AVERAGING OPERATORS ON THE RING OF CONTINUOUS FUNCTIONS ON A COMPACT SPACE

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1. Introduction

In this note we answer the following question: Given $C(X)$ the lattice-ordered ring of real continuous functions on the compact Hausdorff space $X$ and $T$ an averaging operator on $C(X)$, under what circumstances can $X$ be decomposed into a topological product $\mathbb{A} \times \mathbb{B}$ such that $\mathbb{B}$ supports a measure $m$ and $Tf = h$ where

$$h(\alpha, \beta) = \int_\mathbb{B} f(\alpha, \xi)dm(\xi).$$

By an averaging operator we mean a linear transformation $T$ on $C(X)$ such that:

1. $T$ is positive, that is, if $f > 0$ ($f(x) \geq 0$ for all $x \in X$ and $f(x) > 0$ for some $x \in X$), then $Tf > 0$.
2. $T(fg) = (Tf)(Tg)$.
3. $T1 = 1$ where $1(x) = 1$ for all $x \in X$.

If $X$ is the unit square $[0, 1] \times [0, 1]$, $f(x_1, x_2) \in C(X)$, then the linear operator $T$ given by the relation,

$$(Tf)(x_1, x_2) = \int_0^1 f(x_1, \xi)d\xi,$$

is clearly an averaging operator. Our problem in effect is to characterize those pairs $(X, T)$ where $X$ is a compact Hausdorff space and $T$ an averaging operator such that $X$ can be decomposed into a product and $T$ can be written as an integration over one factor of that product.

Averaging operators have been the object of considerable attention. A recent survey of the literature is given by G. Birkhoff in [1]. In [2] the author discusses the above question when the role of $C(X)$ is played by a

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2 The author wishes to express his gratitude to the referee whose suggestions greatly simplified the proof of the results.

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bounded $F$-ring. Bounded $F$-rings are one of the more natural generalizations of $L^\infty$-spaces.

2. When $X$ is known to be a product space

Let $X$ be a topological product $\mathfrak{X} \times \mathfrak{Y}$ of compact Hausdorff spaces $\mathfrak{X}$ and $\mathfrak{Y}$. Let $D$ be the subalgebra of $C(X)$ consisting of all $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on $\alpha \in \mathfrak{X}$, that is $f(\alpha, \beta) = f_1(\alpha)$ for all $\alpha \in \mathfrak{X}$ and $\beta \in \mathfrak{Y}$ where $f_1$ is a continuous function on $\mathfrak{X}$. Let $E$ be the subalgebra consisting of all functions $f$ which depend only on $\beta$.

**Theorem 1.** A necessary and sufficient set of conditions for the existence of a normalized Borel measure $m$ on $\mathfrak{Y}$ such that
\[
(Tf)(\alpha, \beta) = \int_\mathfrak{Y} f(\alpha, \xi) dm(\xi)
\]
for all $f \in C(X)$, $\alpha \in \mathfrak{X}$, $\beta \in \mathfrak{Y}$ is the following:

(i) for all $f \in E$, $Tf$ should be a constant function,

(ii) for all $f \in D$, $g \in E$, $T(fg) = f \cdot T(g)$.

**Proof.** (A) The necessity is trivial.

(B) Sufficiency. For each continuous function $\varphi$ on $\mathfrak{Y}$, let $\varphi^*$ be the continuous function on $X$ given by
\[
\varphi^*(\alpha, \beta) = \varphi(\beta).
\]
Define $U(\varphi)$ to be the value of the constant function $T\varphi^*$. Then, since $T$ is a positive linear operator on $X$, $U$ is a positive Radon measure and hence by the Riesz representation theorem, there exists a normalized Borel measure $m$ on $\mathfrak{Y}$ such that
\[
U(\varphi) = \int_\mathfrak{Y} \varphi(\xi) dm(\xi)
\]
for every continuous function $\varphi$ on $\mathfrak{Y}$.

