Can. J. Math., Vol. XXXV, No. 2, 1983, pp. 300–310

DERIVATIONS WITH INVERTIBLE VALUES

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In this paper we study a question which, although somewhat special, has the virtue that its answer can be given in a very precise, definitive, and succinct way. It shows that the structure of a ring is very tightly determined by the imposition of a special behavior on one of its derivations.

The problem which we shall examine is: Suppose that R is a ring with unit element, 1, and that $d \neq 0$ is a derivation of R such that for every $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R; must R then have a very special structure?

As we shall see, the answer to this question is yes, in particular we show that except for a special case which occurs when $2R = 0$, R must be a division ring D or the ring D_2 of 2×2 matrices over a division ring. More precisely we shall prove:

THEOREM. *Let R be a ring with 1 and $d \neq 0$ a derivation of R such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R. Then R is either*

1. *a division ring D, or*
2. *D_2, or*
3. *$D[x]/(x^2)$, where* char $D = 2$, *$d(D) = 0$ and $d(x) = 1 + ax$ for some a in the center Z of D.*

Furthermore, if $2R \neq 0$ then $R = D_2$ is possible if and only if D does not contain all quadratic extensions of Z, the center of D; equivalently if and only if some element in Z is not a square in D.

We shall also see that if $R = D_2$ then d must be inner, provided $2R \neq 0$; however, d may fail to be inner when $2R = 0$. In addition, we shall see that if $R = D[x]/(x^2)$, then d cannot be inner.

Finally, we consider a similar situation, one in which $d(x) = 0$ or is invertible not for all $x \in R$, but for all x in a suitable subset. In that context we also obtain results that say that $R = D$, $R = D_2$, or $R = D[x]/(x^2)$; however the relationship between d and R will be somewhat different, from that described in the theorem.

In all that follows, unless otherwise specified, R will be a ring with 1 and $d \neq 0$ will be a derivation of R such that $d(x) = 0$ or is invertible, for all $x \in R$.

Received December 22, 1981 and in revised form May 4, 1982. The research of the second author was supported by the NSF grant, NSF MCS 810–2472 at the University of Chicago. This is dedicated to Professor G. Azumaya on his sixtieth birthday.

We begin with

LEMMA 1. *If* $d(x) = 0$ *then either* $x = 0$ *or* x *is invertible.*

Proof. Suppose that $x \neq 0$; since $d \neq 0$ there is a $y \in R$ such that $d(y) \neq 0$. Hence $d(y)$ is invertible. Now $d(yx) = d(y)x \neq 0$ since $x \neq 0$ and $d(y)$ is invertible; therefore $d(yx)$ is invertible, that is, $d(y)x$ is invertible. This forces x to be invertible.

As an easy consequence of Lemma 1 we have

LEMMA 2. *If* $L \neq 0$ *is a one-sided ideal of* R *then* $d(L) \neq 0$.

Proof. Since $d \neq 0$ the lemma is certainly true if $L = R$. Suppose, then, that $L \neq R$. If $0 \neq a \in L$ then, by Lemma 1, $d(a) \neq 0$ since a is not invertible. Thus $d(L) \neq 0$; in fact we saw that d is not zero on the nonzero elements of L.

Another immediate consequence of Lemma 1 is

LEMMA 3. *If* $2x = 0$ *for some* $x \neq 0$ *in* R *then* $2R = 0$.

Proof. Since $2x = 0$, $d(2x) = 2d(x) = 0$. If $d(x) = 0$ then, by Lemma 1, x is invertible, and since $2x = 0$ we get $0 = (2x)x^{-1} = 2$, and so $2R = 0$. On the other hand, if $d(x) \neq 0$ then $d(x)$ is invertible and since $2d(x) = 0$ we get, once again, that $2R = 0$.

What the lemma says is that R can have 2-torsion if and only if R is of characteristic 2.

