D}$, and $x \leq y$ in $\mathcal{L}$. Then $y \in \mathrm{D}$ implies $x \leq y \leq m$ for some $m \in X$, so $x \notin \mathrm{D}$ implies $s = (x, 0) \leq t$. In all other cases, we get $s \leq t$ trivially.

Thus g is an isomorphism. ∎

LEMMA 15. *If* C *is a convex subset of a finite lattice* $\mathcal{L}$, *then* $\rho(\mathcal{L}[\mathrm{C}]) \leq \rho(\mathcal{L}) + 1$.

PROOF. Write C as a disjoint union of connected convex sets, $\mathrm{C} = \bigcup \mathrm{C}_i$. In $\mathbf{Con}\mathcal{L}[\mathrm{C}]$, let $\theta = \ker \gamma$, and let φ_i be the kernel of the natural map $\eta_i \colon \mathcal{L}[\mathrm{C}] \twoheadrightarrow \mathcal{L}[\mathrm{C}_i]$. Then $\varphi_i \prec \theta$ for each i, so $\delta(\varphi_i) \leq \delta(\theta) + 1 = \rho(\mathcal{L}) + 1$ by Lemma 8. But $\bigwedge \varphi_i = 0$, so by Lemma 7(2) $\rho(\mathcal{L}[\mathrm{C}]) = \delta(0) = \max \delta(\varphi_i) \leq \rho(\mathcal{L}) + 1$. ∎

With these lemmas in hand, we can show that (4) holds.

LEMMA 16. $\mathbf{CN}_k$ *is closed under sublattices.*

PROOF. Since $\mathbf{CN}_0$ is the class of finite distributive lattices, the claim is true for $k = 0$. Assume then that $k > 0$ and the claim holds for all $j < k$.

Let $\mathcal{S} \leq \mathcal{L} \in \mathbf{CN}_k$, and first consider the case when $\mathcal{L}$ is subdirectly irreducible. By property (3a), $\mathcal{K} := \mathcal{L}/\mu \in \mathbf{CN}_{k-1}$. Moreover, by Lemma 11(b), there is a convex set $\mathrm{C} \subseteq \mathcal{K}$ such that $\mathcal{L} \cong \mathcal{K}[\mathrm{C}]$, and the natural map $\mathcal{L} \longrightarrow \mathcal{L}/\mu$ corresponds to $\gamma \colon \mathcal{K}[\mathrm{C}] \longrightarrow \mathcal{K}$. By Lemma 14, there exists a convex set $\mathrm{D} \subseteq \mathcal{T} := \gamma(\mathcal{S})$ such that $\mathcal{S} \cong \mathcal{T}[\mathrm{D}]$. Then $\mathcal{K} \in \mathbf{CN}_{k-1}$ implies $\mathcal{T} \in \mathbf{CN}_{k-1}$ by induction, and hence $\mathcal{S} \in \mathbf{CN}_k$ by Lemma 15.

Now let $\mathcal{L} \in \mathbf{CN}_k$ be arbitrary, and again $\mathcal{S} \leq \mathcal{L}$. Let $\mathcal{L} \leq \prod \mathcal{L}_i$ be a subdirect representation of $\mathcal{L}$ into subdirectly irreducible lattices. By Lemma 7(2), $\mathcal{L}_i \in \mathbf{CN}_k$ for each i. Thus in the induced subdirect representation $\mathcal{S} \leq \prod \mathcal{S}_i$ with $\mathcal{S}_i \leq \mathcal{L}_i$, we have $\mathcal{S}_i \in \mathbf{CN}_k$ by the first case. Hence $\mathcal{S} \in \mathbf{CN}_k$ by Lemma 13. ∎

Thus Theorem 6 applies with $\mathbf{K} = \mathbf{CN}$, yielding our main result.

