ON GROUP GRADED RINGS SATISFYING POLYNOMIAL IDENTITIES

by A. V. KELAREV and J. OKNIŃSKI

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A number of classical theorems of ring theory deal with nilness and nilpotency of the Jacobson radical of various ring constructions (see [10], [18]). Several interesting results of this sort have appeared in the literature recently. In particular, it was proved in [1] that the Jacobson radical of every finitely generated PI-ring is nilpotent. For every commutative semigroup ring \( RS \), it was shown in [11] that if \( J(R) \) is nil then \( J(RS) \) is nil. This result was generalized to all semigroup algebras satisfying polynomial identities in [15] (see [16, Chapter 21]). Further, it was proved in [12] that, for every normal band \( B \), if \( J(R) \) is nilpotent, then \( J(RB) \) is nilpotent. A similar result for special band-graded rings was established in [13, Section 6]. Analogous theorems concerning nilpotency and local nilpotency were proved in [2] for rings graded by finite and locally finite semigroups.

This paper is devoted to the radicals of group graded rings, which have been actively investigated by many authors (see [10], [14]). Let \( G \) be a group. An associative ring \( R = \bigoplus_{g \in G} R_g \) is said to be \( G \)-graded (strongly \( G \)-graded) if \( R_g R_h \subseteq R_{gh} \) (respectively, \( R_g R_h = R_{gh} \)) for all \( g, h \in G \).

First, we consider algebras over a field of characteristic zero. In this case our result will be also of interest in connection with the well-known problem of finding necessary and sufficient conditions for the Jacobson radical to be homogeneous. An ideal \( I \) of \( R = \bigoplus_{g \in G} R_g \) is said to be homogeneous if \( I = \bigoplus_{g \in G} I \cap R_g \). This problem has not been solved even for u.p.-groups (see [7], [8], [10]). Polynomial identities give what appears to be the first natural sufficient condition which is applicable to the case of arbitrary groups.

**Theorem 1.** Let \( G \) be a group with identity \( e \), and let \( R = \bigoplus_{g \in G} R_g \) be a \( G \)-graded PI-algebra over a field of characteristic zero. If the Jacobson radical \( J(R_e) \) is nil, then \( J(R) \) is a homogeneous nil ideal of \( R \).

The following corollary to the main theorem is worth mentioning.

**Corollary 2.** Let \( G \) be a group with identity \( e \), and let \( R = \bigoplus_{g \in G} R_g \) be a strongly \( G \)-graded PI-algebra over a field of characteristic zero. If \( J(R_e) \) is nilpotent, then \( J(R) \) is nilpotent.

It is impossible to replace strongly graded algebras by ordinary group graded algebras in Corollary 2. Indeed, let \( A \) be the free commutative algebra with free generators \( a_1, a_2, \ldots \). Denote by \( I \) the ideal of \( A \) generated by \( a_1, a_1^2, a_1^3, \ldots \). Then \( A/I \) is positively graded, and so \( A/I = \bigoplus_{z \in Z} A_z \) where \( Z \) is the infinite cyclic group. Although \( A_0 = 0 \), it is clear that \( J(A/I) = A/I \) is not nilpotent.

One cannot omit the restriction on the characteristic of the field neither in Theorem

THEOREM 3. Let $G$ be a group with identity $e$, and let $R = \bigoplus_{g \in G} R_g$ be a $G$-graded PI-ring. If $J(R_e)$ is nil, then $J(R)$ is nil, too.

Throughout $R = \bigoplus_{g \in G} R_g$ will be a $G$-graded ring, $G$ a group with identity $e$. Let $T$ be a subset of $G$. Put $R_T = \bigoplus_{i \in T} R_i$. For any $r \in R$, say $r = \sum_{g \in G} r_g$ where $r_g \in R_g$, we put $r_T = \sum_{i \in T} r_i$. If $I \subseteq R$ and $g \in G$, then we put $I_g = I \cap R_g$ and $I_T = I \cap R_T$. Further, for $r \neq 0$, denote the set of all non-zero homogeneous summands $r_g$ of $r$ by $H(r)$, and put $\text{supp}(r) = \{g \in G | r_g \neq 0\}$. Put $H(0) = 0$, $\text{supp}(0) = \emptyset$. Clearly, $H(r)$ and $\text{supp}(r)$ are finite sets. Let $H(I) = \bigcup_{r \in I} H(r)$. Then $H(R)$ is the set of all homogeneous elements of $R$.

