NONEXISTENCE OF STABLE CURRENTS IN SUBMANIFOLDS OF A PRODUCT OF TWO SPHERES

Xue-shan Zhang

Dedicated to Professor Yuen-da Wang on his 68th birthday

By using techniques of the calculus of variations in geometric measure theory, we investigate the nonexistence of stable integral currents in $S^n_1 \times S^n_2$ and its immersed submanifolds. Several vanishing theorems concerning the homology group of these manifolds are established.

0. INTRODUCTION

For any compact Riemannian manifold $M$, a theorem of Federer and Fleming [2] tells us that any non-trivial integral homology class in $H_p(M, \mathbb{Z})$ corresponds to a stable integral current. By establishing a second variation formula for minimal integral currents and applying it to different situations of $M$, Lawson and Simons [3] investigated the nonexistence of stable integral currents in $M$ and showed some vanishing theorems concerning the $p$th singular homology group $H_p(M, \mathbb{Z})$ of $M$ with integer coefficients. For an immersed submanifold $M$ of the unit sphere $S^n$, they showed the following theorem.

THEOREM. (Lawson and Simons [3]). Let $M^m$ be a compact submanifold of $S^n$ with the second fundamental form $h$, and $p$ a given integer, $p \in (0, m)$. If for any $x \in M$ and any orthonormal basis $\{e_i, e_\alpha\}$ ($i = 1, \ldots, p; \alpha = p + 1, \ldots, m$) of $T_xM$ the following condition is satisfied

\[(0.1)\quad B(\xi) = \sum_{i, \alpha} |2\|h(e_i, e_\alpha)\|^2 - \langle h(e_i, e_i), h(e_\alpha, e_\alpha) \rangle| < p(m - p),\]

then there is no stable $p$-current in $M$ and hence

$$H_p(M, \mathbb{Z}) = H_{m-p}(M, \mathbb{Z}) = 0.$$ 

In this paper we shall extend the above theorem. We shall introduce a selfadjoint linear operator $Q^A$ on a $p$-subspace $V$ of the tangent space $T_xM$. Replacing $B(\xi)$ in (0.1) by the trace of $Q^A$, we shall prove the following two theorems.

Received 6 November 1990

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325
**Theorem 1.** There is no stable $p$-current in $S^{m_1} \times S^{m_2}$ and

$$H_p(S^{m_1} \times S^{m_2}, Z) = 0,$$

where $0 < p < m_1 + m_2$, $p \neq m_1$ and $p \neq m_2$.

**Theorem 2.** Let $\phi: M^m \to S^{n_1} \times S^{n_2}$ be an isometric immersion of a compact Riemannian manifold $M$ in $S^{n_1} \times S^{n_2}$, and $p$ a given integer, $p \in (0, m)$. If for any $x \in M$ and any $p$-subspace $V$ of $T_xM$

$$\text{tr } Q^A < 0,$$

then there is no stable $p$-current in $M$ and

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

1. **Integral Currents**

For later convenience, in this section we shall give a brief description of integral currents. We refer the reader to [2, 3] for more details.

Let $M^m$ be an $m$-dimensional compact Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle$ and Levi-Civita connection $\nabla$. And let $\mathcal{H}^p$ denote Hausdorff $p$-measure on $M$. A subset $S$ of $M$ is called a $p$-rectifiable set if $S$ is a countable union of disjoint $p$-dimensional $C^1$ submanifolds, up to sets of $\mathcal{H}^p$-measure zero. Consider over $S$ an $\mathcal{H}^p$-measurable section $\xi: S \to \Lambda^p T^*M$ with the property that for $\mathcal{H}^p$-almost all $x \in S$, $\xi_x$ is a simple vector of unit length which represents $T_x S$. Such a pair $(S, \xi)$ is called an oriented, $p$-rectifiable set.

The set of rectifiable $p$-currents is defined by

$$R_p(M) = \{ S = \sum_{n=1}^{\infty} nS_n; S_n = (S_n, \xi_n), M(S) = \sum_{n=1}^{\infty} n\mathcal{H}^p(S_n) < \infty \}.$$ 

It can be thought of as the group of infinite, summable chains of oriented $p$-rectifiable sets.