We continue with the important

LEMMA 4. *If* L *is a proper left ideal of* R *then* L *is both minimal and maximal.*

Proof. It certainly suffices to show that every proper left ideal of R is maximal. Let $L \subset T$ be proper left ideals of R. As is easy to verify, $L + d(L)$ is also a left ideal of R. Since, by Lemma 2, $d(L) \neq 0$, and so $L + d(L)$ contains invertible elements, we must have $L + d(L) = R$. Therefore if $t \in T$ there exist $a, b \in L$ such that $t = a + d(b)$. Consequently, $d(b) = t - a \in T \cap d(L) = 0$; therefore $t = a \in L$. Thus $L = T$ and L is maximal.

We can now narrow in on the structure of R:

LEMMA 5. (a) *If* I *is a proper ideal of* R *then* $I^2 = 0$.
(b) *If* $2R \neq 0$ *then* R *is simple.*

Proof. (a) If $I \neq R$ is an ideal of R then

$$d(I^2) \subset d(I)I + Id(I) \subset I,$$

hence by Lemma 2, $I^2 = 0$ as I cannot contain any invertible elements.

(b) Suppose $2R \neq 0$ and let $I \neq 0$ be a proper ideal of R, then, by Lemma 2, there is a $b \in I$ such that $d(b) \neq 0$, so $d(b)$ is invertible. Now, since $b^2 = 0$,

$$0 = d^2(b^2) = d^2(b)b + 2d(b)^2 + bd(b)^2,$$

in consequence of which, $2d(b)^2 \in I$, hence

$$0 = (2d(b)^2)^2 = 4d(b)^4.$$

Since $d(b)$ is invertible we get $4 = 0$, so, by Lemma 3, $2R = 0$, in contradiction to $2R \neq 0$. Therefore if $2R \neq 0$, R is simple.

By combining Lemmas 4 and 5 we see that if $2R \neq 0$ then $R = D$ or $R = D_2$. For any division ring D and every non-zero derivation, d, of D we certainly have that $d(x) = 0$ or $d(x)$ is invertible for every $x \in R$. For D_2, under what conditions on D, is there a non-zero derivation d with this property? To answer this question we need to analyze the derivations of the 2×2 matrices over an arbitrary ring. In the following two lemmas we assume that S is any ring with 1, $R = S_2$, and d is any derivation of R. The first lemma is well known; since its proof is obtained by a straightforward computation, we omit the proof.

LEMMA 6. *Let S be any ring with* 1 *and let $R = S_2$. If d is a derivation of R then there exist $\alpha, \beta, \gamma \in S$ and a derivation f of S such that:*

$$d(e_{11}) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \quad d(e_{12}) = \begin{pmatrix} -\beta & \gamma \\ 0 & \beta \end{pmatrix}, \quad d(e_{21}) = \begin{pmatrix} -\alpha & 0 \\ -\gamma & \alpha \end{pmatrix}$$

$$d(e_{22}) = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix},$$

and, for $a \in S$,

$$d\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} f(a) & a\alpha - \alpha a \\ -(a\beta - \beta a) & f(a) + a\gamma - \gamma a \end{pmatrix}.$$

We use the formulas in Lemma 6 to prove the following fact interrelating d and f:

LEMMA 7. *Let R, S, d, and f be as in Lemma* 6. *Then d is inner on R if and only if f is inner on S.*

Proof. If d is the inner derivation on R induced by $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$, where $s, t, u, v \in S$, then it is immediate that $f(x) = sx - xs$ for all $x \in S$, hence f is inner on S.

Conversely, if f is the inner derivation on S defined by $f(x) = rx - xr$, where $r \in S$, then

$$d(T) = \begin{pmatrix} r & -\alpha \\ \beta & r - \gamma \end{pmatrix} T - T \begin{pmatrix} r & -\alpha \\ \beta & r - \gamma \end{pmatrix}$$

for all $T \in R$, where α, β, γ are as in Lemma 6. This is verified by noting that d and the inner derivation induced by $\begin{pmatrix} r & -\alpha \\ \beta & r-\gamma \end{pmatrix}$ agree on all matrix units and on the elements of S, hence on all of R.