Evidently, $H(R)$ is a multiplicative subsemigroup of $R$. By the length of $r$ we mean $|\text{supp}(r)|$. Recall that a semigroup $S$ is said to be permutational if there exists $n > 1$ such that, for any $n$ elements $x_1, \ldots, x_n$ of $S$, their product can be rearranged as $x_1 \ldots x_n = x_{\sigma_1} \ldots x_{\sigma_n}$ for a non-trivial permutation $\sigma$. Every PI-ring (or PI-algebra) satisfies a multilinear identity, i.e., an identity of the form

$$x_1 \ldots x_n + \sum_{1 \neq \sigma \in S_n} k_\sigma x_{\sigma_1} \ldots x_{\sigma_n} = 0,$$

(1)

where $S_n$ is the symmetric group, $k_\sigma$ are integers (elements of the field in the case of algebras, see [17]). Let us begin with a known lemma (see [10, Proposition 6.18]), which will be used repeatedly.

**Lemma 4.** Let $R$ be a $G$-graded ring, $H$ a subgroup of $G$. Then $J(R_H) \supseteq R_H \cap J(R)$.

**Lemma 5.** Let $R$ be a $G$-graded PI-ring, $I$ a homogeneous ideal of $R$ contained in $J(R)$. If $I_e$ is nil, then $I$ is nil.

**Proof.** Take any element $r$ in $H(I)$. Since $I$ is homogeneous, $r \in I$. Let $r \in J(R) \cap R_g$, where $g \in G$. If $g$ is a periodic element, then there exists a positive integer $n$ such that $r^n \in I \cap R_e = I_e$, and so $r$ is nilpotent. Further, assume that $g$ is not periodic. Denote by $T$ the infinite cyclic group generated in $G$ by $g$. Lemma 4 shows that $r \in J(R_T)$, and therefore $r$ is nilpotent again in view of [10, Theorem 32.5]. Thus $H(I)$ is a multiplicative nil subsemigroup of $R$. Since $R$ satisfies a polynomial identity, it follows from [17, Theorem 1.6.36], that $I$ is nil, as required.

**Lemma 6.** Let $G$ be a permutational group, $R$ a $G$-graded PI-ring. If $J(R_e)$ is nil, then $J(R)$ is nil.

**Proof.** By [16, Theorem 19.8], $G$ is finite-by-abelian-by-finite. Take any $r \in J(R_G)$. Denote by $S$ the subgroup generated in $G$ by the support of $r$. It is easily seen that $S$ is also finite-by-abelian-by-finite. Lemma 4 implies $r \in J(R_S)$, and so without loss of
generality we may assume that \(G\) is finitely generated itself. Then assertion (2.2) of [4] tells us that \(G\) is abelian-by-finite, i.e. \(G\) has an abelian normal subgroup \(A\) of finite index. Then \(A\) is finitely generated, too (see [9]). Therefore \(G\) contains a torsion-free abelian subgroup \(T\) of finite index. By [10, Corollary 22.8], \(R\) is graded by the finite group \(G/T\) with the identity component \(R_T\). Therefore [15, Lemma 1.1(1)], shows that it suffices to prove that \(J(R_T)\) is nil. However, \(J(R_T)\) is homogeneous by [10, Theorem 30.28], because \(T\) is torsion-free abelian. Lemma 5 completes the proof.

**Lemma 7.** Let \(G\) be a permutational group, \(R\) a \(G\)-graded PI-algebra over a field of characteristic zero. If \(J(R_e)\) is nil, then \(J(R)\) is homogeneous.

