Canad. Math. Bull. Vol. **43** (1), 2000 pp. 37–46

Multiplicative Structure of the Ring $K\big(S(T^*\mathbb{R}P^{2n+1})\big)$

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Abstract. We calculate the additive and multiplicative structure of the ring $K\big(S(T^*\mathbb{R}P^{2n+1})\big)$ using the eta invariant.

1 Introduction

Let $\mathbb{R}P^{2n+1}$ be the real projective space of dimension $2n+1$. In his paper [6, p. 125], V. Snaith calculated the additive structure of $K\big(S(T^*\mathbb{R}P^{2n+1})\big)$ where $S(T^*\mathbb{R}P^{2n+1})$ denotes the total space of the sphere bundle associated to the cotangent bundle of $\mathbb{R}P^{2n+1}$. The problem of determining the multiplicative structure was left open.

In this paper, we give a different proof concerning the above additive structure and use the eta invariant to calculate the multiplicative structure. The proof relies entirely on the methods developed by Gilkey concerning the eta invariant. This tool allows us to find explicit generators which make possible the calculation of the multiplicative structure.

After this work was finished, I was made aware by P. B. Gilkey about his work on this subject, and we refer to [4, Theorem 1.6.6] for a different proof concerning the additive structure.

I am grateful to Professors P. B. Gilkey and V. Snaith for their patience in reading and evaluating this work.

I am finally deeply indebted to Professor A. Mesnaoui for proposing the subject and for his help and advice during the preparation of this work.

2 Preliminaries

The real projective space $\mathbb{R}P^{2n+1}$ is a special case of the so called: spherical space form of odd dimension. They always take the form: $M(\tau) = S^{2l-1}/\tau(G)$ (we suppose $l > 1$), where G is a finite group, $\tau\colon G \to U(l)$ is a fixed point free representation; they have two specific features which will be needed later:

(2.a) $T\big(M(\tau)\big) \oplus \mathbf{1} = \mathrm{real}(V_\tau)$, where V_τ is the complex vector bundle associated to the representation τ. This implies that $M(\tau)$ admits a spin$_c$ structure.

(2.b) $T\big(M(\tau)\big) = V \oplus \mathbf{1}$, which is an immediate consequence of (2.a). As a consequence, $T\big(M(\tau)\big)$ has a nowhere vanishing section

$$s'\colon M(\tau) \to T\big(M(\tau)\big).$$

Received by the editors January 28, 1998; revised November 6, 1998.
AMS subject classification: Primary: 19L64; secondary: 19K56, 55C35.

https://doi.org/10.4153/CMB-2000-005-1 Published online by Cambridge University Press

Then the map defined by:

$$s\colon M(\tau) \to S\Big(T\big(M(\tau)\big)\Big)$$
$$x \mapsto s'(x)/\|s'(x)\|$$

is a section of the fibre bundle: $\begin{pmatrix} S\Big(T\big(M(\tau)\big)\Big) \\ \downarrow \pi \\ M(\tau) \end{pmatrix}$. This yields the result: $s^* \circ \pi^* = 1$ *i.e.*: π^* is one to one.

Let L be the Hopf complex line bundle over $\mathbb{R}P^{2n+1}$; then it is well known that $L = V_{\rho_1}$ where $\rho_1\colon Z/2 \to U(1)$, $\rho(-1) = -1$, and that: $\langle L-1\rangle = Z/2^n = \tilde{K}(\mathbb{R}P^{2n+1})$ satisfying: $(L-1)^2 = -2(L-1)$. For notational convenience, we shall write again L for the image of L in $K\big(S(T^*\mathbb{R}P^{2n+1})\big)$. By (2.a), $\mathbb{R}P^{2n+1}$ admits a spin$_c$ structure. Let P_{Dol} be the tangential operator of the spin$_c$ complex [3, Lemma 2.4, a]. Consider the bundle: $\pi_+\big(\sigma_L(P_{\text{Dol}})\big)$ where $\sigma_L(P_{\text{Dol}})$ is the leading symbol of P_{Dol} and $\pi_+\big(\sigma_L(P_{\text{Dol}})(x,.)\big)$ is the eigenspace of the eigenvalue $+1$; $\pi_+\big(\sigma_L(P_{\text{Dol}})\big)$ is a complex vector bundle over $S(T^*\mathbb{R}P^{2n+1}) =: W_{2n+1}$, of complex dimension 2^{n-1}.