We now return to our original situation, assuming that R is a ring with 1 and a derivation $d \neq 0$ such that for each $x \in R$ either $d(x) = 0$ or $d(x)$ is invertible. We shall characterize those D for which $R = D_2$ has such a derivation, at least when the characteristic of D is not 2. To do so we need

LEMMA 8. *If* $R = D_2$ *and* $2R \neq 0$ *then* d *is inner.*

Proof. Given d let f, α, β, γ be as in Lemma 6. Then, by Lemma 7, it is enough to prove that f is inner on D. If $a, b, c, e \in D$, then, by Lemma 6 and by the multiplicative law for derivations we have

$$(1)\quad d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} f(a) - b\beta - \alpha c & f(b) + a\alpha + b\gamma - \alpha e \\ f(c) + \beta a - e\beta - \gamma c & f(e) + e\gamma - \gamma e + \beta b + c\alpha \end{pmatrix}.$$

By Lemma 2, $d(e_{11})$ is invertible, therefore $\alpha \neq 0$. By (1) we have for $a \in D$ that

$$d\begin{pmatrix} a & 0 \\ \alpha^{-1}f(a) & \alpha^{-1}a\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}$$

where

$$u = f(\alpha^{-1}f(a)) + \beta a - \alpha^{-1}a\alpha\beta - \gamma\alpha^{-1}f(a) \quad \text{and}$$
$$v = f(\alpha^{-1}a\alpha) + \alpha^{-1}a\alpha\gamma - \gamma\alpha^{-1}a\alpha + \alpha^{-1}f(a)\alpha.$$

Since $\begin{pmatrix} 0 & 0 \\ u & v \end{pmatrix}$ is not invertible we must have $u = v = 0$. Since f is a derivation,

$$f(\alpha^{-1}) = -\alpha^{-1}f(\alpha)\alpha^{-1};$$

thus $v = 0$ gives us

$$(2)\quad 0 = v = -\alpha^{-1}f(\alpha)\alpha^{-1}a\alpha + \alpha^{-1}f(a)\alpha + \alpha^{-1}af(a) + \alpha^{-1}a\alpha\gamma - \gamma\alpha^{-1}a\alpha + \alpha^{-1}f(a)\alpha.$$

Thus relation (2) can be re-written as

$$2\alpha^{-1}f(a)\alpha = \alpha^{-1}f(\alpha)\alpha^{-1}a\alpha + \gamma\alpha^{-1}a\alpha - \alpha^{-1}af(\alpha) - \alpha^{-1}a\alpha\gamma,$$

which gives us

$$2f(a) = (f(\alpha)\alpha^{-1} + \alpha\gamma\alpha^{-1})a - a(f(\alpha)\alpha^{-1} + \alpha\gamma\alpha^{-1}).$$

Since char $D \neq 2$, dividing by 2 we see that f is the inner derivation on D induced by $\frac{1}{2}(f(\alpha)\alpha^{-1} + \alpha\gamma\alpha^{-1})$. This proves the lemma.

We now completely characterize those division rings D (independent of characteristic) such that D_2 has an inner derivation with our special property. In doing so, in light of the results we have obtained so far, we shall completely describe all rings R such that $2R \neq 0$ for which there is a derivation $d \neq 0$ with our special property.

The condition: "D does not contain all quadratic extensions of Z" will come up. By this we mean that there are elements δ and σ in Z such that the polynomial $t^2 + \delta t + \sigma$ has no root in D.

LEMMA 9. *If D is a division ring then $R = D_2$ has an inner derivation $d \neq 0$ such that for $x \in R$ either $d(x) = 0$ or $d(x)$ is invertible if and only if D does not contain all quadratic extensions of Z.*

Proof. Suppose that R has such an inner derivation induced by the matrix $M \in R$. We claim that M cannot be a diagonal matrix; for if $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in D$, computing

$$Me_{12} - e_{12}M = \begin{pmatrix} 0 & a-b \\ 0 & 0 \end{pmatrix}$$

we have, by our basic hypothesis, that $a = b$. Computing

$$M\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}M = \begin{pmatrix} ac - ca & 0 \\ 0 & 0 \end{pmatrix},$$

for all $c \in D$, we get that $a \in Z$. Hence $M \in Z$, whence $d = 0$, contrary to hypothesis.