**Proof.** We shall verify that \(H(J(R))\) consists of nilpotent elements. Then [17, Theorem 1.6.36], will show that \(H(J(R))\) generates a homogeneous nil ideal \(I\) in \(R\), and so \(J(R) = I\) is homogeneous.

Pick any \(r \in J(R)\) and \(g \in \text{supp}(r)\). We claim that \(r_g\) is nilpotent. As in the beginning of the proof of Lemma 6, we may assume that \(G\) has a torsion-free abelian subgroup \(T\) of finite index. If we look at the natural \(G/T\)-gradation of \(R\) and apply [10, Theorem 30.28 (b)], and the fact that our field has characteristic zero, then we conclude that \(J(R)\) is \(G/T\)-homogeneous. We may assume that the whole \(\text{supp}(r)\) is contained in one \(T\)-coset of \(G\) (otherwise we would pass to the \(G/T\)-homogeneous summand of \(r\) corresponding to the coset containing \(g\)). Since \(G/T\) is finite, there exists a positive integer \(n\) such that \(\text{rs}_g^n \in R_T\). Given that \(J(R_e)\) is nil, [10, Theorem 32.5], implies that all the homogeneous summands of \(\text{rs}_g^n\) are nilpotent. Therefore \(r_g\) is nilpotent, as required.

**Proof of Theorem 1.** By Lemma 5 the largest homogeneous ideal \(I\) of \(R\) contained in \(J(R)\) is nil. Obviously, \(R/I\) is a \(G\)-graded ring, and \(J(R/I) = J(R)/I\). Therefore it suffices to prove Theorem 1 for \(R/I\). To simplify the notation we may assume that from the very beginning \(I = 0\).

Suppose that \(J(R) \neq 0\). Choose a non-zero element \(r\) with a minimal length in \(J(R)\). Denote by \(T\) and \(S\) the subgroup and, respectively, subsemigroup generated in \(G\) by \(\text{supp}(r)\). Let \(M = M(r)\) be the multiplicative subsemigroup generated in \(R\) by \(H(r)\). We claim that \(H(r)\) consists of nilpotent elements.

If \(S\) is permutational, then the group \(T\) is permutational too, by [16, Theorem 19.8], and so all elements in \(H(r)\) are nilpotent in view of Lemmas 6 and 7.

Further, consider the case where \(S\) is not permutational. Let \(n\) be the degree of a multilinear identity (1) of \(R\). There exist elements \(s_1, \ldots, s_n\) in \(S\) such that \(s_1 \ldots s_n \neq s_{\sigma_1} \ldots s_{\sigma_n}\) for all \(\sigma \in S_n\) such that \(\sigma \neq 1\). Clearly, there exist \(x_1, \ldots, x_n \in M\) such that \(x_i \in R_s\) for all \(i = 1, \ldots, n\). Applying (1) to the elements \(x_1, \ldots, x_n\) we get

\[
x_1 \ldots x_n \in \bigcap_{1 \neq \sigma \in S_n} R_{x_{\sigma_1} \ldots x_{\sigma_n}},
\]

whence \(x_1 \ldots x_n = 0\). It follows that \(y_1 \ldots y_m = 0\) for some \(y_1, \ldots, y_m \in H(r)\). Then we can choose \(m\) and the \(y_1, \ldots, y_m\) such that \(y_1 \ldots y_{m-1} \neq 0\). Now look at the product \(y_1 \ldots y_{m-1}r\). It also belongs to \(J(R)\) but has fewer homogeneous summands than \(r\). Hence \(y_1 \ldots y_{m-1}r = 0\) by the choice of \(r\). Since \(G\) is a group, we get \(y_1 \ldots y_{m-1}H(r) = 0\). Further, we can look at \(y_1 \ldots y_{m-2}rH(r) = 0\) and infer \(y_1 \ldots y_{m-2}(H(r)^2) = 0\). Reasoning like this \(m\) times, we conclude \((H(r))^m = 0\). In particular, all elements in \(H(r)\) are nilpotent, again.
Denote by $L$ the ideal generated in $H(R)$ by $H(r)$. Each non-zero element of $L$ is a homogeneous summand of a certain element of positive minimal length in $J(R)$. It follows from what we have proved that $L$ is a nil ideal of $H(R)$. Hence $L$ is locally nilpotent by [17, Theorem 1.6.36]. Therefore $L$ generates a nil subalgebra $K$ in $R$. Evidently, $K$ is a homogeneous ideal of $R$, and so $K \subseteq I$, contradicting the fact that $I = 0$. Thus $J(R) = I$, and so $J(R)$ is a homogeneous nil ideal of $R$.