For an oriented, $p$-rectifiable set $S = (S, \xi)$ and a smooth $p$-form $\omega \in \Lambda^p(M)$, define

$$S(\omega) = \int_S \omega(\xi_x)dS^p(x).$$

This assigns to $S$ a continuous linear functional on $\Lambda^p(M)$. The boundary of $S$ is defined as the linear functional on $\Lambda^{p-1}(M)$ given by

$$(\partial S)(\omega) = S(d\omega).$$
In the case that $S$ and $\partial S$ are both rectifiable currents, $S$ is called an integral $p$-current. The space of integral $p$-currents is denoted by $T_p(M)$. The direct sum $T_r(M) = \bigoplus T_p(M)$ together with $\partial: T_r(M) \to T_r(M)$ forms a differential chain complex. For this complex there is the following theorem.

**Theorem.** (Federer and Fleming [2]). For each $p \geq 0$ there is a natural isomorphism

$$H_p(T_r(M)) \cong H_p(M, \mathbb{Z}).$$

And for each $\alpha \in H_p(T_r(M))$ there exists a current $S \in \alpha$ of "least mass", that is,

$$M(S) \leq M(S')$$

for all $S' \in \alpha$.

Consider a current $S \in \mathcal{R}_p(M)$ and a smooth vector field $X \in C(TM)$. Let $\phi_t: M \to M$ be the 1-parameter group of local diffeomorphisms generated by $X$. Then the rectifiable current $\phi_t^*S$ is given by

$$\phi_t^*(S)(\omega) = S(\phi_t^*\omega).$$

Its "mass" is

$$M(\phi_t^*S) = \int_M \|\phi_t^*S\| d\|S\|,$$

where $\tilde{S}$ is the field of oriented tangent planes of $S = \sum_n nS_n$, for each $x \in S_n$, $\tilde{S}_x = \xi_n(x)$.

A current $S \in \mathcal{R}_p(M)$ is said to be stable if for each vector field $X$ there is an $\varepsilon > 0$ such that

$$M(\phi_t^*S) \geq M(S)$$

for $|t| < \varepsilon$. This implies that for each $X$ we have

$$\frac{d}{dt} M(\phi_t^*S)\big|_{t=0} = 0, \quad \frac{d^2}{dt^2} M(\phi_t^*S)\big|_{t=0} \geq 0.$$

The following variation formulas have been derived by Lawson and Simons [3].

$$\frac{d}{dt} M(\phi_t^*S)\big|_{t=0} = \int_a X(\tilde{S}, S) d\|S\|,$$

$$\left(1.1\right) \quad \frac{d^2}{dt^2} M(\phi_t^*S)\big|_{t=0} = \int \left\{ -a(X(\tilde{S}, S))^2 + (aX(\tilde{S}, S) + \|aX(\tilde{S})\|^2 + \langle \nabla_{X,S} X, S \rangle \right\} d\|S\|,$$
where \( a^X : \wedge^p T_x M \to \wedge^p T_x M \) is a linear map given by
\[
a^X(X_1 \wedge \ldots \wedge X_p) = \sum_{j} X_1 \wedge \ldots \wedge a^X(X_j) \wedge \ldots \wedge X_p,
\]
and \( \nabla_{X_1} : \wedge^p T_x M \to \wedge^p T_x M \) is another linear map defined by
\[
\nabla_{X_1} X_1 \wedge \ldots \wedge X_p X = \sum_{j} X_1 \wedge \ldots \wedge \left( \nabla_{X_1} X_j \right) \wedge \ldots \wedge X_p,
\]
\[
\nabla_{X_1} X_j X = \nabla X \nabla_{X_1} X - \nabla \nabla_{X_1} X.
\]

To any simple \( p \)-vector \( \xi \in \wedge^p T_x M \) and \( X \in C(TM) \), let \( \phi_t \) be the flow generated by \( X \), and define
\[
Q_\xi(X) = \frac{d^2}{dt^2} \| \phi_t \|_t = 0.
\]
Then the expression (1.1) can be denoted by
\[
(1.2) \quad \frac{d^2}{dt^2} M(\phi_t \cdot S) |_{t=0} = \sum_n \int_{S_n} Q_{\xi_n}(X) d\mathcal{H}^p(x).
\]
If \( X = \nabla f \) for some \( f \in C^3(M) \), from [3, p.436] we have
\[
(1.3) \quad Q_\xi(X) = \left[ \sum_j \langle a^X(e_j), e_j \rangle \right]^2 + 2 \sum_{i, \alpha} \langle a^X(e_j), e_\alpha \rangle^2 + \sum_i \langle \nabla X, e_j X, e_j \rangle,
\]
where \( \{e_i, e_\alpha\} \ (i = 1, \ldots, p; \alpha = p + 1, \ldots, m) \) is an orthonormal basis of \( T_x M \) and \( \xi = e_1 \wedge \ldots \wedge e_p \).