Since M is not diagonal there exists an invertible matrix $T \in D_2$ such that

$$TMT^{-1} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} \quad \text{where } \alpha, \beta \in D.$$

The inner derivation induced by TMT^{-1} also has the property that all its values are 0 or invertible. So, without loss of generality, we may assume that d is induced by $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$, $\alpha, \beta \in D$.

If $\gamma \in D$ then

$$d\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha\gamma - \gamma\alpha & \beta\gamma - \gamma\beta \end{pmatrix}$$

which is not invertible, therefore $\alpha\gamma = \gamma\alpha$, $\beta\gamma = \gamma\beta$. In short, α and β are both in Z. Since $d\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} = 0$, by Lemma 1 we have that $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$ is invertible, hence $\alpha \neq 0$. For $\gamma \in D$,

$$d\begin{pmatrix} 0 & 1 \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} -\alpha & \gamma - \beta \\ -\alpha\gamma & \alpha \end{pmatrix}$$

cannot be 0 by Lemma 1, so is invertible. This gives us that

$$\alpha(\gamma^2 - \beta\gamma - \alpha) \neq 0 \quad \text{for all } \gamma \in D.$$

In other words the quadratic polynomial $t^2 - \beta t - \alpha$ over Z has no root in D, and so D does not contain all quadratic extensions of Z.

Conversely, if D does not contain all quadratic extensions of Z there exist $\alpha, \beta \in Z$, with $\alpha \neq 0$, such that $\alpha x^2 - \beta x - 1$ has no solutions in D. Let d be the inner derivations of D_2 induced by $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$. We claim that every non-zero value of d is invertible. Let a, b, c, and e be in D; then

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} c - \alpha b & e - a - \beta b \\ \alpha(a - e) + \beta c & \alpha b - c \end{pmatrix}.$$

If we let $m = c - \alpha b$ and $n = e - a - \beta b$ then

$$\alpha(a - e) + \beta c = -\alpha n + \beta m, \quad \text{and}$$

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} m & n \\ -\alpha n + \beta m & -m \end{pmatrix}.$$

Suppose, for the moment, that $m = 0$; in that case

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} 0 & n \\ -\alpha n & 0 \end{pmatrix}$$

which is either 0 or invertible, since $\alpha \neq 0$.

If, on the other hand, $m \neq 0$ then

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} m & n \\ -\alpha n + \beta m & -m \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 1 & w \\ -\alpha w + \beta & -1 \end{pmatrix}$$

where $w = m^{-1}n$. Since $m \neq 0$, $d\begin{pmatrix} a & b \\ c & e \end{pmatrix}$ is invertible if and only if $\begin{pmatrix} 1 & w \\ -\alpha w + \beta & -1 \end{pmatrix}$ is invertible, that is, if and only if

$$-1 - w(-\alpha w + \beta) \neq 0.$$

However, by our choice of α and β, $\alpha w^2 - \beta w - 1 \neq 0$ for all $w \in D$. Thus d is an inner derivation of D_2 all of whose non-zero values are invertible.

The only piece that remains in order to prove our main theorem is the case where $2R = 0$ and R is neither D nor D_2. We handle this case with

LEMMA 10. *If R is not simple then $R = D[x]/(x^2)$ where* char $D = 2$, *$d(D) = 0$, $d(x) = 1 + ax$ for some a in Z, the center of D; moreover, d is not inner.*

https://doi.org/10.4153/CJM-1983-016-0 Published online by Cambridge University Press

Proof. By Lemmas 4 and 5, $2R = 0$, all proper ideals of R have square zero, and all proper one-sided ideals of R are both minimal and maximal. As a result, we easily obtain that R contains a unique (left, right, two-sided) ideal M and $M^2 = 0$. Therefore, as in the proof of Lemma 4, $R = M + d(M)$, hence if $r \in R$ there exist $m, n \in M$ such that $d(r) = m + d(n)$. Consequently, $d(r - n) = m \in M \cap d(R) = 0$ and so, if $D = \ker d$ then, by Lemma 1, D is a division ring and $R = D + M$.