The main result of this work is:

Theorem 2.1

$$\begin{aligned}\tilde{K}(W_{2n+1}) &= \langle L-1\rangle \oplus \Big\langle \Big(\pi_+\big(\sigma_L(P_{\text{Dol}})\big) - 2^{n-1}\Big) \otimes (L-1)\Big\rangle \\ &\oplus \langle \pi_+\big(\sigma_L(P_{\text{Dol}})\big) - 2^{n-1}\rangle \\ &= \mathbf{Z}/2^n \oplus \mathbf{Z}/2^n \oplus \mathbf{Z} \\ &= \text{Tors}\big(\tilde{K}(W_{2n+1})\big) \oplus \text{Free}\big(\tilde{K}(W_{2n+1})\big).\end{aligned}$$

If $x = L-1$, $y = \Big(\pi_+\big(\sigma_L(P_{\text{Dol}})\big) - 2^{n-1}\Big) \otimes (L-1)$ and $z = \pi_+\big(\sigma_L(P_{\text{Dol}})\big) - 2^{n-1}$, then the generators: x, y and z are subject to the relations:

$$\begin{cases} x^2 = -2x \\ x.z = y \\ x.y = -2y \\ z^2 = y^2 = z.y = 0 \end{cases}$$

3 Proof of Theorem 2.1

First, we recover the results of V. Snaith [6, p. 125].

We shall use the following notation: $K := K^\circ$, D_{2n+1} is the disk bundle of $T^*\mathbb{R}P^{2n+1}$, $\sum(X)$ the suspension of the space X and $T(V)$ the Thom space of the vector bundle V. Now, $T^*\mathbb{R}P^{2n+1} \oplus \mathbf{1}$ admits a spin$_c$ structure via $T\mathbb{R}P^{2n+1} \simeq T^*\mathbb{R}P^{2n+1}$ and is of even

https://doi.org/10.4153/CMB-2000-005-1 Published online by Cambridge University Press

dimension. Then, by [5, Theorem IV.5.14], we have:

$$\begin{aligned} K^{\alpha}(D_{2n+1}, W_{2n+1}) &= \tilde{K}^{\alpha}\big(T(T^*\mathbb{R}P^{2n+1})\big) \\ &= \tilde{K}^{\alpha+1}\Big(\sum\big(T(T^*\mathbb{R}P^{2n+1})\big)\Big) \\ &= \tilde{K}^{\alpha+1}\big(T(T^*\mathbb{R}P^{2n+1} \oplus \mathbf{1})\big) \\ &= K^{\alpha+1}(\mathbb{R}P^{2n+1}) \quad \text{for any integer } \alpha. \end{aligned}$$

By (2.b), there exists a smooth section $s\colon \mathbb{R}P^{2n+1} \to W_{2n+1}$. Then the exact sequence, in K-theory, of the pair (D_{2n+1}, W_{2n+1}) yields the following short exact sequence:

$$0 \to \tilde{K}(\mathbb{R}P^{2n+1}) \xrightarrow{\pi^*} \tilde{K}(W_{2n+1}) \to K(\mathbb{R}P^{2n+1}) \to 0$$

This sequence splits, and the splitting is given by: $\tilde{K}(W_{2n+1}) \xrightarrow{s^*} \tilde{K}(\mathbb{R}P^{2n+1})$. Then,

$$\begin{aligned} \tilde{K}(W_{2n+1}) &\simeq \tilde{K}(\mathbb{R}P^{2n+1})) \oplus K(\mathbb{R}P^{2n+1}) \\ &\simeq \mathbf{Z}/2^n \oplus \mathbf{Z}/2^n \oplus \mathbf{Z} \\ &= \operatorname{Tors}\big(\tilde{K}(W_{2n+1})\big) \oplus \operatorname{Free}\big(\tilde{K}(W_{2n+1})\big). \end{aligned}$$

Next, we need to prove several lemmas: Let us consider the eta invariant:

$$\operatorname{ind}(*, *)\colon R_0(\mathbf{Z}/2) \otimes \frac{K(W_{2n+1})}{\pi^* K(\mathbb{R}P^{2n+1})} \to \mathbb{R}/\mathbf{Z}.$$

together with its restriction:

$$\operatorname{ind}(*.*)|\colon R_0(\mathbf{Z}/2) \otimes \operatorname{Tors}\left(\frac{K(W_{2n+1})}{\pi^* K(\mathbb{R}P^{2n+1})}\right) \to \mathbf{Q}/\mathbf{Z}.$$

where π stands for the projection of the fibre bundle: $\begin{pmatrix} W_{2n+1} \\ \downarrow \pi \\ \mathbb{R}P^{2n+1} \end{pmatrix}$ then, we have:

Lemma 3.1 *a)* $\operatorname{ind}(*.*)|$ *extends to a bilinear map:*

$$\operatorname{ind}(*.*)|\colon \tilde{K}(\mathbb{R}P^{2n+1}) \otimes \left(\frac{\operatorname{Tors}\tilde{K}(W_{2n+1})}{\pi^*\tilde{K}(\mathbb{R}P^{2n+1})}\right) \to \mathbf{Q}/\mathbf{Z}.$$

b) $\operatorname{Tors}\big(\tilde{K}(W_{2n+1})\big) = \langle x \rangle \oplus \langle y \rangle = \mathbf{Z}/2^n \oplus \mathbf{Z}/2^n$.