For all $f \in C(X)$, let
\[
(Vf)(\alpha, \beta) = \int_\mathfrak{Y} f(\alpha, \xi) dm(\xi).
\]
Since for fixed $\alpha$, $f(\alpha, \beta)$ is continuous in $\beta$ and hence a continuous function on $\mathfrak{Y}$, it follows that $Vf(\alpha, \beta)$ is a well defined function on $X$. Since $m$ is a normalized Borel measure, it follows that $Vf \in C(X)$ and $V$ is a bounded linear operator.

Now for $f \in D$ and $g \in E$,

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* A normalized Borel measure on $X$ is a non-negative countably additive measure $\mu$ on the Borel sets of $X$ for which $\mu(X) = 1$. 

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{\mathit{T}(\mathit{f} \cdot \mathit{g})}(\alpha, \beta) = \mathit{f}(\alpha, \beta) \cdot \{(\mathit{T}g)(\alpha, \beta)\}
\tag{by (ii)}

\begin{align*}
= \mathit{f}(\alpha, \beta) \int g(\alpha, \xi)\mathit{d}m(\xi)
\end{align*}

because \(g\) does not depend on \(\alpha\). Since \(\mathit{f}(\alpha, \beta)\) is independent of \(\beta\)

\begin{align*}
\mathit{T}(\mathit{f} \cdot \mathit{g})(\alpha, \beta) &= \int g(\alpha, \beta)g(\alpha, \xi)\mathit{d}m(\xi)
\end{align*}

Thus \(\mathit{T}(\mathit{f} \cdot \mathit{g}) = \mathit{V}(\mathit{f} \cdot \mathit{g})\) for all \(\mathit{f} \in \mathcal{D}, \mathit{g} \in \mathcal{E}\). However, the subalgebra of \(C(X)\) generated by \(\mathcal{D} \cup \mathcal{E}\) is dense in \(C(X)\) and by linearity \(\mathit{T}\) and \(\mathit{V}\) coincide on this subalgebra. Therefore \(\mathit{T} = \mathit{V}\) on \(C(X)\).

3. The main result

In order to prove our main result (Theorem 3) we employ certain results of MacDowell [3]. MacDowell makes the following definitions:

(i) A subalgebra \(\mathcal{A}\) of \(C(X)\) is analytic if it contains the constant functions.

(ii) Two subalgebras \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are additively related if for \(\mathit{f} \in \mathcal{A}_1\) and \(\mathit{g} \in \mathcal{A}_2\) either 
\(||\mathit{f} + \mathit{g}|| = ||\mathit{f}|| + ||\mathit{g}||\) or 
\(||\mathit{f} - \mathit{g}|| = ||\mathit{f}|| + ||\mathit{g}||\) \footnote{By \(||\mathit{f}||\) we mean the usual sup norm: \(||\mathit{f}|| = \mathit{sup}_{x \in X} |\mathit{f}(x)|\).}

We define \(\mathcal{A}_1\) and \(\mathcal{A}_2\) to be multiplicatively related if for \(\mathit{f} \in \mathcal{A}_1\) and \(\mathit{g} \in \mathcal{A}_2\),
\(||\mathit{g}|| = ||\mathit{f}|| \cdot ||\mathit{g}||\).

It is a matter of direct verification that subalgebras \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are additively related if and only if they are multiplicatively related.

THEOREM 2. Let \(X\) be an arbitrary compact Hausdorff space. If \(\mathcal{A}\) is a subalgebra of \(C(X)\) such that for all \(\mathit{g} \in \mathcal{A}\), \(\mathit{Tg}\) is a constant function, then \(\mathit{TC}\) and \(\mathcal{A}\) are multiplicatively (and hence additively) related.

PROOF. First note that \(\mathit{TC}\) is closed under multiplication, because if \(\mathit{f}_1, \mathit{f}_2 \in \mathit{TC}\) and \(\mathit{f}_1 = \mathit{T}\mathit{h}_1, \mathit{f}_2 = \mathit{T}\mathit{h}_2\), then
\(\mathit{T}(\mathit{h}_1\mathit{T}\mathit{h}_2) = \mathit{T}\mathit{h}_1\mathit{T}\mathit{h}_2 = \mathit{f}_1\mathit{f}_2\).