By the uniqueness of M, if $0 \neq x \in M$ then $R = D + Dx$ and thus $d(x) = s + tx$ where $s, t \in D$ and $s \neq 0$. Since $d(D) = 0$, if we replace x by $s^{-1}x$, we may assume $d(x) = 1 + ax$ for some $a \in D$.

If $s \in D$, we can use the facts $M = Rx$, $M^2 = 0$, $d(s) = 0$, and $d(x) = 1 + ax$ to obtain

$$0 = d((sx)^2) = sxd(sx) + d(sx)sx = sxs(1 + ax) + s(1 + ax)sx$$
$$= sxs + s^2x = s(xs + sx).$$

If $s \neq 0$, s is invertible, hence $xs = sx$ and x is in the center of R. Therefore $R = D[x]/(x^2)$.

Now, if $s \in D$ then $sx + xs = 0$, thus

$$0 = d(sx + xs) = s(1 + ax) + (1 + ax)s$$
$$= sax + axs = (sa + as)x.$$

Since all non-zero elements of D are invertible in R, $sa + as = 0$, hence a is in the center of D.

Finally, since $x \in M$ and $d(x) \notin M$, it is clear that d is not inner.

We can now prove our main result, which is the theorem stated at the outset, and which we record as

THEOREM 1. *Let R be a ring with 1 and $d \neq 0$ a derivation of R such that, for each $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R. Then R is either*

1. *a division ring D, or*
2. *D_2, or*
3. *$D[x]/(x^2)$, where* char $D = 2$, *$d(D) = 0$, and $d(x) = 1 + ax$, for some a in the center Z of D.*

Furthermore, if $2R \neq 0$ then $R = D_2$ is possible if and only if D does not contain all quadratic extensions of Z, the center of D; equivalently if and only if some element in Z is not a square in D.

Proof. If R is simple, then by Lemma 4 either $R = D$ or $R = D_2$. Furthermore if $2R \neq 0$, by Lemma 8 D_2 has such a derivation if and only if it has an inner derivation with the special property. However Lemma 9 tells us that D_2 has such an inner derivation if and only if D does not contain all quadratic extensions of Z.

If R is not simple, then by applying Lemma 10 we obtain our result. Theorem 1 is now proved.

One question concerning Theorem 1 remains. Namely, in the case $R = D_2$ is it necessary to assume $2R \neq 0$ in order to prove that d is inner? We now present an example that shows if $2R = 0$ then $R = D_2$ can have an outer derivation d such that $d(x) = 0$ or $d(x)$ is invertible, for all $x \in R$.

Example. Take $R = M_2(F)$ for $F = GF(2)(x)\langle\langle y\rangle\rangle$, the field of (finite) Laurent series with coefficients in the rational function field in one variable over $GF(2)$. Define a derivation δ on F by extending the action $\delta(f(x)) = 0$ and $\delta(y) = xy$. If $a \in F$ is written $a = a_E + a_0$, where a_E is the series of even powers of y appearing in a, and $a_0 = a - a_E$, then $\delta(a) = xa_0$. Let $A = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \in M_2(F)$ and set $d = d_A + \bar{\delta}$ where d_A is the inner derivation of $M_2(F)$ induced by A and $\bar{\delta}$ is the derivation of $M_2(F)$ defined by componentwise application of δ. Note that d is not inner since

$$d\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix}.$$

An easy computation shows

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} b + c + xa_0 & a + e + xb_E \\ a + e + xc_E & b + c + xe_0 \end{pmatrix}.$$

It can now be shown by a direct, if somewhat tedious computation that d has invertible values; and we omit the details.

