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ASYMPTOTIC BEHAVIOR FOR THE SUM OF PARTIAL QUOTIENTS IN CONTINUED FRACTION EXPANSIONS

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ABSTRACT. Let $[a_1(x), a_2(x), a_3(x), \dots]$ be the continued fraction expansion of an irrational number $x \in (0,1)$. Denote by $S_n(x) := \sum_{k=1}^n a_k(x)$ the sum of partial quotients of x. From the results of Khintchine (1935), Diamond and Vaaler (1986), and Philipp (1988), it follows that for almost every $x \in (0,1)$,

$$\liminf_{n\to\infty} \frac{S_n(x)}{n\log n} = \frac{1}{\log 2} \quad \text{and} \quad \limsup_{n\to\infty} \frac{S_n(x)}{n\log n} = \infty.$$

We investigate the Baire category and Hausdorff dimension of the set of points for which the above limit inferior and limit superior assume any prescribed values. We also conduct analogous analyses for the sum of products of consecutive partial quotients.

1. Introduction

The *continued fraction expansion* of a real number $x \in \mathbb{I} := (0,1) \setminus \mathbb{Q}$ can be written in the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} =: [a_1(x), a_2(x), a_3(x), \dots], \tag{1.1}$$

where $a_1(x), a_2(x), a_3(x), \dots$ are positive integers, known as the *partial quotients* of x. See [6] for more details on continued fractions.

In this paper, we study the asymptotic behavior for the sum of partial quotients. For any $x \in \mathbb{I}$ and $n \in \mathbb{N}$, let

$$S_n(x) := \sum_{k=1}^n a_k(x)$$

be the sum of the first n partial quotients of x. In 1935, Khintchine [7] proved that

$$\lim_{n\to\infty} \frac{S_n(x)}{n\log n} = \frac{1}{\log 2} \quad \text{in Lebesgue measure,} \tag{1.2}$$

and pointed out that (1.2) cannot hold almost everywhere. In 1986, Diamond and Vaaler [1] explained that the obstacle to almost everywhere convergence of (1.2) is

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the occurrence of the largest term $\max_{1\le k\le n}\{a_k(x)\}$ in the sum $S_n(x)$. Precisely, they showed that for Lebesgue almost every $x \in \mathbb{I}$,

$$\lim_{n\to\infty} \frac{S_n(x) - \max_{1\le k\le n}\{a_k(x)\}}{n\log n} = \frac{1}{\log 2}. \tag{1.3}$$

Combining (1.2) and (1.3), we see that for Lebesgue almost every $x \in \mathbb{I}$,

$$\liminf_{n\to\infty} \frac{S_n(x)}{n\log n} = \frac{1}{\log 2}. \tag{1.4}$$

For its limit superior behavior, Philipp [12] established that for any sequence $\{b_n\}_{n\ge1}$ of positive numbers with $\{b_n/n\}_{n\ge1}$ being non-decreasing, the following holds for Lebesgue almost every $x \in \mathbb{I}$,

$$\limsup_{n\to\infty} \frac{S_n(x)}{b_n} = 0 \quad \text{or} \quad \limsup_{n\to\infty} \frac{S_n(x)}{b_n} = \infty$$

according to whether the series $\sum_{n=1}^{\infty} 1/b_n$ converges or diverges. Consequently, for Lebesgue almost every $x \in \mathbb{I}$,

$$\limsup_{n\to\infty} \frac{S_n(x)}{n\log n} = \infty. \tag{1.5}$$

We summarize the results of (1.4) and (1.5) in the following theorem.

Theorem 1.1 (Khintchine-Diamond-Vaaler-Philipp's Theorem). *For Lebesgue almost every $x \in \mathbb{I}$,*

$$\liminf_{n\to\infty} \frac{S_n(x)}{n\log n} = \frac{1}{\log 2} \quad \text{and} \quad \limsup_{n\to\infty} \frac{S_n(x)}{n\log n} = \infty$$

Theorem 1.1 demonstrates that, for almost all real numbers, the limit inferior in (1.4) and the limit superior in (1.5) take values of $\frac{1}{\log 2}$ and infinity, respectively. That is to say, the set of points where the limit inferior and the limit superior assume other values has Lebesgue measure zero. Hence, it is natural to investigate the size of such null sets.