2. A SELFADJOINT LINEAR OPERATOR

For a \( p \)-rectifiable set \( S \) in \( M \), we know that at \( \mathcal{H}^p \)-almost all points \( x \in S \), there exists an approximate \( p \)-space \( T_x S \subset T_x M \), to \( S \). In this section we shall introduce a selfadjoint linear operator on \( T_x S \). Its trace is equal to the trace of \( Q_\xi \) given by (1.3).

Let \( \phi : M^m \to N^n \) be an isometric immersion of a Riemannian manifold \( M \) into a Riemannian manifold \( N \). The Levi-Civita connections of \( M \) and \( N \) are denoted by \( \nabla \) and \( \nabla \) respectively. For any \( X, Y \in C(TM) \), we have
\[
\nabla_{X} Y = \nabla_{X} Y + h(X, Y),
\]
where $h$ is the second fundamental form of the immersion $\phi$. If $V(N, M)$ is the normal bundle of $M$ in $N$, for a smooth section $\nu \in C(V(N, M))$ we have

$$\nabla_X \nu = -A_\nu X + \nabla^N_X \nu,$$

where $A_\nu$ is the so-called the shape operator determined by $\nu$. We know that

(2.1) \hspace{1cm} \langle A_\nu X, Y \rangle = \langle h(X, Y), \nu \rangle.$$

For a given integer $p \in (0, m)$ let $V$ be a $p$-dimensional subspace in $T_{x}M$. Define a map $B_\nu : V \to V$ associated with $A_\nu$ by

$$B_\nu X = \text{orthogonal projection of } A_\nu X \text{ onto } V,$$

where $X \in V$. If $\{e_i\}$ is an orthonormal basis of $V$, we have

(2.2) \hspace{1cm} B_\nu X = \sum_i \langle A_\nu X, e_i \rangle e_i.$$

It may be seen that $B_\nu$ is a selfadjoint linear operator on $V$ because $A_\nu$ is selfadjoint linear. Let $\{\nu_\lambda\}$ be an orthonormal basis of the normal space $V_\nu(N, M)$ and $A_\lambda = A_{\nu_\lambda}$. Define a selfadjoint linear map $Q^A : V \to V$ associated with the immersion $\phi$ by

(2.3) \hspace{1cm} Q^A X = \sum_\lambda \left[ 2 \left( \sum_i \langle A^2_\lambda X, e_i \rangle e_i - B^2_\lambda X \right) - \langle \text{tr} A_\lambda - \text{tr} B_\lambda \rangle B_\lambda X \right],$$

where $X \in V$ and $\{e_i\}$ is an orthonormal basis of $V$. $Q^A$ is independent of the choice of orthonormal bases of $V_\nu(N, M)$ and $V$. And its trace is

(2.4) \hspace{1cm} \text{tr} Q^A = \sum_i \langle Q^A e_i, e_i \rangle

= \sum_\lambda \left[ 2 \left( \sum_i \langle A^2_\lambda e_i, e_i \rangle - \text{tr} B^2_\lambda \right) - \langle \text{tr} A_\lambda - \text{tr} B_\lambda \rangle \text{tr} B_\lambda \right].$$

Let $\{e_\alpha\}$ be an orthonormal basis of $V^\perp$ which is the orthogonal complement of $V$ in $T_{x}M$. Then $\{e_i, e_\alpha\}$ is an orthonormal basis of $T_{x}M$ and

$$\sum_i \langle A^2_\lambda e_i, e_i \rangle = \sum_{i, j} \langle A_\lambda e_i, e_j \rangle^2 + \sum_{i, \alpha} \langle A_\lambda e_i, e_\alpha \rangle^2,$$

$$\text{tr} B^2_\lambda = \sum_i \langle B^2_\lambda e_i, e_i \rangle = \sum_{i, j} \langle A_\lambda e_i, e_j \rangle^2.$$
Hence (2.4) becomes

\[(2.5) \quad \text{tr } Q^A = \sum_\lambda \left[ 2 \sum_{i, \alpha} (A_{\alpha} e_i, e_\alpha)^2 - (\text{tr } A_\lambda - \text{tr } B_\lambda) \text{tr } B_\lambda \right].\]