We shall now consider a situation closely related to the one we have been discussing. Let R be a ring with 1 and $d \neq 0$ a derivation R. Suppose that $L \neq 0$ is a left ideal of R such that $d(L) \neq 0$, and such that for every $x \in L$ either $d(x) = 0$ or $d(x)$ is invertible in R. Since we already know the answer when $L = R$, we suppose that $L \neq R$. We wish to determine the structure of R. Since the arguments will be similar to the ones we have given earlier we give then more sketchily here.

Let $x \neq 0 \in R$ be such that $d(x) = 0$; then, since $xL \subset L$ and $d(xL) = xd(L)$ we easily get the result of Lemma 1, namely, that x is invertible in R. This immediately implies the results of Lemmas 2 and 3, that is, that if $d(W) = 0$ for some left ideal W of R then $W = 0$, and if R has 2-torsion then $2R = 0$.

As before, from our assumptions on L, $L + d(L) = R$, hence if T is a proper left ideal of R containing L and $t \in T$ then $t = a + d(b)$, for some $a, b \in L$. Once again,

$$t - a = d(b) \in T \cap d(L) = 0$$

and so, $T = L$. By this argument and our analog to Lemma 2, L and every non-zero left ideal of R contained in L are maximal, hence L is both minimal and maximal.

We now examine $l(L) = \{x \in R | xL = 0\}$. Since $1 = a + d(b)$, for $a, b, \in L$, if $x \in l(L)$ then

$$x = x(a + d(b)) = xd(b) = d(xb) - d(x)b = -d(x)b \in L$$

and so, by the minimality of L, $l(L) = 0$ or $l(L) = L$.

Suppose $l(L) = 0$, then R is semiprime for if $I^2 = 0$ and $I \neq 0$ we obtain the contradiction $0 = I^2L = I(IL) = IL = L$. It easily follows that R is simple, for if $I \neq 0$ then

$$0 \neq d(I^2L) \subset d(L) \cap I,$$

hence $I = R$. By Wedderburn's theorem, $R = D$ or $R = D_2$.

On the other hand, suppose $l(L) = L$, that is $L^2 = 0$. By repeated use of the maximality and minimality of L we obtain that L is the unique left ideal of R, for if $I \neq L$ is a left ideal of R then $R = I + L$ and so,

$$L = LR = LI + L^2 = LI \subset I,$$

a contradiction. It is now clear that L is the unique (left, right, two-sided) ideal of R. Now, as in Lemma 5, if $b \in L$ such that $d(b) \neq 0$ then

$$0 = d^2(b^2) = bd^2(b) + 2d(b)^2 + d^2(b)b,$$

hence $2d(b)^2 \in L$, and so $4d(b)^4 = 0$. Once again, $2R = 0$. Let $x \in R$ and $y \in L$ such that $d(x) \in L$ and $d(y) \neq 0$; in this case

$$d(xy) = d(x)y + xd(y) = xd(y)$$

and so, x is 0 or invertible. Therefore $D = \{x \in R | d(x) \in L\}$ is a division ring and by the identical argument used in the proof of Lemma 10 we obtain that $R = D[x]/(x^2)$ where $d(x) = 1 + ax$ for some a in the center of D. The only difference we obtain is that although $d(D) \subset L$, it need not be the case that $d(D) = 0$. In fact, it is easy to see that for any $s \in D$, $d(s) = s'x$ where $'$ is a derivation of D.

We have now proved

THEOREM 2. *Let R be a ring with 1 and suppose that $d \neq 0$ is a derivation of R such that $d(L) \neq 0$ for some left ideal of R and $d(x) = 0$ or $d(x)$ is invertible for every $x \in L$. Then $R = D$, $R = D_2$, or, $R = D[x]_{/(x^2)}$ where $2R = 0$ for some division ring D.*

We note that in the case $R = D[x]/(x^2)$, the hypothesis of d on L does not necessarily carry over to the behavior of d on all of R. We conclude this paper by showing that in the case $R = D_2$, not only does the behavior of d on L necessarily not carry over to all of R, but R may fail to have any derivation $\delta \neq 0$ such that $\delta(x) = 0$ or $\delta(x)$ is invertible for all $x \in R$.