Suppose \(\mathit{TC}\) and \(\mathcal{A}\) are not multiplicatively related. Then there exists \(\mathit{f} \in \mathit{TC}\) and \(\mathit{g} \in \mathcal{A}\) such that
\(||\mathit{g}|| < ||\mathit{f}|| \cdot ||\mathit{g}||\).

Thus \(|\mathit{f}(\mathit{x})|| < 1\) when \(|\mathit{g}(\mathit{x})|| = 1\). Therefore there is an \(\mathit{x} \in X\) for which
\(|\mathit{f}(\mathit{x})|^2 \cdot |\mathit{g}(\mathit{x})|^2 < |\mathit{g}(\mathit{x})|^2\),

and so

\(\mathit{T}(\mathit{f} \cdot \mathit{g}) = \mathit{V}(\mathit{f} \cdot \mathit{g})\) for all \(\mathit{f} \in \mathcal{D}, \mathit{g} \in \mathcal{E}\). However, the subalgebra of \(C(X)\) generated by \(\mathcal{D} \cup \mathcal{E}\) is dense in \(C(X)\) and by linearity \(\mathit{T}\) and \(\mathit{V}\) coincide on this subalgebra. Therefore \(\mathit{T} = \mathit{V}\) on \(C(X)\).
(2) $0 \leq T(fg^2) < T(g^2)$

because $T$ is a positive operator. Now since $g^2 \in A$, $T(g^2)$ is constant and

(3) $||T(fg^2)|| < T(g^2)$.

On the other hand $f^2 \in TC$ and so $f^2 = Th$ for some $h \in C(X)$. Thus

\[ T(fg^2) = T(g^2)f^2, \]

and

\[ ||T(fg^2)|| = T(g^2) ||f^2|| = T(g^2) \]

which contradicts Equation (3). Thus $TC$ and $A$ are multiplicatively related.

**Lemma.** Let $X = \mathcal{A} \times \mathcal{B}$ be a topological product of compact Hausdorff spaces $\mathcal{A}$, $\mathcal{B}$. Let $D$ consist of all $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on $\alpha$ and let $E$ consist of those $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on $\beta$. If $A$, $B$ are analytic subalgebras of $C(X)$ such that $A \subseteq D$ and $B \subseteq E$ and if the subalgebra generated by $A \cup B$ is dense in $C(X)$, then $A$ is dense in $D$ and $B$ is dense in $E$.

**Proof.** Follows from the Stone-Weierstrass theorem.

Now we come to prove our main result and answer the question posed in the Introduction.

**Theorem 4.** Let $T$ be an averaging operator on $C(X)$ where $X$ is a compact Hausdorff space and let $A$ be an analytic subalgebra of $C(X)$ such that for all $g \in A$, $Tg$ is a constant function. If $TC \cup A$ separates points of $X$, then $X$ can be decomposed into a topological product $\mathcal{A} \times \mathcal{B}$ of compact Hausdorff spaces and there is a normalized Borel measure $m$ on $\mathcal{B}$ such that

(i) for every $f \in C(X)$

(4) \[ (Tf)(\alpha, \beta) = \int_\mathcal{B} f(\alpha, \xi)d\mu(\xi) \]

and

(ii) $A$ is contained in the subalgebra $E$ of $C(X)$ consisting of all $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on $\beta$.

**Proof.** $TC$ is clearly an analytic subalgebra of $C(X)$. By Theorem 2, $TC$ and $A$ are additively related. Since $TC \cup A$ separates points of $X$, the algebra generated by it is dense in $C$. Therefore by [3] T14, $X$ is a product space $\mathcal{A} \times \mathcal{B}$. Let $D$ be the subalgebra of $C$ consisting of all $f \in C$ such that $f(\alpha, \beta)$ depends only on $\alpha$ and let $E$ be the subalgebra of $C$ consisting of $f \in C$ such that $f(\alpha, \beta)$ depends only on $\beta$.

The spaces $\mathcal{A}$, $\mathcal{B}$ are evidently compact. Since they have for their points certain closed subsets of $C$ and for their topologies the usual topologies for spaces of closed sets, they are Hausdorff.