Proof a) Let $\rho_1, \rho_2 \in R(\mathbf{Z}/2)$ such that: $V_{\rho_1} \simeq V_{\rho_2}$ and let: $\tau\colon V_{\rho_1} \to V_{\rho_2}$ be an isomorphism of fibre bundles.

Let W be a complex vector bundle over $\mathbb{R}P^{2n+1}$ and let: $P\colon C^\infty(W) \to C^\infty(W)$ be a pseudo-differential operator. Use τ to regard ρ_1 and ρ_2 as two different locally flat structures on the same bundle: $V_{\rho_1} = V_{\rho_2}$.

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Now, consider the operators: $P_{\rho_i}\colon C^\infty(W\otimes V_{\rho_i})\to C^\infty(W\otimes V_{\rho_i})$, $i=1,2$. Let $P(a)=aP_{\rho_1}+(1-a)P_{\rho_2}$ ($a\in[0.1]$) be a smooth one parameter family of operators and define:

$$\operatorname{ind}(\rho_1,\rho_2,\tau,P)=\int_0^1\frac{d\eta\big(P(a)\big)}{da}\,da\in\mathbb{R}.$$

Then according to [3, Lemma 1.3], $\operatorname{ind}(\rho_1,\rho_2,\tau,*)$ extends to a linear map:

$$\operatorname{ind}(\rho_1,\rho_2,\tau,*)\colon \frac{K(W_{2n+1})}{\pi^*K(\mathbb{R}P^{2n+1})}\to\mathbb{R}.$$

so,

$$\operatorname{ind}(\rho_1,\rho_2,\tau,*)\colon \operatorname{Tors}\left(\frac{K(W_{2n+1})}{\pi^*K(\mathbb{R}P^{2n+1})}\right)\to\operatorname{Tors}(\mathbb{R})=0.$$

Then, for every element V of $\operatorname{Tors}(\frac{K(W_{2n+1})}{\pi^*K(\mathbb{R}P^{2n+1})})$, we have:

$$\operatorname{ind}(\rho_1-\rho_2,V)=\overline{\operatorname{ind}(\rho_1,\rho_2,\tau,V)}=\bar{0}\in\mathbb{R}/\mathbf{Z}$$

so $\operatorname{ind}(\rho_1,V)=\operatorname{ind}(\rho_2,V)$. Then $\operatorname{ind}(*.*)|$ extends to a bilinear map:

$$\operatorname{ind}(*.*)|\colon \tilde{K}_{\mathrm{Flat}}(\mathbb{R}P^{2n+1})\otimes\operatorname{Tors}\left(\frac{K(W_{2n+1})}{\pi^*K(\mathbb{R}P^{2n+1})}\right)\to\mathbf{Q}/\mathbf{Z}.$$

Now, we have obviously:

$$\frac{K(W_{2n+1})}{\pi^*K(\mathbb{R}P^{2n+1})}\cong\frac{\tilde{K}(W_{2n+1})}{\pi^*\tilde{K}(\mathbb{R}P^{2n+1})}=\mathbf{Z}/2^n\oplus\mathbf{Z}.$$

Then,

$$\operatorname{Tors}\left(\frac{K(W_{2n+1})}{\pi^*K(\mathbb{R}P^{2n+1})}\right)=\operatorname{Tors}\left(\frac{\tilde{K}(W_{2n+1})}{\pi^*\tilde{K}(\mathbb{R}P^{2n+1})}\right)=\mathbf{Z}/2^n=\frac{\operatorname{Tors}\big(\tilde{K}(W_{2n+1})\big)}{\pi^*\tilde{K}(\mathbb{R}P^{2n+1})}\tag{3.1.1}$$

By (3.1.1) and according to [2, Theorem 3.1], we have:

$$\operatorname{ind}(*.*)|\colon \tilde{K}(\mathbb{R}P^{2n+1})\otimes\frac{\operatorname{Tors}\tilde{K}(W_{2n+1})}{\pi^*\tilde{K}(\mathbb{R}P^{2n+1})}\to\mathbf{Q}/\mathbf{Z}$$

b) Now, $(L-1)\in\tilde{K}(W_{2n+1})$ is a torsion element for $2^n(L-1)=0$, and so is: $\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)-2^{n-1}\Big)\otimes(L-1)$.