Let D be a division ring and suppose that $\sigma \in D$, $\sigma \notin Z$, is such that σ is not a square in D. Define d on D_2 by:

$$d\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} - \begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma t - s & \sigma u - r\sigma \\ r - u & s - t\sigma \end{pmatrix}.$$

If $L = \left\{ \begin{pmatrix} r & 0 \\ t & 0 \end{pmatrix} \middle| r, r \in D \right\}$ then L is a left ideal of D_2 and, for $\begin{pmatrix} r & 0 \\ t & 0 \end{pmatrix} \neq 0 \in L$,

$$d\begin{pmatrix} r & 0 \\ t & 0 \end{pmatrix} = \begin{pmatrix} \sigma t & -r\sigma \\ r & -t\sigma \end{pmatrix}.$$

If $\begin{pmatrix} \sigma t & -r\sigma \\ r & -t\sigma \end{pmatrix}$ is not invertible then $\begin{pmatrix} \sigma t & r \\ r & t \end{pmatrix}$ is not invertible, hence

$$\begin{pmatrix} \sigma t & r \\ r & t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for some $x, y \in D$, not both 0. This would imply that

$$\sigma t x + r y = 0 = r x + t y;$$

since not both r, t are 0 we get $x \neq 0$, $y \neq 0$, so without loss of generality, $x = -1$. Thus $r = ty$ and $\sigma t = ry = ty^2$; this latter implies that

$$\sigma = ty^2t^{-1} = (tyt^{-1})^2,$$

a square in D. Thus we see that the non-zero values of d on L are invertible.

However, since $\sigma \in Z$, $\sigma a \neq a\sigma$ for some $a \in D$, hence

$$d\begin{pmatrix} a & 0 \\ 0 & \sigma^{-1}a\sigma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a - \sigma^{-1}a\sigma & 0 \end{pmatrix} \neq 0,$$

and is not invertible.

We have shown that a sufficient condition that there exist a derivation $d \neq 0$ on D_2 and a left ideal L of D_2 such that $d(L) \neq 0$ and the non-zero elements of $d(L)$ invertible, is that some element of D not be a square in D. Compare this to Theorem 1, where the element in D which is not a square must be in Z.

Finally we take a special D in the discussion above. Let $\mathbf{C}$ be the field of complex numbers and F the field of rational functions in x over $\mathbf{C}$. Consider the set of Laurent series, D, of all $\sum_{-n}^{\infty} f_i y^i$ in y over F, where $yr(x)y^{-1} = r(2x)$, for any $r(x) \in F$. D is a division ring with center $\mathbf{C}$, hence all elements of $\mathbf{C}$ are squares in $\mathbf{C}$, hence in D. Thus, by Theorem 1, there is no derivation of D_2 with the property that all its non-zero values are invertible. However, as is easily verified, x is not a square in D. Thus by the above there is a derivation $d \neq 0$ on D_2 which on the left ideal $\left\{\begin{pmatrix} r & 0 \\ t & 0 \end{pmatrix}\right\}$ has all its non-zero values invertible, yet there is no derivation $\delta \neq 0$ on D_2 all of whose non-zero values are invertible.

One can give a rather awkward necessary and sufficient condition on D such that D_2 have a derivation $d \neq 0$ and a left ideal L such that $d(L) \neq 0$ and all non-zero $d(x)$ be invertible for $x \in L$. For instance, to have an inner derivation with this property, for which the f of Lemma 6 is 0, is that there exist $\alpha, \beta, \gamma \in D$ such that

$$t^2 - \beta t^{-1}\gamma\beta + \alpha\beta \neq 0 \quad \text{for all } t \in D.$$

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