For any $0 \le \alpha \le \beta \le \infty$, let

$$S(\alpha,\beta) := \left\{ x \in \mathbb{I} : \liminf_{n\to\infty} \frac{S_n(x)}{n\log n} = \alpha,\ \limsup_{n\to\infty} \frac{S_n(x)}{n\log n} = \beta \right\}.$$

Baire category (see [11]) and Hausdorff dimension (see [2]), apart from Lebesgue measure, are common methods for examining the size of sets. A set is said to be of *first category* if it is represented as a countable union of nowhere dense sets, and a set is called *residual* if its complement is of first category. In this context, a set of first category is considered "small", while a residual set is regarded as "large". We will show that $S(\alpha,\beta)$ is residual for $\alpha = 0$ and $\beta = \infty$, otherwise it is of first category. A similar result concerning the behavior of partial quotients is discussed in [13].

Theorem 1.2. *The set $S(\alpha,\beta)$ is residual if and only if $\alpha = 0$ and $\beta = \infty$.*

From Theorems 1.1 and 1.2, we conclude that $S(\frac{1}{\log 2}, \infty)$ has full Lebesgue measure but is of first category. This indicates that $S(\frac{1}{\log 2}, \infty)$ is "large" in terms of Lebesgue measure but "small" from the perspective of Baire category. In contrast, the set $S(0,\infty)$ is "small" in the sense of Lebesgue measure, but "large" with respect to Baire category.

In the follow theorem, we will see that each $S(\alpha,\beta)$ is "large" from the viewpoint of Hausdorff dimension.

Theorem 1.3. *For any $0 \le \alpha \le \beta \le \infty$, we have $\dim_{\mathrm{H}} S(\alpha,\beta) = 1$.*

Denote by $S(\alpha) := S(\alpha,\alpha)$ for any $0 \le \alpha \le \infty$. As a consequence of Theorem 1.3, we have $S(\alpha)$ has full Hausdorff dimension. This was obtained by Wu and Xu [15, Theorem 1.4]. Actually, the Hausdorff dimension of $S_n(x)$ has been extensively studied by many authors, see [3, 5, 9, 15, 16] and references therein. Let

$$S_\varphi := \left\{ x \in \mathbb{I} : \lim_{n\to\infty} \frac{S_n(x)}{\varphi(n)} = 1 \right\},$$

where $\varphi : \mathbb{N} \to \mathbb{R}_{>0}$ is an increasing function satisfying $\varphi(n) \to \infty$ as $n \to \infty$. For the linear case $\varphi(n) = \gamma n$ with $\gamma \ge 1$, Iommi and Jordan [5] established that with respect to γ, the Hausdorff dimension of S_φ is analytic and increasing from 0 to 1, and tends to 1 as γ goes to infinity. In [15], Wu and Xu proved that if $\varphi(n) = n^r$ with $r \in (1,\infty)$ or $\varphi(n) = \exp(n^r)$ with $r \in (0, 1/2)$, then S_φ has full Hausdorff dimension. Later, it was shown by Xu [16] that if $\varphi(n) = \exp(n)$, then $\dim_{\mathrm{H}} S_\varphi = 1/2$. We remark that his proof also applies to the case $\varphi(n) = \exp(n^r)$ with $r > 1$, and and it implies that the Hausdorff dimension of S_φ remains 1/2. In 2016, Liao and Rams [9] investigated the Hausdorff dimension of S_φ for the remaining case $\varphi(n) = \exp(n^r)$ with $r \in [1/2, 1)$ and showed that there is a jump of the Hausdorff dimensions from 1 to 1/2 at $r = 1/2$. Recently, the dimension gap was filled by Fang, Moreira and Zhang [3].

The rest of the paper is organized as follows. In Section 2, we provide the proof of Theorem 1.2. Section 3 is devoted to proving Theorem 1.3. At last, we do similar analyses for the sum of products of consecutive partial quotients in Section 4.

2. Proof of Theorem 1.2

In this section, we consider $\mathbb{I}$ equipped with the induced topology. Since $\mathbb{I}$ is a Baire space, it suffices to prove that a set is residual by showing that it contains a dense G_δ subset (see [11, Theorem 9.2]).