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Furthermore,

$$
\begin{aligned}
&\operatorname{ind}\Big(V_{\rho_1-\rho_0},\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)-2^{n-1}\Big)\otimes(L-1)\Big)\\
&\quad=\operatorname{ind}\Big(V_{\rho_1-\rho_0},\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)\otimes(L-1)\Big)\\
&\quad=\operatorname{ind}\Big(\rho_1-\rho_0,\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)\otimes(L-1)\Big)\\
&\quad=\operatorname{ind}\Big((\rho_1-\rho_0)^2,\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)\Big)\\
&\quad=2\operatorname{ind}\Big(\rho_0-\rho_1,\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)\Big)\qquad(3.1.2)\\
&\quad=2\operatorname{ind}(\rho_0-\rho_1,P_{\mathrm{Dol}})\\
&\quad=\sum_{\substack{j\in\mathbf{Z}/2\\ j\neq 1}}\operatorname{Tr}\big((\rho_0-\rho_1)(g)\big).\mathrm{def}\big(\tau(g),\mathrm{Dol}\big)(\operatorname{mod}\mathbf{Z})\quad\text{by [2, Lemma 2.1,b)]}\\
&\quad=\frac{1}{2^n}\operatorname{mod}\mathbf{Z}
\end{aligned}
$$

where: $\tau\colon\mathbf{Z}/2\to U(n+1)$; $\tau=(n+1)\rho_1$, $S^{2n+1}/\tau(\mathbf{Z}/2)=\mathbb{R}P^{2n+1}$. (3.1.2) and the fact that: $\tilde K(\mathbb{R}P^{2n+1})=\mathbf{Z}/2^n=\langle L-1\rangle=\langle V_{\rho_1-\rho_0}\rangle$ implies: $\operatorname{ind}\Big(*,\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)-2^{n-1}\Big)\otimes(L-1)\Big)\colon\tilde K(\mathbb{R}P^{2n+1})\to Z/2^n$ is a ring isomorphism. So

$$\operatorname{ind}(\rho_1-\rho_0,*)|\colon\frac{\operatorname{Tors}\big(\tilde K(W_{2n+1})\big)}{\pi^*\tilde K(\mathbb{R}P^{2n+1})}\to Z/2^n\quad\text{is onto.}$$

This, added to the fact: $|\frac{\operatorname{Tors}(\tilde K(W_{2n+1}))}{\pi^*\tilde K(\mathbb{R}P^{2n+1})}|=2^n$, yield:

$$\operatorname{ind}(\rho_1-\rho_0,*)|\colon\frac{\operatorname{Tors}\big(\tilde K(W_{2n+1})\big)}{\pi^*\tilde K(\mathbb{R}P^{2n+1})}\to Z/2^n\quad\text{is a ring isomorphism.}$$

Then, the exact sequence:

$$0\to\tilde K(\mathbb{R}P^{2n+1})\xrightarrow{\pi^*}\operatorname{Tors}\big(\tilde K(W_{2n+1})\big)\to\frac{\operatorname{Tors}\big(\tilde K(W_{2n+1})\big)}{\pi^*\tilde K(\mathbb{R}P^{2n+1})}=\mathbf{Z}/2^n\to 0$$

splits and the splitting is given by the element: $\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)-2^{n-1}\Big)\otimes(L-1)$. So, we have

$$
\begin{aligned}
\operatorname{Tors}\big(\tilde K(W_{2n+1})\big)&=\langle L-1\rangle\oplus\Big\langle\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)-2^{n-1}\Big)\otimes(L-1)\Big\rangle\\
&=\langle x\rangle\oplus\langle y\rangle\\
&=\mathbf{Z}/2^n\oplus\mathbf{Z}/2^n.
\end{aligned}
$$

■

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Lemma 3.2

a) $(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1})$ *is a free element.*
b) *If* $(P_{\mathrm{Dol}})_V$ *denote the operator* P_{Dol} *with coefficient in the bundle* $V \in K(\mathbb{R}P^{2n+1};\mathbf{Q})$*, then:* $\frac{K(W_{2n+1};\mathbf{Q})}{\pi^* K(\mathbb{R}P^{2n+1};\mathbf{Q})}$ *is generated by the bundles:* $\pi_+\Big(\sigma_L\big((P_{\mathrm{Dol}})_V\big)\Big)$ *as* V *runs over* $K(\mathbb{R}P^{2n+1};\mathbf{Q})$.
c) $\langle\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1}\rangle = \mathrm{Free}\big(\tilde{K}(W_{2n+1})\big) = \mathbf{Z}$.