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In his statement of T14, MacDowell does not say that
\[(5)\quad TC \subset D, \quad A \subset E,\]
but these relations can easily be deduced from the proof of his T13 in the following manner. To use his notation, put \(F_1 = TC, \ F_2 = A, \ \mathcal{E}_1 = \emptyset, \ \mathcal{E}_2 = \emptyset.\) In his proof ([3]T12) a homeomorphism \(\varphi\) of \(X\) onto a space \(\mathcal{S}\) is constructed by letting
\[
\varphi(x) = \{f \in C| |f(x)| = ||f||\}.
\]
The points of \(\mathcal{S}\) are there subsets of \(C.\) A homeomorphism \(F\) of \(\mathcal{S}\) onto \(\mathcal{S}_1 \times \mathcal{S}_2\) is then constructed by putting
\[
F(S) = (S \cap F_1, S \cap F_2).
\]
If \(h \in F_1\) then \(h\varphi^{-1}F^{-1}(\mu_1, \mu_2)\) depends only on \(U_1,\) because if
\[
x_0 = \varphi^{-1}F^{-1}(\mu_1, \mu_2), \quad x'_0 = \varphi^{-1}F^{-1}(\mu_1, \mu'_2)
\]
then for
\[
S_0 = \{f \in C| |f(x_0)| = ||f||\},
\]
\[
S'_0 = \{f \in C| |f(x'_0)| = ||f||\}
\]
we have
\[
\mu_1 = S_0 \cap F_1 = S'_0 \cap F_1,
\]
and so for every \(f \in F_1\) either
\[
(a) \quad |f(x_0)| = ||f|| \quad \text{and} \quad |f(x'_0)| = ||f||
\]
or
\[
(b) \quad |f(x_0)| < ||f|| \quad \text{and} \quad |f(x'_0)| < ||f||.
\]
Now, using \(h \in F_1,\) construct the non-negative function
\[
g = ||\{h - h(x_0)\}^2|| - \{h - h(x_0)\}^2
\]
in \(F_1.\) It is clear that
\[
0 \leq g(x) \leq ||\{h - h(x_0)\}^2||
\]
for all \(x \in X\) and that
\[
g(x_0) = ||\{h - h(x_0)\}^2||.
\]
Therefore
\[
||g|| = g(x_0),
\]
and from (a) it follows that \(g(x'_0) = ||g||.\) Thus
\[
\{h(x'_0) - h(x_0)\}^2 = 0 \quad \text{so} \quad h(x_0) = h(x'_0).
Since for each $h \in TC = F_1$, 

$$(h \varphi^{-1} F^{-1})(\mu_1, \mu_2)$$

depends only on $\mu_1$, it follows that $TC \subseteq D$. Similarly $A \subseteq E$. Therefore statement (5) is valid.

By the Lemma, $TC$ is dense in $D$, and $A$ is dense in $E$. Hence every $f \in D$ is a limit of functions in $TC$ and every $g \in E$ is a limit of functions in $A$. However, for $h \in A$, $Th$ is constant, and so $Tg$ is a constant for each $g \in E$. Similarly if $f \in D$ and $g \in E$, then by the density of $TC$ in $D$

$$T(fg) = fTg.$$ 

Thus the conditions for the application of Theorem 1 are satisfied and so there exists a normalized Borel measure $m$ on $\mathcal{B}$ for which Equation (4) is valid.

Remarks. (1) It is clear that any operator on $C(X)$ of the form given in Equation (4) is an averaging operator. Then $TC$ is the set of functions which depend only on $\alpha$, and $A$ can be taken as the set of functions depending only on $\beta$. In this case $TC \cup A$ generates an algebra dense in $C(X)$.

(2) For an averaging operator $T$, it is possible that the only subalgebra $A$ of $C(X)$ for which $g \in A$ implies $Tg$ is constant is exactly the algebra of constant functions. This occurs when $X = \{1, 2, 3\}$ and $Tf = g$ where

$$g(1) = g(2) = \frac{f(1)+f(2)}{2} \quad \text{and} \quad g(3) = f(3).$$

(3) It is also possible that an analytic subalgebra $A$ of $C(X)$ exists which is maximal relative to the condition that for each $g \in A$, $Tg$ is constant while $TC \cup A$ does not separate points in $X$. See, for example, Example 11.1 in [2].

References


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