For any $(a_1,\dots,a_n) \in \mathbb{N}^n$, let

$$I_n(a_1,\dots,a_n) := \{x \in (0,1) : a_1(x) = a_1, \dots, a_n(x) = a_n\}.$$

It was shown in Theorem 1.2.2 of [6] that $I_n(a_1,\dots,a_n) \cap \mathbb{I}$ is an open interval with rational endpoints in $\mathbb{I}$.

For any $\gamma \in (0,\infty)$, define

$$S_*(\gamma) = \left\{ x \in \mathbb{I} : \liminf_{n\to\infty} \frac{S_n(x)}{n\log n} \le \gamma \right\}$$

and

$$S^*(\gamma) = \left\{ x \in \mathbb{I} : \limsup_{n\to\infty} \frac{S_n(x)}{n\log n} \ge \gamma \right\},$$

and let $S(\gamma) := S_*(\gamma) \cap S^*(\gamma)$.

Lemma 2.1. *For any $\gamma \in (0,\infty)$, the set $S(\gamma)$ is dense in $\mathbb{I}$.*

Proof. For any $\gamma \in (0,\infty)$, let $\theta_n := \lfloor \gamma \log n \rfloor + 1$ for all $n \in \mathbb{N}$. Then $\{\theta_n\}_{n\geq 1}$ corresponds to a unique irrational number $x_0 \in [0,1)$ such that $a_n(x_0) = \theta_n$ for all $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} \frac{S_n(x_0)}{n \log n} = \gamma,$$

that is, $x_0 \in S(\gamma)$.

Set

$$D(x_0) := \bigcup_{N=1}^{\infty} \{x \in \mathbb{I} : a_n(x) = a_n(x_0), \forall n \geq N\}.$$

Then $D(x_0) \subseteq S(\gamma)$ is dense in $\mathbb{I}$. To see this, for any $x \in \mathbb{I}$ and $n \in \mathbb{N}$, define x_n in terms of the continued fraction expansion as

$$x_n := [a_1(x), \ldots, a_n(x), a_{n+1}(x_0), a_{n+2}(x_0), \ldots].$$

Then $x_n \in D(x_0)$ and $|x_n - x| \to 0$ as $n \to \infty$, which implies that $D(x_0)$ is dense in $\mathbb{I}$. Hence $S(\gamma)$ is also dense in $\mathbb{I}$. □

Lemma 2.2. *For any $\gamma \in (0,\infty)$, the sets $S_*(\gamma)$ and $S^*(\gamma)$ are residual.*

Proof. Let $\gamma \in (0,\infty)$ be fixed. It follows from Lemma 2.1 that $S_*(\gamma)$ and $S^*(\gamma)$ are dense. To prove that $S^*(\gamma)$ and $S_*(\gamma)$ are residual, it suffices to show that they are all G_δ sets.

For $S_*(\gamma)$, we deduce that

$$S_*(\gamma) = \bigcap_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n(\gamma,k),$$

where $E_n(\gamma,k)$ is given by

$$E_n(\gamma,k) := \left\{x \in \mathbb{I} : S_n(x) < \left(\gamma + \frac{1}{k}\right) n \log n\right\}.$$

We observe that for sufficiently large n, $E_n(\gamma,k)$ is nonempty and can be written as a countable union of open sets in $\mathbb{I}$. More precisely,

$$E_n(\gamma,k) = \bigcup_{\sum_{i=1}^{n} k_i < (\gamma+1/k) n \log n} I_n(k_1,k_2,\ldots,k_n) \cap \mathbb{I},$$

where $(k_1,k_2,\ldots,k_n) \in \mathbb{N}^n$. Since $I_n(k_1,k_2,\ldots,k_n)$ is an open set in $\mathbb{I}$, we have $E_n(\gamma,k)$ is open in $\mathbb{I}$, and so $S_*(\gamma)$ is a G_δ set.