THEOREM 17. *Let* $\mathcal{L}$ *be a finite lattice.*

(1) In the lattice of lattice varieties, $\mathbf{V}(\mathcal{L})$ *has only finitely many covering varieties of the form* $\mathbf{W} = \mathbf{V}(\mathcal{L}, \mathcal{K})$ *with* $\mathcal{K} \in \mathbf{CN}$. *For each such* $\mathcal{K}$, $\rho(\mathcal{K}) \leq \rho\big(\mathbf{V}(\mathcal{L})\big)$.

(2) For all other varieties $\mathbf{W} \succ \mathbf{V}(\mathcal{L})$, $\mathbf{W} \cap \mathbf{CN} \subseteq \mathbf{V}(\mathcal{L})$.

In the Appendix to [3], a simple test is given for determining whether a finite lattice $\mathcal{L}$ is in $\mathbf{CN}_k$. Let

$$\lambda_{\mathcal{L}} = \{(m, u) \in M(\mathcal{L}) \times J(\mathcal{L}) : m \vee u = m^* \text{ and } m \wedge u = u_*\}.$$

For $p, q \in J(\mathcal{L})$ we write $p \sim q$ if there exists $m \in M(\mathcal{L})$ such that (m, p) and $(m, q) \in \lambda_{\mathcal{L}}$. Let $\approx$ denote the transitive closure of $\sim$. Note $p \approx q$ implies $\Phi_p = \Phi_q$; for lattices in $\mathbf{CN}$, the converse is also true.

For subsets $A, B \subseteq \mathcal{L}$, define $A \ll B$ if for each $a \in A$ there exists $b \in B$ with $a \leq b$. We define subsets $F_k(\mathcal{L}) \subseteq J(\mathcal{L})$ as follows. Let $F_0(\mathcal{L})$ be the set of all join-prime

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elements of $\mathcal{L}$. Given $F_k(\mathcal{L})$, let $F_{k+1}(\mathcal{L})$ be the set of all $p \in J(\mathcal{L})$ such that $p \leq \bigvee B$ implies there exists $A \ll B$ such that $p \leq \bigvee A$ and $A \subseteq F_k(\mathcal{L}) \cup (p/\approx)$.

We also need a generalized semidistributivity condition:

(T) $\qquad$ for all $p, q \in J(\mathcal{L}),\ p \approx q$ implies $q \leq q_* \vee p$.

By keeping track of the ranks in the proof of the main result in the Appendix of [3], we obtain a characterization of lattices in $\mathbf{CN}_k$.

THEOREM 18. *Let $\mathcal{L}$ be a finite lattice and $k \in \omega$. Then $\mathcal{L} \in \mathbf{CN}_k$ if and only if $\mathcal{L}$ satisfies (T) and $F_k(\mathcal{L}) = J(\mathcal{L})$.*

5. **More general doubling classes.** **CN** is the smallest class of lattices containing all finite distributive lattices ($\mathbf{CN}_0$) and closed under the doubling construction. The natural generalization of this is to start with an arbitrary locally finite variety **U**, and let **DU** be the smallest class of lattices containing all finite lattices in **U** and closed under the doubling of convex sets. The natural rank function for this class is $\rho_{\mathbf{U}}$, so that $\mathbf{DU}_0$ is the class of all finite lattices in **U**.

In order to apply Theorem 6, we want to show that $\mathbf{DU}_k$ is a pseudovariety for each $k \in \omega$, from which it will follow that **DU** is also a pseudovariety.

LEMMA 19. *$\mathbf{DU}_k$ is closed under finite direct products.*

PROOF. Note that if C is a convex subset of $\mathcal{L}$, then

$$\mathcal{L}[\mathrm{C}] \times \mathcal{K} \cong (\mathcal{L} \times \mathcal{K})[\mathrm{C} \times \mathcal{K}],$$

because both sides consist of $(\mathcal{L} - \mathrm{C}) \times \mathcal{K} \dot{\cup} \mathrm{C} \times \mathcal{K} \times \mathbf{2}$ ordered naturally. Repeated application of this observation shows that **DU** is closed under finite direct products.