Now assume that \( \psi: N^n \to R^l \) is an isometric immersion of the Riemannian manifold \( N \) in the Euclidean space \( R^l \). Let \( D \) be the Levi-Civita connection on \( R^l \). Associated with the isometric immersion \( x = \psi \circ \phi: M^m \to R^l \), the shape operator \( A'_\nu \) determined by \( \nu \in C(V(R^l, M)) \) is given by

\[(2.6) \quad A'_\nu Y = -(D_Y \nu)^T,\]

where \( Y \in C(TM) \). In particular, if \( \nu \in C(V(N, M)) \),

\[(2.7) \quad A'_\nu Y = -(D_Y \nu)^T = -[\nabla_Y \nu + h(\nu, T)]^T = -(-A_\nu Y + \nabla_Y^\perp \nu)^T = A_\nu Y,\]

and if \( \nu \in C(V(R^l, N)) \),

\[(2.8) \quad A'_\nu Y = (A_\nu Y)^T.\]

For a given vector \( v \in R^l \), we define two vector fields \( v^T \) and \( v^\perp \) on \( M \) by

\[(2.9) \quad v^T(x) = \text{orthogonal projection of } v \text{ onto } T_x M, \quad v^\perp(x) = \text{orthogonal projection of } v \text{ onto } V_x(R^l, M).\]

To any unit, simple \( p \)-vector \( \xi \in \wedge^p T_x M \), we shall calculate the quadratic form \( Q_\xi(v^T) \) given by (1.3). Using (2.6), we have

\[(2.10) \quad \alpha^T(Y) = \nabla_Y v^T = (D_Y v - D_Y v^\perp)^T = A'_\perp Y, \quad \nabla_Y^\perp v^\perp = (D_Y v - D_Y v^T)^\perp = -h'(v^T, Y).\]

These imply

\[(2.11) \quad \nabla_{v^T, Y} v^T = (\nabla_{v^T} A')_{v^\perp} Y - A'_h(v^T, v^T) Y,\]

where \( \nabla_{v^T} A' \) is the derivative with respect to the connection of Van der Waerden-Bortolotti ([1, p.65]). Putting (2.10) and (2.11) into (1.3), we obtain

\[(2.12) \quad Q_\xi(V^T) = \left[ \sum_j \langle A'_{\perp} e_j, e_j \rangle \right]^2 + 2 \sum_{j, \alpha} \langle A'_{\perp} e_j, e_\alpha \rangle^2 + \sum_j \langle (\nabla_{v^T} A')_{v^\perp} e_j, e_j \rangle - \sum_j \langle A'_{h}(v^T, v^T) e_j, e_j \rangle.\]
Let \((S, \xi)\) be an oriented, \(p\)-rectifiable set. With a point \(x \in S\) associate a tangent \(p\)-space \(V = T_xS \subset T_xM\). Choose an orthonormal basis \(\{e_i, e_\alpha\}\) of \(T_xM\) such that \(\{e_i\}\) is a basis of \(V\) and \(\xi = e_1 \wedge \ldots \wedge e_p\). Suppose that \(\{\nu_\sigma\}\) is an orthonormal basis of \(V_x(R^l, M)\) associated with the immersion \(\psi \circ \phi: M \to R^l\), \(A_\sigma' = A_\nu_\sigma\), and \(Q_A'\) is the selfadjoint linear operator on \(V\) defined by (2.3). Consider \(Q_\xi\) as a quadratic form defined on the set

\[
\theta = \{v^T \in C(TM); v \in R^l, v^T \text{ is defined by (2.9)}\}.
\]

There is the following relation between the quadratic form \(Q_\xi\) and the operator \(Q_A'\).

**Proposition 1.** \(\text{tr} \, Q_\xi = \text{tr} \, Q_A'\), where

\[
\text{tr} \, Q_A' = \sum_\sigma \left[2 \sum_{i, \alpha} (A_\alpha' e_i, e_\alpha)^2 - (\text{tr} \, A_\sigma' - \text{tr} \, B_\sigma') \text{tr} \, B_\sigma' \right].
\]