For $S^*(\gamma)$, we conclude that

$$S^*(\gamma) = \bigcap_{k=\lceil 1/\gamma \rceil}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{x \in \mathbb{I} : S_n(x) > \left(\gamma - \frac{1}{k}\right) n \log n\right\},$$

and $\{x \in \mathbb{I} : S_n(x) > (\gamma - 1/k) n \log n\}$ is a countable union of open sets in $\mathbb{I}$. Hence $S^*(\gamma)$ is a G_δ set. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. For any $K \in \mathbb{N}$, it follows form Lemma 2.2 that $S_*(1/K)$ and $S^*(K)$ are residual. By the definition of residual set, we see that the countable intersection of residual sets is also residual. Since

$$\left\{x \in \mathbb{I} : \liminf_{n\to\infty} \frac{S_n(x)}{n \log n} = 0\right\} = \bigcap_{K=1}^{\infty} S_*(1/K) \tag{2.1}$$

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and

$$\left\{x \in \mathbb{I} : \limsup_{n\to\infty} \frac{S_n(x)}{n \log n} = \infty\right\} = \bigcap_{K=1}^{\infty} S^*(K), \tag{2.2}$$

we derive that the sets on the left-hand side of (2.1) and (2.2) are residual, and so and their intersection $S(0,\infty)$ is residual.

Note that every subset of a set of first category is also of first category. For any $\alpha > 0$ or $\beta < \infty$, each $S(\alpha,\beta)$ is a subset of the complement of $S(0,\infty)$, and so $S(\alpha,\beta)$ is of first category. □

Remark 2.3. As suggested by the referee, we provide the following refinement of the result concerning the set in (2.1): for a residual set of values of $x \in \mathbb{I}$, it holds that

$$\liminf_{n\to\infty} \frac{S_n(x)}{n} = 1.$$

In fact, using the arguments in this section, we obtain an analogue of Theorem 1.2 for the arithmetic mean of partial quotients: the set

$$\left\{x \in \mathbb{I} : \liminf_{n\to\infty} \frac{S_n(x)}{n} = \alpha,\ \limsup_{n\to\infty} \frac{S_n(x)}{n} = \beta\right\}$$

is residual if and only if $\alpha = 1$ and $\beta = \infty$.

3. Proof of Theorem 1.3

In this section, we prove that $\dim_{\mathrm{H}} S(\alpha,\beta) = 1$ for all $0 \le \alpha \le \beta \le \infty$. To this end, we require the following lemma.

Let $\{n_k\}_{k\ge1}$ and $\{u_k\}_{k\ge1}$ be sequences of positive integers satisfying $n_k/k \to \infty$ and $u_k \to \infty$ as $k \to \infty$. For each $M \in \mathbb{N}$, define

$$E_M(\{n_k\},\{u_k\}) := \{x \in \mathbb{I} : a_{n_k}(x) = u_k,\ 1 \le a_j \le M\,(j \ne n_k)\,, \forall k \in \mathbb{N}\}\,. \tag{3.1}$$

Shang and Wu [14] determined the Hausdorff dimension of $E_M(\{n_k\},\{u_k\})$, stated as follows. See [3, 10] for similar results.

Lemma 3.1 ([14, Lemma 3.1])**.** *Let $E_M(\{n_k\},\{u_k\})$ be defined as above. Suppose that*

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{j=1}^{k} \log u_j = 0. \tag{3.2}$$

Then

$$\lim_{M\to\infty} \dim_{\mathrm{H}} E_M(\{n_k\},\{u_k\}) = 1$$

We will use Lemma 3.1 to show $\dim_{\mathrm{H}} S(\alpha,\beta) = 1$ by choosing suitable sequences $\{n_k\}_{k\ge1}$ and $\{u_k\}_{k\ge1}$ according to the values of α and β.

Proof of Theorem 1.3. The proof is divided into six cases. For convenience, we first introduce some notation. For any $n \in \mathbb{N}$, let $t(n) := \max\{k \in \mathbb{N} : k^2 \le n\}$ and $\phi(n) := 2^{2^n}$. Then $\sqrt{n} - 1 \le t(n) \le \sqrt{n}$ and $(\phi(n))^2 = \phi(n+1)$.

In the following, we always take $n_k := k^2$ for all $k \in \mathbb{N}$. The definition of $\{u_k\}_{k\ge1}$ depends on the values of α and β.