On the other hand, $M(\mathbf{Con}\mathcal{L} \times \mathcal{K}) \cong M(\mathbf{Con}\mathcal{L}) \dot{\cup} M(\mathbf{Con}\mathcal{K})$ implies $\rho_{\mathbf{U}}(\mathcal{L} \times \mathcal{K}) = \max\{\rho_{\mathbf{U}}(\mathcal{L}), \rho_{\mathbf{U}}(\mathcal{K})\}$, so in fact $\mathbf{DU}_k$ is closed under finite products. ■

The argument for closure under H is based on the next lemma.

LEMMA 20. *Let $\mathcal{L} = \bar{\mathcal{L}}[\mathrm{C}]$ with C connected and convex, and let $\varphi = \ker\gamma$ where $\gamma\colon \mathcal{L} \twoheadrightarrow \bar{\mathcal{L}}$ is the canonical homomorphism. If $\psi \in \mathbf{Con}\mathcal{L}$ and $\psi \wedge \varphi = 0$, then*

$$\mathcal{L}/\psi \cong \mathcal{K}[\mathrm{D}]$$

where $\mathcal{K} = \mathcal{L}/(\psi \vee \varphi)$ and $\mathrm{D} = \mathrm{C}/(\psi \vee \varphi)$.

PROOF. First note that for $x, y \in \mathrm{C}$, we have $(x,0)\,\psi\,(y,0)$ iff $(x,1)\,\psi\,(y,1)$. Therefore $\varphi \circ \psi \circ \varphi \subseteq \varphi \circ \psi \cup \psi \circ \varphi$, whence $\psi \vee \varphi = \psi \circ \varphi \circ \psi$.

Let $A = \bigcup_{x \in \mathrm{C}} \uparrow (x,1)$, $B = \bigcup_{x \in \mathrm{C}} \downarrow (x,0)$ and $R = L - (A \cup B)$, so that $\mathcal{L} = A \dot{\cup} B \dot{\cup} R$. We claim that if $x \in \mathrm{C}$, $y \in \mathcal{L}$ and $(x,1)\psi y$, then $y \in A$. For $(x,1)\psi y$ implies $(x,1)\psi(x,1) \wedge y$, and hence $(x,1) \wedge y \not\leq (x,0)$ since $\psi \wedge \varphi = 0$. Therefore $(x,1) \wedge y = (z,1)$ for some $z \in \mathrm{C}$, and $y \in A$. This in turn implies that if $u\psi v$ and $u \in A$, then $v \in A$. Of course, the dual statements hold for B.

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Combining what we have so far, we see that each $\psi \vee \varphi$-block E which intersects $\mathrm{C} \times \mathbf{2}$ nontrivially splits into two parts, $E = (E \cap A) \dot{\cup} (E \cap B)$. Moreover, if $x \in R$, then $x/(\psi \vee \varphi) = x/\psi$.

Also, observe that if C is connected and convex, and h is an epimorphism, then $h(\mathrm{C})$ is connected and convex. Hence D is connected and convex in $\mathcal{K}$.

It is now straightforward to check that the isomorphism $g : \mathcal{L}/\psi \cong \mathcal{K}[\mathrm{D}]$ is provided by

$$g(x/\psi) = \begin{cases} x/(\psi \vee \varphi) & \text{if } x/(\psi \vee \varphi) \cap (\mathrm{C} \times \mathbf{2}) = \emptyset, \\ \big(x/(\psi \vee \varphi), 1\big) & \text{if } x/(\psi \vee \varphi) \cap (\mathrm{C} \times \mathbf{2}) \neq \emptyset \text{ and } x \in A, \\ \big(x/(\psi \vee \varphi), 0\big) & \text{if } x/(\psi \vee \psi) \cap (\mathrm{C} \times \mathbf{2}) \neq \emptyset \text{ and } x \in \mathrm{B}\,. \end{cases}$$

■

LEMMA 21. $\mathbf{DU}_k$ *is closed under homomorphic images.*

PROOF. First we show that $\mathbf{DU}$ is closed under H. Let $\mathcal{L} \in \mathbf{DU}$, so that in $\mathbf{Con}\mathcal{L}$ there exist $\varphi_0, \ldots, \varphi_n$ such that

