ON THE PROJECTIVE COVER OF THE STONE-ČECH COMPACTIFICATION OF A COMPLETELY REGULAR HAUSDORFF SPACE

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The main object of this paper is to give an explicit object in the study of projective covers in the category of compact Hausdorff spaces and continuous maps studied in [2] and [5]. Let $\phi : K \longrightarrow \beta X$ be a projective cover of the Stone-Čech compactification βX of a completely regular Hausdorff space X. Here, it will be shown that the maximal ideal space endowed with the Stone topology of the maximal ring of quotients of the ring $C(X)$ of all real valued continuous functions on X is homeomorphic to K.

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1. For a completely regular Hausdorff space E, let $O(E)$ be its topology, i.e., the collection of its open sets, and $\wedge(E)$ be the space of maximal filters $M \subseteq O(E)$ whose topology is generated by the set $\wedge_W(E) = \{M \mid W \in M, M \in \wedge(E)\}$ for each $W \in O(E)$. It is known in [2] that $\wedge(E)$ is an extremally disconnected compact Hausdorff space. Moreover, as a particular case of [2, Proposition 3], if E is compact then the mapping $\lim_E : \wedge(E) \longrightarrow E$ which assigns to each $M \in \wedge(E)$ its limit is a projective cover of E in the category of all compact Hausdorff spaces and their continuous mappings.

LEMMA 1. <u>Let</u> X <u>and</u> Y <u>be topological spaces such that</u> X <u>is a dense subspace of</u> Y. <u>Then</u> $\wedge(Y)$ <u>is homeomorphic to</u> $\wedge(X)$ <u>under the mapping</u> $M' \longrightarrow M'|X$, $M' \in \wedge(Y)$.

<u>Proof</u>. Evidently $M'|X$ is a filter in $O(X)$. To show $M'|X$ is maximal, let $M \supset M'|X$ be a filter in $O(X)$. Take $U \in M$, then $U \cap V \neq \emptyset$ for all $V \in M'|X$. Let $U = U' \cap X$, $V = V' \cap X$ where $U' \in O(Y)$ and $V' \in M'$. Then also $U' \cap V' \neq \emptyset$ for all $V' \in M'$. The maximality of M' implies that $U' \in M'$ and hence

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$U \in M' | X$, i.e., $M = M' | X$. Clearly the mapping $M' \longrightarrow M' | X$ is one to one, since $M' \neq M''$ implies $M' | X \neq M'' | X$. Now we show this mapping is onto. For any $M \in \Lambda(X)$, define a set $M^* = \{U' \in O(Y) | U' \cap X \in M\}$. Clearly $\emptyset \notin M^*$ and $U' \cap V' \in M^*$ whenever $U', V' \in M^*$. Let $U' \in M^*$ and $U' \subseteq W'$, $W' \in O(Y)$. Then $W' = U' \cup W'$, and hence $W' \cap X = (U' \cap X) \cup (W' \cap X)$; thus $U' \cap X \subseteq W' \cap X$ and hence $W' \cap X \in M$, i.e., $W' \in M^*$. Hence M^* is a filter in $O(Y)$. To show that M^* is maximal, let $M' \supset M^*$ be a filter in $O(Y)$. Take any member $U' \in M'$, then $U' \cap V' \neq \emptyset$ for all $V' \in M^*$. Hence $(U' \cap X) \cap (V' \cap X) \neq \emptyset$ for all $V' \in M^*$. In particular, $(U' \cap X) \cap V \neq \emptyset$ for all $V \in M$; the maximality implies that $U' \cap X \in M$. Thus $U' \in M^*$, i.e., $M' = M^*$, and clearly $M^* | X = M$. Hence the mapping $M' \longrightarrow M' | X$ is onto. Finally, since $\Lambda_W(Y) | X = \Lambda_{W \cap X}(X)$ for $W \in O(Y)$ where $\Lambda_W(Y) | X =_{df} \{M' | X | M' \in \Lambda_W(Y)\}$, the mapping is a homeomorphism.

Notation. For a topological space E, Γ_E, I_E and C_E will denote the closure operator, the interior operator and the complement respectively with respect to the space E.

LEMMA 2. For any $M \in \Lambda(X)$, if $U \in M$, then $M \in \Gamma_{\Lambda(X)} \lim_{\beta X}^{-1}(U)$.

Proof. Let W be any member of M; then $U \cap W \neq \emptyset$, and $\Lambda_W(X)$ is an open neighborhood of M. Let $a \in U \cap W$. Take a member N in $\Lambda(X)$ which converges to the point a. Then every member of N intersects with W. Since N is a maximal filter, it contains W, i.e., N is a member of $\Lambda_W(X)$. On the other hand N converges to the point a of U, hence $N \in \lim_{\beta X}^{-1}(U)$. Thus we have $\Lambda_W(X) \cap \lim_{\beta X}^{-1}(U) \neq \emptyset$.

LEMMA 3. Let $\phi: K \longrightarrow \beta X$ be a projective cover in the category of compact Hausdorff spaces and continuous mappings. Then for each dense subset D of X, $\phi^{-1}(D)$ is dense in K.

Proof. Note that D is also dense in βX. Since K is a compact Hausdorff space, $\Gamma_K \phi^{-1}(D)$ is also a compact subset of K. Since ϕ is onto, $D = \phi(\phi^{-1}(D)) \subset \phi(\Gamma_K \phi^{-1}(D))$. Hence $\phi(\Gamma_K \phi^{-1}(D))$ is dense in βX. But $\phi(\Gamma_K \phi^{-1}(D))$ is compact, hence is closed in βX, i.e., $\phi(\Gamma_K \phi^{-1}(D)) = \beta X$. Since K is the projective cover, $\Gamma_K \phi^{-1}(D)$ can not be a proper closed subset of K, i.e., $\Gamma_K \phi^{-1}(D) = K$.