Case 1: $0 < \alpha \le \beta < \infty$. Let $u_1 := 1$. For all $k \ge 2$, let

$$u_k := \begin{cases} \lceil 4\alpha k \log k \rceil, & \text{if } \lceil \log_2 \log_2 k \rceil \text{ is even;} \\ \lceil 4\beta k \log k \rceil, & \text{if } \lceil \log_2 \log_2 k \rceil \text{ is odd.} \end{cases} \tag{3.3}$$

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Then $\{n_k\}$ and $\{u_k\}$ satisfy condition (3.2). Moreover, we claim that the set $S(\alpha,\beta)$ defined in (3.1) is a subset of $E_M(\{n_k\},\{u_k\})$. In fact, for any $x\in E_M(\{n_k\},\{u_k\})$ and for all $k\in\mathbb{N}$, we have $4\alpha k\log k\le a_{k^2}(x)<4\beta k\log k+1$ and $1\le a_j(x)\le M$ for all $j\ne k^2$. For any $n\in\mathbb{N}$, we see that

$$S_n(x)\ge\sum_{k=2}^{t(n)}4\alpha k\log k\ge 4\alpha\int_1^{t(n)}x\log x dx=\alpha(t(n))^2\big(2\log t(n)-1\big)$$

and

$$\begin{aligned}S_n(x)&<\sum_{k=2}^{t(n)}(4\beta k\log k+1)+nM\\&<4\beta\int_1^{t(n)+1}x\log x dx+t(n)+nM\\&=\beta(t(n)+1)^2\big(2\log(t(n)+1)-1\big)+t(n)+nM.\end{aligned}$$

Combining this with the fact that $\sqrt{n}-1\le t(n)\le\sqrt{n}$, we derive that

$$\alpha\le\liminf_{n\to\infty}\frac{S_n(x)}{n\log n}\le\limsup_{n\to\infty}\frac{S_n(x)}{n\log n}\le\beta. \tag{3.4}$$

Next, we show that the limit superior and the limit inferior in (3.4) are β and α, respectively. To this end, consider

$$\limsup_{m\to\infty}\frac{S_{\phi(2m)}(x)}{\phi(2m)\log\phi(2m)}$$

and

$$\liminf_{m\to\infty}\frac{S_{\phi(2m+1)}(x)}{\phi(2m+1)\log\phi(2m+1)}.$$

Note that for all k,

(1) if $\phi(2m-2)<k\le\phi(2m-1)$, then $\lceil\log_2\log_2 k\rceil$ is odd, and so $a_{k^2}(x)=\lceil 4\beta k\log k\rceil$;
(2) if $\phi(2m-1)<k\le\phi(2m)$, then $\lceil\log_2\log_2 k\rceil$ is even, and so $a_{k^2}(x)=\lceil 4\alpha k\log k\rceil$,

and $1\le a_j(x)\le M$ for all $j\ne k^2$. On the one hand, it follows from (1) that

$$\begin{aligned}S_{\phi(2m)}(x)&\ge\sum_{k=\phi(2m-2)+1}^{\phi(2m-1)}4\beta k\log k\\&\ge 4\beta\int_{\phi(2m-2)}^{\phi(2m-1)}x\log x\,\mathrm{d}x\\&=\beta(\phi(2m-1))^2\left(2\log\phi(2m-1)-1\right)\\&\quad-\beta(\phi(2m-2))^2\left(2\log\phi(2m-2)-1\right).\end{aligned}$$

Note that $(\phi(n))^2=\phi(n+1)$, we have

$$S_{\phi(2m)}(x)\ge\beta\phi(2m)(\log\phi(2m)-1)-\beta\phi(2m-1)(\log\phi(2m-1)-1).$$

Since

$$\lim_{m\to\infty}\frac{\phi(2m-1)(\log\phi(2m-1)-1)}{\phi(2m)(\log\phi(2m)-1)}=0,$$

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we derive that

$$\limsup_{m\to\infty} \frac{S_{\phi(2m)}(x)}{\phi(2m)\log\phi(2m)} \geq \beta. \tag{3.5}$$