**Proof of Corollary 2.** By Theorem 1 we get $J(R) = \bigoplus_{g \in G} I_g$, where $I_g = I \cap R_g$. Denote by $n$ the nilpotency index of $J(R_e)$. Lemma 4 implies $I_e \subseteq J(R_e)$, and so $(I_e)^n = 0$. We claim that $(J(R))^n = 0$.

To this end we need only to verify that $x_1 \ldots x_{2n} = 0$ for arbitrary homogeneous elements $x_1, \ldots, x_{2n} \in J(R)$. Let $x_i \in I_{g_i}$, where $i = 1, \ldots, 2n$. Given that $x_2 \in R_{g_2} = R_{g_1}R_{g_{2,1}}$, we can find $y_{j_1}^{(1)} \in R_{g_1}$ and $z_{j_1}^{(1)} \in R_{g_{1,2}}$ such that $x_2 = \sum_{j_1} y_{j_1}^{(1)} z_{j_1}^{(1)}$. Then $x_1 y_{j_1}^{(1)} \in I_e$ for all $j_1$. Further, suppose that for some $i = 2, \ldots, n$ elements $y_{j_{i-1}}^{(i-1)}$ and $z_{j_{i-1}}^{(i-1)}$ have been introduced such that $x_{2i-2} = \sum_{j_{i-1}} y_{j_{i-1}}^{(i-1)} z_{j_{i-1}}^{(i-1)}$ and all $z_{j_{i-1}}^{(i-1)} \in R_{g_{i-2,i-1}}$. Then $x_{2i} \in R_{g_{i}} = R_{(g_1 \ldots g_{i-1})^{-1}} R_{g_{i-1}g_{i+1}}$, and so there exist homogeneous elements $y_{j_i}^{(i)} \in R_{(g_1 \ldots g_{i-1})^{-1}}$ and $z_{j_i}^{(i)} \in R_{g_{i-1}g_{i+1}}$ such that $x_{2i} = \sum_{j_i} y_{j_i}^{(i)} z_{j_i}^{(i)}$ and all $z_{j_{i-1}}^{(i-1)} x_{2i-1} y_{j_i}^{(i)}$ belong to $I_e$. Therefore

$$x_1 \ldots x_{2n} = \sum_{j_1, \ldots, j_n} (x_1 y_{j_1}^{(1)})(z_{j_1}^{(1)} x_3 y_{j_2}^{(2)})(z_{j_2}^{(2)} x_5 y_{j_3}^{(3)}) \ldots (z_{j_{n-1}}^{(n-1)} x_{2n-1} y_{j_n}^{(n)}) z_{j_n}^{(n)} \in I_e \sum_{j_n} z_{j_n}^{(n)} = 0,$$

which completes the proof.

**Proof of Theorem 3.** We shall prove that $J(R) = B(R)$, where $B(R)$ denotes the Baer radical of $R$. To this end we show that $R/B(R)$ is semisimple. Suppose to the contrary that $J(R/B(R)) \neq 0$. Since $R/B(R)$ is a subdirect product of prime PI-rings, there exists a prime PI-ring $\bar{R}$ and a homomorphism $f$ of $R$ onto $\bar{R}$ such that $f(J(R)) \neq 0$. By Posner's theorem (see [17]) $\bar{R}$ is contained in a matrix ring $D_m$, where $D$ is a division ring. For any $r \in R$ and $T \subseteq R$, put $\bar{r} = f(r)$, $\bar{T} = f(T)$.