Remark 1 The proof of a) and b) will be omitted as it is completely the analog of [1, Lemma 4.3.4] done for the tangential operator of the spin$_c$ complex instead of the tangential operator of the signature complex.

Proof c) Part b) of the lemma above implies that $\Big\{\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)-2^{n-1}\Big)\otimes V\Big\}_{V\in K(\mathbb{R}P^{2n+1};\mathbf{Q})}$ generates $\frac{\tilde{K}(W_{2n+1};\mathbf{Q})}{\pi^*\tilde{K}(\mathbb{R}P^{2n+1};\mathbf{Q})}$. This implies, using the facts: $K(\mathbb{R}P^{2n+1};\mathbf{Q}) = K(\mathbb{R}P^{2n+1}) \otimes \mathbf{Q} = (\mathbf{Z}/2^n \oplus \mathbf{Z})\otimes\mathbf{Q} = \mathbf{Z}\otimes\mathbf{Q} = \mathbf{Q}\,\tilde{K}(W_{2n+1};\mathbf{Q}) = \tilde{K}(W_{2n+1})\otimes\mathbf{Q} = \mathrm{Free}\big(\tilde{K}(W_{2n+1})\big)\otimes\mathbf{Q} = \mathbf{Q}$, that $\{(\pi_+(\sigma_L(P_{\mathrm{Dol}}) - 2^{n-1})\otimes V\}_{V\in K(\mathbb{R}P^{2n+1};\mathbf{Q})=\mathbf{Q})}$ generate $\mathrm{Free}\big(\tilde{K}(W_{2n+1})\big)\otimes\mathbf{Q}$. This gives: $\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1} \in \mathrm{Free}\big(\tilde{K}(W_{2n+1})\big)$. Furthermore, if $\langle\mu\rangle = \mathrm{Free}\big(\tilde{K}(W_{2n+1})\big) = \mathbf{Z}$, then: $\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1} = m\mu$ ($m \in \mathbf{Z}$) and $\mu = \Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1}\Big)\otimes V$ where $V \in \pi^* K(\mathbb{R}P^{2n+1})$. $\Rightarrow \pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1} = \Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1}\Big)\otimes mV \Rightarrow mV = \mathbf{1} \Rightarrow (m = \pm 1$ and $V = \pm\mathbf{1}) \Rightarrow \pi_+\big(\sigma(P_{\mathrm{Dol}})\big) - 2^{n-1} = \pm\mu$ *i.e.* $\langle z\rangle = \langle\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1}\rangle = \langle\mu\rangle = \mathrm{Free}\big(\tilde{K}(W_{2n+1})\big)$. ■

A crucial point in determining the multiplicative structure is the calculus of $\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)\otimes\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)$. For this purpose, we need the following:

Lemma 3.3 *The notation are those of [1, Lemma 2.1.5]. Then, if* $\mathrm{Tr}(e_0\cdots e_m) \neq 0$*, there exists an integer* $\alpha(j;e_0,\dots,e_m)$ *such that:*

$$\pi_+\otimes\pi_+ - \dim(\pi_+\otimes\pi_+) = \alpha(j;e_0,\dots,e_m).(\pi_+ - \dim\pi_+),$$

where

$$\alpha(j;e_0,\dots,e_m) = \frac{j!}{2(m!).\,\mathrm{Tr}(e_0\cdots e_m)}.\Bigg\{\sum_{a+b=j}\frac{(2a)!}{a!}\frac{(2b)!}{b!}\Bigg[\sum_{\substack{\sigma\in S_m\\ 1\le\sigma(1)<\cdots<\sigma(2a)\le m\\ 1\le\sigma(2a+1)<\cdots<\sigma(m)\le m}}\varepsilon(\sigma)\,.\,\mathrm{Tr}(e_0.e_{\sigma(1)}\cdots e_{\sigma(2a)}).\,\mathrm{Tr}(e_0.e_{\sigma(2a+1)}\cdots e_{\sigma(m)})\Bigg]\Bigg\}.$$

Proof According to [1, Lemma 2.1.5], we have:

$$\Omega_+(x_0) = \frac{1}{2}(1+e_0).\Big(\frac{1}{2}\sum_{i=1}^m dx_i.e_i\Big)^2.$$