**Proof:** Observing that at the given point \(x \in M\), \(\{e_i, e_\alpha, \nu_\sigma\}\) is an orthonormal basis of \(R^l\) and \((\nabla'_{x'} A')_\perp = 0\) as \(v^T = 0\) or \(v_\perp = 0\), from (2.12) we have

\[
\text{tr} \, Q_\xi = \sum_i Q_\xi(e_i) + \sum_\alpha Q_\xi(e_\alpha) + \sum_\sigma Q_\xi(\nu_\sigma)
\]

\[
= -\sum_{i, j} \langle A'_{\sigma}(e_i, e_i) e_j, e_j \rangle - \sum_{\alpha, j} \langle A'_{\sigma}(e_\alpha, e_\alpha) e_j, e_j \rangle
\]

\[
+ \sum_\sigma \left\{ \left[ \sum_j \langle A'_{\alpha} e_j, e_j \rangle \right]^2 + 2 \sum_{j, \alpha} (A'_{\alpha} e_j, e_\alpha)^2 \right\}
\]

\[
= -\sum_\sigma \left[ \sum_{i, j} \langle A'_{\alpha} e_i, e_i \rangle \langle A'_{\alpha} e_j, e_j \rangle + \sum_{\alpha, i} \langle A'_{\alpha} e_\alpha, e_\alpha \rangle \langle A'_{\alpha} e_i, e_i \rangle \right]
\]

\[
+ \sum_\sigma \left[ \left[ \sum_j \langle A'_{\alpha} e_j, e_j \rangle \right]^2 + 2 \sum_{j, \alpha} (A'_{\alpha} e_j, e_\alpha)^2 \right]
\]

\[
= \sum_{\sigma, i, \alpha} [2 \langle A'_{\alpha} e_i, e_\alpha \rangle^2 - \langle A'_{\alpha} e_\alpha, e_\alpha \rangle \langle A'_{\alpha} e_i, e_i \rangle].
\]

Since

\[
\text{tr} \, A_\sigma' = \sum_i \langle A'_{\alpha} e_i, e_i \rangle + \sum_\alpha \langle A'_{\alpha} e_\alpha, e_\alpha \rangle
\]

and

\[
\text{tr} \, B_\sigma' = \sum_i \langle A'_{\alpha} e_i, e_i \rangle
\]

we obtain \(\text{tr} \, Q_\xi = \text{tr} \, Q_A'\).
From the above proof, expression (2.14) can also be written as

\[(2.15) \quad \text{tr} \, Q^A' = \sum_{\sigma, i, \alpha} [2(A_\sigma^i e_i, e_\alpha)^2 - (A_\sigma^i e_\alpha, e_\alpha)(A_\sigma^i e_i, e_i)].\]

At a point \(x \in M\), we take an orthonormal basis \(\{\nu_{\lambda}, \eta_\alpha\}\) of \(V_x(R^l, M)\) so that \(\{\nu_{\lambda}\}\) and \(\{\eta_\alpha\}\) are bases of \(V_x(N, M)\) and \(V_x(R^l, N)\) respectively. From (2.7), (2.8) and (2.15) we obtain

\[(2.16) \quad \text{tr} \, Q^A' = \text{tr} \, Q^A + \overline{A}(V),\]

where \(\text{tr} \, Q^A\) is given by (2.5) and

\[(2.17) \quad \overline{A}(V) = \sum_{\alpha, i, \alpha} [2(A_\alpha^i e_i, e_\alpha)^2 - (A_\alpha^i e_\alpha, e_\alpha)(A_\alpha^i e_i, e_i)].\]

Note that \(\overline{A}(V) \neq \text{tr} \, Q^A\).

**Theorem.** Let \(\phi : M^m \to N^n\) be an isometric immersion of a compact Riemannian manifold \(M\) in a submanifold \(N\) of \(R^l\), and \(p\) a given integer, \(p \in (0, m)\). Suppose that for any \(x \in M\) and any \(p\)-subspace \(V\) of \(T_xM\),

\[(2.18) \quad \text{tr} \, Q^A < -\overline{A}(V).\]

Then there is no stable \(p\)-current in \(M\) and

\[H_p(M, Z) = H_{m-p}(M, Z) = 0.\]

**Proof:** Let \(\theta\) be the set given by (2.13). If \(v^T \in \theta\), \(v^T\) is the gradient \(\nabla f\) of the function \(f(x) = \langle v, x \rangle\) on \(M\). To each \(S \in R_p(M)\) associate a quadratic form \(Q_S\) on \(\theta\) as follows. For \(X \in \theta\) let \(\phi_t\) be the flow generated by \(X\) and set

\[Q_S(X) = \frac{d^2}{dt^2} M(\phi_{t*}S)|_{t=0}.\]

From (1.2) we have

\[\text{tr} \, Q_S = \sum_n n \int_{S_n} \text{tr} \, Q_{\xi_{tn}} d\mathcal{H}^p(x).\]