Consider the set $L$ of all $x \in H(R)$ such that either $\bar{x} = 0$ or $\bar{x}$ has the smallest non-zero rank in $D_m$. By [16, Theorem 1.6], all non-zero elements of $\bar{L}$ lie in the same completely 0-simple factor $F$ of the multiplicative semigroup $D_m$. Obviously, $\bar{L}$ is a multiplicative ideal of $H(R)$, and so $L$ is a multiplicative ideal of $H(R)$.

Put $M = LJ(R)L = \{xay \mid a \in J(R), x, y \in L\}$. Clearly, $M$ is a multiplicative subsemigroup of $R$, because $J(R)$ is an ideal of $R$. Denote by $I$ the subring generated in $R$ by $L$. It follows that $I$ is an ideal of $R$ and $\bar{I}$ is an ideal of $\bar{R}$. Since $\bar{I} \neq 0$ and $\bar{R}$ is prime, we get $IJ(R) \neq 0$, and so $\bar{IJ}(R) \bar{I} \neq 0$. Since $I$ consists of all finite sums of elements of $L$, we get $\bar{M} \neq 0$. We shall show that every element in $\bar{M}$ is nilpotent.

Fix any non-zero $\bar{w} \in \bar{M}$. There exists $w = axb$, such that $a, b \in L$, $x \in J(R)$, $x = \sum_{k=1}^s x_k$, and all $x_k$ are homogeneous. Elements $\bar{a}$ and $\bar{b}$ belong to the same completely 0-simple factor $F$ of $D_m$. By [16, Theorem 1.3], we can represent $F$ as a Rees
matrix semigroup $F = \mathcal{M}(G^0, I, \Lambda; P)$. (The definitions and standard properties of completely 0-simple semigroups and Rees matrix semigroups can be also found in each of the following monographs [3], [5], [6].) Let $\vec{a} = (i, g, \lambda)$ and $\vec{b} = (j, h, \mu)$, where $i, j \in I$, $\lambda, \mu \in \Lambda$, $g, h \in G$. It is routine to verify that $ax_k b \in (i, G^0, \mu)$ for every $k = 1, \ldots, s$.

If $p_{\mu \iota} = 0$, then $(i, G^0, \mu)^2 = 0$ in $F$, and therefore $(axb)^2 = 0$ in $F$. Hence every element $(ax_k b)(ax_k b)$ has a smaller rank in $D_m$ than $\vec{a}$. Therefore $(ax_k b)(ax_k b) = 0$ where $1 \leq k, \iota \leq s$. Thus $w^2 = 0$, as required.

Further, consider the case where $p_{\mu \iota} \neq 0$. By [16, Lemma 1.4], $P = (i, G, \mu)$ is a multiplicative subgroup of $D_m$. Put $T = \{ g \in G \mid R_{g^k} \cap P \neq 0 \}$. Clearly, $T$ is a subsemigroup of $G$. Let $n$ be the degree of a multilinear identity (1) satisfied in $R$. For any $g_1, \ldots, g_n \in T$, we can choose $r_k \in R_{g_k}$ such that $r_k \in P$, where $k = 1, \ldots, n$. Applying (1) we get (2), which implies that $T$ is permutational. By [16, Theorem 19.8], $T$ generates a permutational subgroup $Q$ in $G$. Lemma 6 shows that $J(R_Q)$ is nil.

Since all non-zero summands $ax_k b$ belong to $P$, where $k = 1, \ldots, s$, we get $ax_k b \in R_Q$, and so $axb \in J(R_Q)$ by Lemma 4. Therefore $w = axb$ is a nilpotent element.

Thus $M$ is a multiplicative nil subsemigroup of $D_m$. Hence $M^q = 0$ for some $q > 1$ by [18, Proposition 2.6.30]. Since $M$ is, evidently, closed under multiplication by elements of $H(R)$, it follows that $M$ generates a nilpotent ideal in $\bar{R}$. This contradicts the primeness of $R$ and completes the proof.

REFERENCES