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Then, for $a \le j$, we have:

$$\Omega_+^a(x_0) = \frac{1}{2^{2a+1}}(2a)!\,(1+e_0) \sum_{1\le i_1<\cdots<i_{2a}\le m} e_{i_1}\cdots e_{i_{2a}} dx_{i_1}\wedge\cdots\wedge dx_{i_{2a}}(x_0).$$

$$\begin{aligned}\Rightarrow \mathrm{ch}_a(\Omega_+)(x_0) &= \frac{1}{a!}\,\mathrm{Tr}\left(\left(\frac{i\Omega_+}{2\pi}\right)^a\right)(x_0)\\ &= \frac{(2a)!}{a!}\left(\frac{i}{2\pi}\right)^a \frac{1}{2^{2a+1}} \sum_{1\le i_1<\cdots<i_{2a}\le m} \mathrm{Tr}(e_0.e_{i_1}\cdots e_{i_{2a}}) dx_{i_1}\wedge\cdots\wedge dx_{i_{2a}}(x_0).\end{aligned}$$

because:

$$\mathrm{Tr}(e_{i_1}\cdots e_{i_{2a}}) = -\,\mathrm{Tr}(e_{i_{2a}}.e_{i_1}\cdots e_{i_{2a-1}}) = -\,\mathrm{Tr}(e_{i_1}\cdots e_{i_{2a-1}}.e_{i_{2a}}) \text{ i.e., } \mathrm{Tr}(e_{i_1}\cdots e_{i_{2a}}) = 0.$$

Then,

$$\begin{aligned}&\mathrm{ch}_j(\pi_+\otimes\pi_+)(x_0)\\ &\quad= \sum_{a+b=j} \mathrm{ch}_a(\pi_+)(x_0)\wedge \mathrm{ch}_b(\pi_+)(x_0)\\ &\quad= \sum_{a+b=j} \mathrm{ch}_a(\Omega_+)(x_0)\wedge \mathrm{ch}_b(\Omega_+)(x_0)\\ &\quad= \left(\frac{i}{2\pi}\right)^j \frac{1}{2^{m+2}} \left\{ \sum_{a+b=j} \frac{(2a)!}{a!}\frac{(2b)!}{b!} \left[\sum_{\substack{\sigma\in S_m\\ 1\le\sigma(1)<\cdots<\sigma(2a)\le m\\ 1\le\sigma(2a+1)<\cdots<\sigma(m)\le m}} \varepsilon(\sigma) \right.\right.\\ &\qquad\qquad \left.\left. .\,\mathrm{Tr}(e_0.e_{\sigma(1)}\cdots e_{\sigma(2a)}).\,\mathrm{Tr}(e_0.e_{\sigma(2a+1)}\cdots e_{\sigma(m)}) \right]\right\}\\ &\qquad \cdot\, dx_1\wedge\cdots\wedge dx_m(x_o).\end{aligned}$$

The curvature of $\pi_+\otimes\pi_+$ being tensorial, we have: for every $x\in S^m$:

$$\begin{aligned}&\mathrm{ch}_j(\pi_+\otimes\pi_+)(x)\\ &\quad= \left(\frac{i}{2\pi}\right)^j . \frac{1}{2^{m+2}} \left\{ \sum_{a+b=j} \frac{(2a)!}{a!}\frac{(2b)!}{b!} \left[\sum_{\substack{\sigma\in S_m\\ 1\le\sigma(1)<\cdots<\sigma(2a)\le m\\ 1\le\sigma(2a+1)<\cdots<\sigma(m)\le m}} \varepsilon(\sigma) \right.\right.\\ &\qquad\qquad \left.\left. .\,\mathrm{Tr}(e_0.e_{\sigma(1)}\cdots e_{\sigma(2a)}).\,\mathrm{Tr}(e_0.e_{\sigma(2a+1)}\cdots e_{\sigma(m)}) \right]\right\}\\ &\qquad \cdot\, dx_1\wedge\cdots\wedge dx_m(x)\end{aligned}$$

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$\Rightarrow \int_{S^m} \mathrm{ch}_j(\pi_+ \otimes \pi_+) = (\frac{i}{2\pi})^j . \frac{1}{2^{m+2}}\{\cdots\}. \mathrm{Vol}(S^m)$ with: $\mathrm{Vol}(S^m) = j!.2^{m+1}\frac{\pi^j}{m!}$. This, added to the fact that: $\int_{S^m} \mathrm{ch}_j(\pi_+) = i^j.2^{-j}.T(e_0.\cdots.e_m)$ [1, Lemma 2.1.5] yield:

$$
\begin{aligned}
&\int_{S^m} \mathrm{ch}_j(\pi_+ \otimes \pi_+) \\
&\quad = \frac{j!}{2.(m!)\,\mathrm{Tr}(e_0.\cdots.e_m)} \\
&\qquad .\Big\{ \sum_{a+b=j} \frac{(2a)!}{a!}\frac{(2b)!}{b!} \Big[\sum_{\substack{\sigma\in S_m \\ 1\le\sigma(1)<\cdots<\sigma(2a)\le m \\ 1\le\sigma(2a+1)<\cdots<\sigma(m)\le m}} \varepsilon(\sigma) \\
&\qquad\qquad . \,\mathrm{Tr}(e_0.e_{\sigma(1)}\cdots e_{\sigma(2a)}).\,\mathrm{Tr}(e_0.e_{\sigma(2a+1)}\cdots e_{\sigma(m)}) \Big] \Big\} \int_{S^m} \mathrm{ch}_j(\pi_+) \\
&\quad = \alpha(j; e_0,\dots,e_m). \int_{S^m} \mathrm{ch}_j(\pi_+).
\end{aligned} \tag{3.3.1}
$$