But from (1.3), Proposition 1 and (2.16), (2.18) implies \(\text{tr} \, Q_{\xi_{tn}} < 0\) for any \(n\). Therefore \(\text{tr} \, Q_S < 0\). This implies that there is no stable \(p\)-current in \(M\). By using Federer-Fleming's theorem, we have

\[H_p(M, Z) = H_{m-p}(M, Z) = 0.\]

If \(N^n\) in the above theorem is a totally umbilical submanifold immersed in \(R^l\), \(N^n\) is of constant curvature \(c \geq 0\). In this case \(\overline{A}(V)\) given by (2.17) becomes

\[\overline{A}(V) = -p(m-p)c.\]

Hence we obtain
**COROLLARY 1.** Let $M^m$ be a compact submanifold immersed in a totally umbilical submanifold $N^n$ of $R^l$. If for any $x \in M$ and any p-subspace $V$ of $T_x M$,

$$\text{tr } Q^A < p(m - p)c,$$

where $c$ is the sectional curvature of $N$, then there is no stable p-current in $M$ and

$$H_p(M, Z) = H_{m-p}(M, Z) = 0.$$

**REMARK 1.** In the case $N^n = S^n$ we have $c = 1$, and Corollary 1 becomes Lawson and Simons’ theorem. And when $N^n = R^n$, Corollary 1 is due to Xin [4, Theorem 1].

3. **MAIN RESULTS**

Let $m_1 + m_2 = m$ and

$$M^m = S^{m_1} \times S^{m_2} = \{(x_1, x_2) \in R^{m+2}; x_\lambda \in R^{m\lambda+1} \text{ and } \|x_\lambda\| = 1, \lambda = 1, 2\}.$$

Then $M^m$ is a submanifold of $R^{m+2}$. At $x = (x_1, x_2) \in M^m$ we take an orthonormal basis $\{\nu_\lambda\}$ of $V_x(R^{m+2}, M)$ as follows

$$\nu_1 = (x_1, 0), \quad \nu_2 = (0, x_2).$$

It may be seen that the shape operators $A_\lambda$ can be denoted by the matrices

$$A_1 = -\left( \begin{array}{cc} I_1 & 0 \\ 0 & 0 \end{array} \right), \quad A_2 = -\left( \begin{array}{cc} 0 & 0 \\ 0 & I_2 \end{array} \right),$$

where $I_\lambda$ is the $m_\lambda \times m_\lambda$ identity matrix for each $\lambda = 1, 2$. Hence for any $X \in T_x M$ we have $A_\lambda X = -X_\lambda$, where $X_\lambda$ is the orthogonal projection of $X$ onto $T_x S^{m_\lambda}$.

At $x \in M$, we take an orthonormal basis $\{e_i, e_\alpha\}$ of $T_x M$ so that $\{e_i\}$ is an orthonormal basis of the p-subspace $V$. Denoting the orthogonal projection of $e_i$ (respectively $e_\alpha$) onto $T_x S^{m_\lambda}$ by $e_{i\lambda}$ (respectively $e_{\alpha\lambda}$), we have

$$\langle A_\lambda e_i, e_\alpha \rangle = -\langle e_{i\lambda}, e_{\alpha\lambda} \rangle,$$

$$\text{tr } A_\lambda = \sum_i \langle A_\lambda e_i, e_i \rangle + \sum_\alpha \langle A_\lambda e_\alpha, e_\alpha \rangle = -\sum_i \|e_{i\lambda}\|^2 - \sum_\alpha \|e_{\alpha\lambda}\|^2,$$

$$\text{tr } B_\lambda = \sum_i \langle B_\lambda e_i, e_i \rangle = \sum_i \langle A_\lambda e_i, e_i \rangle = -\sum_i \|e_{i\lambda}\|^2.$$

Substituting these into (2.5) we obtain

$$\text{tr } Q^A = \sum_{i, \alpha} \left[ 2(\langle e_{i1}, e_{\alpha1} \rangle^2 + \langle e_{i2}, e_{\alpha2} \rangle^2) - (\|e_{\alpha1}\|^2 \|e_{i1}\|^2 + \|e_{\alpha2}\|^2 \|e_{i2}\|^2) \right].$$