(1) $\mathcal{L}/\varphi_0 \in \mathbf{U}$,

(2) $\varphi_0 \succ \varphi_1 \succ \cdots \succ \varphi_n = 0$,

(3) $\mathcal{L}/\varphi_{j+1}$ is obtained from $\mathcal{L}/\varphi_j$ by doubling a connected convex set. Let $\psi \in$ $\mathbf{Con}\ \mathcal{L}$, and assume $\theta > \psi$ implies $\mathcal{L}/\theta \in \mathbf{DU}$.

If $\varphi_0 \leq \psi$, then $\mathcal{L}/\psi \in \mathbf{U}$, and we are done. Otherwise, we can find $j > 0$ such that $\varphi_{j-1} \not\leq \psi$ and $\varphi_j \leq \psi$. In this case $\psi \wedge \varphi_{j-1} = \varphi_j$, so we can apply Lemma 20 to $\mathcal{L}/\varphi_j$. This yields $\mathcal{L}/\psi \cong \mathcal{K}[\mathrm{D}]$ with $\mathcal{K} = (\mathcal{L}/\varphi_j)/(\psi \vee \varphi_{j-1}) \in \mathbf{DU}$ and D convex. Hence $\mathcal{L}/\psi \in \mathbf{DU}$, as desired.

Now $\zeta^{\mathcal{L}/\psi}(\mathbf{U}) = \zeta^{\mathcal{L}}(\mathbf{U}) \vee \psi$ and $M(\mathbf{Con}\ \mathcal{L}/\psi) \cong M(\mathbf{Con}\ \mathcal{L}) \cap \uparrow \psi$, from which it follows readily that $\rho_{\mathbf{U}}(\mathcal{L}/\psi) \leq \rho_{\mathbf{U}}(\mathcal{L})$. ■

The proof that $\mathbf{CN}_k$ is closed under S (Lemma 16) used only Lemma 7 and its consequences, and Lemma 11(b). Of course, Lemma 7 applies equally well in our present situation, with $z = \zeta(\mathbf{U})$, while the analogue of Lemma 11(b) for $\mathbf{DU}$ follows immediately from the proof of Lemma 21. Thus, with the appropriate modifications, we obtain S-closure for $\mathbf{DU}_k$.

LEMMA 22. $\mathbf{DU}_k$ *is closed under sublattices.*

Combining the lemmas, we obtain the main result of this section.

THEOREM 23. *For any locally finite lattice variety* $\mathbf{U}$ *and* $k \in \omega$, $\mathbf{DU}_k$ *is a pseudovariety.*

In particular, Theorem 6 holds for covering varieties generated by lattices in $\mathbf{DU}$. We could also prove the analogue of Theorem 23 for classes closed under the doubling of intervals, or of upper or lower pseudo-intervals.

The most interesting applications are probably when $\mathbf{U}$ is a small lattice variety; *e.g.*, for $\mathbf{CN}$ we take $\mathbf{U}$ to be the variety of distributive lattices. However, by taking $\mathbf{U} = \mathbf{V}(\mathcal{L})$, we obtain a corollary much in the spirit of Theorem 1.

Let us say that a subdirectly irreducible lattice $\mathcal{K}$ is *doubling reducible* if its monolith μ is a doubling congruence, *i.e.*, $\mathcal{K} \cong (\mathcal{K}/\mu)[\mathrm{C}]$ for some convex set C. Otherwise, $\mathcal{K}$ is *doubling irreducible*.

THEOREM 24. *Let $\mathcal{L}$ be a finite lattice. Then in the lattice of lattice varieties, $\mathbf{V}(\mathcal{L})$ has only finitely many covering varieties of the form $\mathbf{W} = \mathbf{V}(\mathcal{L}, \mathcal{K})$ with $\mathcal{K}$ a doubling irreducible, subdirectly irreducible lattice.*

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