$\alpha(j; e_0,\dots,e_m)$ is an integer. Indeed: - if $j = 1$, then, trivially we have: $\alpha(1; e_0, e_1, e_2) = 2\dim(\pi_+) \in \mathbf{Z}$ - if $j \geq 2$, then S^m is a spin manifold with trivial Pontrjagin classes except p_0. Then, by [1, Theorem 3.4.4,c], we have:

$$
\begin{aligned}
\mathrm{index}(\pi_+ \otimes \pi_+, \mathrm{spin}) &= \int_{S^m} \mathrm{ch}_j(\pi_+ \otimes \pi_+) \\
&= \alpha(j; e_0,\dots,e_m) \int_{S^m} \mathrm{ch}_j(\pi_+) \\
&= \alpha(j; e_0,\dots,e_m)\,\mathrm{index}(\pi_+, \mathrm{spin}) \quad \text{is an integer}
\end{aligned}
$$

so, $\alpha(j; e_0,\dots,e_m)$ is an integer. Now, according to [1, Lemma 3.8.9], the application:

$$
\begin{aligned}
\tilde{K}(S^m) &\to \mathbf{Z} \\
V &\mapsto \int_{S^m} \mathrm{ch}_j(V)
\end{aligned}
$$

is a group isomorphism. This added to (3.3.1) and to the fact that $\alpha(j; e_0,\dots,e_m) \in \mathbf{Z}$, yield:

$$\pi_+ \otimes \pi_+ - \dim(\pi_+ \otimes \pi_+) = \alpha(j; e_0,\dots,e_m).(\pi_+ - \dim \pi_+). \qquad \blacksquare$$

Remark 2 The expression of $\alpha(j; e_0,\dots,e_m)$, despite its complicated appearance, will simplify considerably in our case.

- We know from [3, p.p 256 $\to$ 258 and Lemma 4.4] that:

$$P_{\mathrm{Dol}} : C^\infty\big(\Delta(\mathbb{R}P^{2n+1})\big) \to C^\infty\big(\Delta(\mathbb{R}P^{2n+1})\big)$$

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$$
\begin{aligned}
*\sigma_L(P_{\text{Dol}})(x,\xi) &= i\overline{c_n}(\xi)\\
&= i\overline{c_n}\Big(\sum_{j=1}^{2n+1}\xi_j s_j\Big) \quad \text{where } (s_j)_{j=1,\dots,2n+1} \text{ is local frame for } T^*\mathbb{R}P^{2n+1}\\
&= \sum_{j=1}^{2n+1}\xi_j f_j \quad \text{where } f_j = i\overline{c_n}(s_j).
\end{aligned}
$$

$*(f_j)_{j=1,\dots,2n+1}$ is a collection of Clifford matrices: $f_i.f_j + f_j.f_i = 2\delta_{ij}$.

- $f_1,\dots,f_{2n+1}$ are our $e_0,\dots,e_m$ of Lemma 3.3 with a translation of indices, but that does not matter.

- We compute: $\text{Tr}(f_1\cdots f_{2n+1}) = i^n.2^n \neq 0$, so, by [3, Lemma 4.5], for every $x\in\mathbb{R}P^{2n+1}$, we have:

$$
\begin{aligned}
&\pi_+\big(\sigma_L(P_{\text{Dol}})(x,.)\big)\otimes\pi_+\big(\sigma_L(P_{\text{Dol}})(x,.)\big) - 2^{2n-2}\\
&\quad = \alpha(n;,f_1,\dots,f_{2n+1})\Big(\pi_+\big(\sigma_L(P_{\text{Dol}})(x,.)\big) - 2^{n-1}\Big)
\end{aligned}
$$