https://doi.org/10.1017/S0004972700029762 Published online by Cambridge University Press
Since $e_i = e_{i1} + e_{i2}$ and $e_\alpha = e_{\alpha 1} + e_{\alpha 2}$, we have
\[
\|e_{i1}\|^2 + \|e_{i2}\|^2 = 1, \quad \|e_{\alpha 1}\|^2 + \|e_{\alpha 2}\|^2 = 1,
\]
\[
\langle e_{i1}, e_{\alpha 1}\rangle + \langle e_{i2}, e_{\alpha 2}\rangle = 0.
\]
So (3.1) becomes
\[
\text{(3.2)} \quad \text{tr} Q^A = \sum_{i, \alpha} \left[ 4\langle e_{i1}, e_{\alpha 1}\rangle^2 + \|e_{i1}\|^2 + \|e_{\alpha 1}\|^2 - 2 \|e_{i1}\|^2 \|e_{\alpha 1}\|^2 \right] - p(m - p).
\]

**LEMMA.** For each pair of fixed indices $i, \alpha$, let
\[
\text{(3.3)} \quad f_{i\alpha} = 4\langle e_{i1}, e_{\alpha 1}\rangle^2 + \|e_{i1}\|^2 + \|e_{\alpha 1}\|^2 - 2 \|e_{i1}\|^2 \|e_{\alpha 1}\|^2.
\]
Then $f_{i\alpha} \leq 1$ and equality holds if and only if $e_i \in T_{x_1} S^{m_1}$ and $e_\alpha \in T_{x_2} S^{m_2}$, or $e_\alpha \in T_{x_1} S^{m_1}$ and $e_i \in T_{x_2} S^{m_2}$.

**PROOF:** Let $e_i^s (s = 1, 2, \ldots, m_i)$ (respectively $e_\alpha^s$) be the components of $e_{i1}$ (respectively $e_{\alpha 1}$) with respect to an orthonormal basis of $T_{x_1} S^{m_1}$. Then (3.3) becomes
\[
\text{(3.4)} \quad f_{i\alpha} = 4 \left( \sum_{s} e_i^s e_\alpha^s \right)^2 + \sum_{s} (e_i^s)^2 + \sum_{s} (e_\alpha^s)^2 - 2 \sum_{s} (e_i^s)^2 (e_\alpha^s)^2,
\]
where
\[
\text{(3.5)} \quad 0 \leq \sum_{s} (e_i^s)^2 \leq 1, \quad 0 \leq \sum_{s} (e_\alpha^s)^2 \leq 1.
\]
In order to seek the maximum of $f_{i\alpha}$ under the condition (3.5), partially differentiating (3.4) with respect to each variable and equating to zero, we obtain
\[
4 \left( \sum_{t} e_t^s e_\alpha^s \right) e_\alpha^s + e_i^s - 2 \sum_{t} (e_\alpha^s)^2 e_t^s = 0,
\]
\[
4 \left( \sum_{t} e_t^s e_\alpha^s \right) e_t^s + e_\alpha^s - 2 \sum_{t} (e_i^s)^2 e_\alpha^s = 0.
\]
These equations can be expressed by
\[
\text{(3.6)} \quad u e_i^s = 4w e_\alpha^s,
\]
\[
\text{(3.7)} \quad v e_\alpha^s = 4w e_i^s,
\]
where $u = 2 \sum (e^*_\alpha)^2 - 1$, $v = 2 \sum (e^\epsilon_i)^2 - 1$, $w = \sum e^*_\alpha e^\epsilon_i$. From (3.6) we obtain

\begin{align*}
(3.8) & \quad \frac{1}{2} (1 + v) u = 4 w^2, \\
(3.9) & \quad uw = 2(\bar{y} + u)w.
\end{align*}

Similarly, from (3.7) we have

\begin{align*}
(3.10) & \quad \frac{1}{2} (1 + u) v = 4 w^2, \\
(3.11) & \quad vw = 2(1 + v)w.
\end{align*}

(3.8) and (3.10) give $u = v$. If $w \neq 0$, from (3.9) we have $u = 2(1 + u)$. So $u = -2$ and thus $\sum (e^*_\alpha)^2 = -1/2$; this is impossible. Therefore $w$ must be zero. And from (3.8) we have $(1 + v)u = 0$, that is,

\begin{enumerate}
  \item $1 + v = 0$; this gives $e^\epsilon_i = 0$, and $e^*_\alpha = 0$ from $u = v$; or
  \item $u = v = w = 0$; this implies $\sum (e^\epsilon_i)^2 = \sum (e^*_\alpha)^2 = 1/2$ and $\sum e^*_\alpha e^\epsilon_i = 0$.
\end{enumerate}