It is well known in [4, p. 96] that a compact space K is extremally disconnected if and only if $K = \beta S$ for every dense subspace S of K. Hence we have the following:

COROLLARY. $K = \beta\phi^{-1}(D)$ <u>for each dense subset</u> D <u>of</u> X.

2. Let $\mathfrak{D}$ be a filter base of dense subsets of X. Then the system $(C^*(D))_{D \in \mathfrak{D}}$, where * denotes the boundedness, is a direct system with respect to the restriction homomorphisms $f \longrightarrow f|D$, $f \in C^*(E)$, $D \subseteq E$ in $\mathfrak{D}$. Let $Q^*_{\mathfrak{D}}(X)$ be the direct limit of the system $(C^*(D))_{D\in\mathfrak{D}}$ with $(\phi_D)_{D\in\mathfrak{D}}$ as a family of the limit homomorphisms [1]. It is evident that, for a member D of $\mathfrak{D}$, a function $f \in C^*(D)$ defines a continuous function $f \circ \phi$ on $\phi^{-1}(D)$. Since $K = \beta\phi^{-1}(D)$, the function $f \circ \phi$ has a unique continuous extension $\tilde{f}$ to K. Let $u_f \in Q^*_{\mathfrak{D}}(X)$ with $u_f = \phi_D(f)$ and $f \in C^*(D)$ for some $D \in \mathfrak{D}$. Define a mapping $Q^*_{\mathfrak{D}}(X) \longrightarrow C(K)$ by $u_f \longrightarrow \tilde{f}$. Clearly this mapping is well defined and a norm preserving monomorphism. We also note that, for each maximal ideal M in $Q^*_{\mathfrak{D}}(X)$, $Q^*_{\mathfrak{D}}(X)/M = R$ [3, p. 39]. Finally, let $\mathfrak{M}(Q^*_{\mathfrak{D}}(X))$ be the set of all maximal ideals in $Q^*_{\mathfrak{D}}(X)$. For each $u \in Q^*_{\mathfrak{D}}(X)$, define a real-valued function $\hat{u}$ on $\mathfrak{M}(Q^*_{\mathfrak{D}}(X))$ by $\hat{u}(M) = u + M \in R$, $M \in \mathfrak{M}(Q^*_{\mathfrak{D}}(X))$. The previous lemmas are now used to obtain the main result.

PROPOSITION 4. <u>If</u> $\mathfrak{D}$ <u>contains all disconnected dense open subsets of</u> X, <u>then the maximal ideal space</u> $\mathfrak{M}(Q^*_{\mathfrak{D}}(X))$ <u>endowed with the weak topology determined by the functions</u> $\hat{u}$, $u \in Q^*_{\mathfrak{D}}(X)$ <u>is homeomorphic to</u> K.

<u>Proof</u>. Since the mapping $u_f \longrightarrow \tilde{f}$ is a norm preserving monomorphism of $Q^*_{\mathfrak{D}}(X)$ into C(K), it is enough to show that the family of all $\tilde{f}$ separates the points of K. Take any a, b in K with $a \neq b$. Since $\Lambda(\beta X) \cong \Lambda(X)$ ($\cong K$), we may assume that a, b are members of $\Lambda(X)$, and hence $\phi = \lim_{\beta X}$. Since $a \neq b$, there exist open sets U and V in βX such that $U \cap V = \emptyset$ and $U \cap X \in a$, $V \cap X \in b$. Then by Lemma 2, $a \in \Gamma_K \phi^{-1}(U \cap X)$ and

$b \in \Gamma_K \phi^{-1}(V \cap X)$. Let

$$D = (U \cap X) \cup (X \cap I_{\beta X} C_{\beta X} U);$$

then clearly $D \in \mathfrak{D}$; and define a function f on D by

$$f(x) = \begin{cases} 0 & \text{if } x \in U \cap X \\ 1 & \text{if } x \in X \cap I_{\beta X} C_{\beta X} U . \end{cases}$$

Then $f \in C^*(D)$. Thus $f \circ \phi$ has an extension $\tilde{f}$ on K, and

$$\tilde{f}(a) = \lim_{\substack{z \to a \\ z \in \phi^{-1}(U \cap X)}} (f \circ \phi)(z) = 0,$$

$$\tilde{f}(b) = \lim_{\substack{z \to b \\ z \in \phi^{-1}(V \cap X)}} (f \circ \phi)(z) = 1.$$

Thus the family of $\tilde{f}$ separates the points of K. By the Stone-Weierstrass theorem the proposition holds.

Let $Q_{\mathfrak{D}}(X)$ be the direct limit of the direct system $(C(D))_{D \in \mathfrak{D}}$ with respect to the homomorphisms $f \longrightarrow f|D$, $f \in C(E)$, $D \subseteq E$ in $\mathfrak{D}$. It is known in [3, p. 40] that the Stone topology on the maximal ideal space of $Q^*_{\mathfrak{D}}(X)$ coincides with the weak topology, and moreover the maximal ideal space of $Q_{\mathfrak{D}}(X)$ with the Stone topology is homeomorphic to that of $Q^*_{\mathfrak{D}}(X)$. Hence we have the following:

COROLLARY. *The maximal ideal space of the maximal ring of quotients of* $C(X)$ *endowed with the Stone topology is homeomorphic to* K.

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