With:

$$
\begin{aligned}
&\alpha(n;f_1,\dots,f_{2n+1})\\
&\quad = \frac{n!}{2\big((2n)!\big).\,\text{Tr}(f_1\cdots f_{2n+1})}\\
&\qquad .\Bigg\{\sum_{a+b=n}\frac{(2a)!}{a!}\frac{(2b)!}{b!}\Bigg[\sum_{\substack{\sigma\in S_{2n}\\ 2\le\sigma(2)<\cdots<\sigma(2a+1)\le 2n+1\\ 2\le\sigma(2a+2)<\cdots<\sigma(2n+1)\le 2n+1}}\varepsilon(\sigma)\\
&\qquad\qquad .\,\text{Tr}(f_1.f_{\sigma(2)}\cdots e_{\sigma(2a+1)}).\,\text{Tr}(f_1.f_{\sigma(2a+2)}\cdots e_{\sigma(2n+1)})\Bigg]\Bigg\}.
\end{aligned}
$$

■

Lemma 3.4 *For $a\neq 0$ and $a\neq n$ and any σ such that $2\le\sigma(2)<\cdots<\sigma(2a+1)\le 2n+1$, $2\le\sigma(2a+2)<\cdots<\sigma(2n+1)\le 2n+1$, we have*

$$
\text{Tr}(f_1.f_{\sigma(2)}\cdots f_{\sigma(2a+1)}).\,\text{Tr}(f_1.f_{\sigma(2a+2)}\cdots f_{\sigma(2n+1)}) = 0.
$$

Proof Suppose first that σ is the identity in S_{2n} then:

$$
\begin{aligned}
f_1.f_{2a+2}\cdots f_{2n+1} &= i^{2n-2a}c_n(s_1)c_n(s_{2a+2})\cdots c_n(s_{2n}).c_n(i^n.s_1 * \cdots * s_{2a+2} * \cdots * s_{2n})\\
&= \pm i^{3n-2a}c_n(s_2)\cdots c_n(s_{2a+1})
\end{aligned}
$$

$\Rightarrow \text{Tr}(f_1.f_{2a+2}\cdots f_{2n+1}) = \pm i^{3n-2a}\,\text{Tr}\big(c_n(s_2)\cdots c_n(s_{2a+1})\big) = 0$. General case: let σ be an element of S_{2n} and k an index such that $f_{\sigma(k)} = f_{2n+1}$. Suppose that $2\le k\le q2a+1$ (the

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other case is similar). Then, as in the first case, we compute:

$$\begin{aligned}\operatorname{Tr}(f_1.f_{\sigma(2)}\cdots f_{\sigma(k)}\cdots f_{\sigma(2a+1)}) &= \operatorname{Tr}(f_1.f_{\sigma(2)}\cdots f_{2n+1}\cdots f_{\sigma(2a+1)})\\ &= \pm\operatorname{Tr}(f_{\sigma(2a+2)}\cdots f_{\sigma(2n+1)}) = 0.\end{aligned}$$ ■

Proof of Theorem 2.1 The additive structure is calculated in Lemmas 3.1 and 3.2. For the multiplicative structure, we have: According to Lemma 3.4

$$\begin{aligned}\alpha(n; f_1,\ldots,f_{2n+1}) &= \frac{n!}{2.\big((2n)!\big).\operatorname{Tr}(f_1\cdots f_{2n+1})}.\frac{(2n)!}{n!}.[2\operatorname{Tr}(\mathrm{Id}_{\Delta(\mathbb{R}P^{2n+1})}).\operatorname{Tr}(f_1\cdots f_{2n+1})]\\ &= \operatorname{Tr}(\mathrm{Id}_{\Delta(\mathbb{R}P^{2n+1})}) = \dim_{\mathbb{C}}\big(\Delta(\mathbb{R}P^{2n+1})\big) = 2^n.\end{aligned}$$

Then $\pi_+\big(\sigma_L(P_{\mathrm{Dol}})(x,.)\big)\otimes\pi_+\big(\sigma_L(P_{\mathrm{Dol}})(x,.)\big) - 2^{2n-2} = 2^n\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})(x,.)\big) - 2^{n-1}\Big)$ for every $x\in\mathbb{R}P^{2n+1}$, so

$$\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)\otimes\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{2n-2} = 2^n\Big(\pi_+\big(\sigma(P_{\mathrm{Dol}})\big) - 2^{n-1}\Big).$$

This last result allows us to complete the multiplicative structure. Indeed: $z^2 = \Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big)\otimes\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{2n-2}\Big) - 2^n\Big(\pi_+\big(\sigma_L(P_{\mathrm{Dol}})\big) - 2^{n-1}\Big) = 0$. Finally, we deduce:

$$\begin{gathered} y.z = z^2.x = 0\\ y^2 = (z.x)^2 = z^2.x^2 = 0\\ xy = -2y \quad \text{because } x^2 = -2x.\end{gathered}$$

This completes the proof of Theorem 2.1. ■

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