From (3.4),

\begin{enumerate}
  \item[(i)] implies $f_{i\alpha} = 0$, and (ii) implies $f_{i\alpha} = 1/2$.
\end{enumerate}

Besides, if $\sum (e^\epsilon_i)^2 = 1$, that is, $\|e_{i1}\| = 1$, then $e_{i2} = 0$ and thus

$$\sum e^*_\alpha e^\epsilon_i = \langle e_{i1}, e_{\alpha1} \rangle = -\langle e_{i2}, e_{\alpha2} \rangle = 0.$$ 

From (3.4) we have

$$f_{i\alpha} = 1 - \sum (e^*_\alpha)^2 \leq 1,$$

equality holds if and only if $\sum (e^*_\alpha)^2 = 0$. Combining (*), we see that under the condition $f_{i\alpha} \leq 1$. Clearly equality holds if and only if $e_i \in T_{x_1}S^{m_1}$ and $e_\alpha \in T_{x_2}S^{m_2}$, or $e_\alpha \in T_{x_1}S^{m_1}$ and $e_i \in T_{x_2}S^{m_2}$. 

From this lemma, (3.2) gives

$$\text{tr} Q^A = \sum_{i, \alpha} f_{i\alpha} - p(m - p) \leq 0.$$ 

It is easy to check that equality holds if and only if $\{e_i\} \subset T_{x_1}S^{m_1}$ and $\{e_\alpha\} \subset T_{x_2}S^{m_2}$, or $\{e_\alpha\} \subset T_{x_1}S^{m_1}$ and $\{e_i\} \subset T_{x_2}S^{m_1}$. These imply the $p$-subspace $V = T_{x_1}S^{m_1}$ and $V^\perp = T_{x_2}S^{m_2}$, or $V = T_{x_2}S^{m_2}$ and $V^\perp = T_{x_1}S^{m_1}$. Hence we have
PROPOSITION 2. For the isometric immersion $S^{m_1} \times S^{m_2} \to R^{m+2}$ $(m = m_1 + m_2)$, $\text{tr } Q^A \leq 0$. Furthermore, if $p \in \{m_1, m_2\}$, $\text{tr } Q^A < 0$.

PROOF OF THEOREM 1: In the Theorem of Section 2 we take $M^m = S^{m_1} \times S^{m_2}$ and $N^n = R^{m+2}$; then $A(V) = 0$ from (2.17). Combining Proposition 2 and the Theorem in Section 2, we obtain Theorem 1.

PROOF OF THEOREM 2: Let $\{e_i, e_a\}$ be an orthonormal basis of $T_xM$ so that $\{e_i\}$ is a basis of the $p$-subspace $V$. Note that the shape operators of $S^{m_1} \times S^{m_2} \to R^{m_1+n_2+2}$ are $A_a (a = 1, 2)$, $A_a X = -X_a$ where $X \in T_xM$ and $X_a$ is the orthogonal projection of $X$ onto $T_xS^m$. So $\langle A_a e_i, e_a \rangle = -\langle e_i, e_a \rangle$, $\langle A_a e_i, e_i \rangle = -\|e_i\|^2$, and $\langle A_a e_a, e_a \rangle = -\|e_a\|^2$. Thus (2.17) becomes

$$A(V) = \sum_{i, a} \left[2((e_{i1}, e_{a1})^2 + (e_{i2}, e_{a2})^2) - (\|e_{a1}\|^2 \|e_{i1}\|^2 + \|e_{a2}\|^2 \|e_{i2}\|^2)\right].$$

So from the Lemma we have $A(V) = \sum_{i, a} f_{ia} - p(m - p) \leq 0$. Combining this with the Theorem in Section 2 we obtain Theorem 2.

COROLLARY 2. Let $M^m$ be a compact submanifold isometrically immersed in $S^{m_1} \times S^{m_2}$. If for any point $x \in M$ and any $p$-subspace $V$ of $T_xM (0 < p < m)$ the selfadjoint linear operator $Q^A$ on $V$ is negative definite, then there is no stable $p$-current in $M$.

REMARK 2. Theorems and corollaries in this paper are true if one replaces the integers by any finitely generated abelian coefficient group because the Federer-Fleming theorem remains true in the latter case. Besides, one can easily generalise these theorems and corollaries to arbitrary varifolds on $M$ from [3, p.436, Remark 4].

REFERENCES


Department of Mathematics
Xian Institute of Metallurgy and
Construction Engineering
Xian 710055
Peoples Republic of China