On the other hand, we deduce from (2) that

$$\begin{aligned} S_{\phi(2m+1)}(x) &< \sum_{k=2}^{\phi(2m)} (4\alpha k\log k + 1) + \sum_{k=2}^{\phi(2m-1)} (4\beta k \log k + 1) \\ &\quad + \phi(2m+1)M \\ &< 4\alpha \int_{k=1}^{\phi(2m)+1} x\log x dx + 4\beta \int_{k=1}^{\phi(2m-1)+1} x \log x dx \\ &\quad + \phi(2m+1)(M+2) \\ &< 2\alpha(\phi(2m)+1)^2 \log(\phi(2m)+1) \\ &\quad + 2\beta(\phi(2m-1)+1)^2\log(\phi(2m-1)+1)) \\ &\quad + \phi(2m+1)(M+2). \end{aligned}$$

By the definition of $\phi(n)$, we see that

$$\lim_{m\to\infty} \frac{2(\phi(2m)+1)^2\log(\phi(2m)+1)}{\phi(2m+1)\log\phi(2m+1)} = 1$$

and

$$\lim_{m\to\infty} \frac{(\phi(2m-1)+1)^2\log(\phi(2m-1)+1))}{(\phi(2m)+1)^2\log(\phi(2m)+1)} = 0,$$

and so

$$\liminf_{m\to\infty} \frac{S_{\phi(2m+1)}(x)}{\phi(2m+1)\log\phi(2m+1)} \leq \alpha. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we obtain

$$\liminf_{n\to\infty} \frac{S_n(x)}{n\log n} = \alpha \quad \text{and} \quad \limsup_{n\to\infty} \frac{S_n(x)}{n\log n} = \beta,$$

i.e., $x \in S(\alpha,\beta)$. Hence, $\dim_{\mathrm{H}} S(\alpha,\beta) \geq \dim_{\mathrm{H}} E_M(\{n_k\},\{u_k\})$. Letting $M\to\infty$, we conclude from Lemma 3.1 that $\dim_{\mathrm{H}} S(\alpha,\beta) = 1$.

For other cases, when $\alpha = 0$, replace $\lceil 4\alpha k\log k\rceil$ in (3.3) with k; when $\beta=\infty$, replace $\lceil 4\beta k\log k\rceil$ in (3.3) with k^2. More precisely,

Case 2: $\alpha=\beta=0$. For all $k\geq 1$, let $u_k := k$.

Case 3: $\alpha=\beta=\infty$. For all $k\geq 1$, let $u_k := k^2$.

Case 4: $\alpha = 0$ and $\beta=\infty$. Let $u_1 := 1$. For all $k \geq 2$, let

$$u_k := \begin{cases} k, & \text{if } \lceil \log_2\log_2 k\rceil \text{ is even;} \\ k^2, & \text{if } \lceil \log_2\log_2 k\rceil \text{ is odd.} \end{cases}$$

Case 5: $\alpha = 0$ and $0<\beta<\infty$. Let $u_1 := 1$. For all $k \geq 2$, let

$$u_k := \begin{cases} k, & \text{if } \lceil \log_2\log_2 k\rceil \text{ is even;} \\ \lceil 4\beta k\log k\rceil, & \text{if } \lceil \log_2\log_2 k\rceil \text{ is odd.} \end{cases}$$

Case 6: $0<\alpha<\infty$ and $\beta=\infty$. Let $u_1 := 1$. For all $k \geq 2$, let

$$u_k := \begin{cases} \lceil 4\alpha k\log k\rceil, & \text{if } \lceil \log_2\log_2 k\rceil \text{ is even;} \\ k^2, & \text{if } \lceil \log_2\log_2 k\rceil \text{ is odd.} \end{cases}$$

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Following the proof in Case 1 step by step, we can similarly derive the same result. The details are omitted here for brevity. □

4. The sum of products of consecutive partial quotients

Our results can be extended to include the sum of products of consecutive partial quotients. For any $x \in \mathbb{I}$ and $n \in \mathbb{N}$, let

$$\widetilde{S}_n(x) := \sum_{k=1}^{n} a_k(x)a_{k+1}(x).$$

The asymptotic behavior of $\{\widetilde{S}_n(x)\}_{n\geq 1}$ was studied by Hu, Hussain and Yu [4]. They obtained the Khintchine-type theorem for $\widetilde{S}_n(x)$:

$$\lim_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \frac{1}{2\log 2} \quad \text{in Lebesgue measure,}$$

and the Diamond-Vaaler-type theorem for $\widetilde{S}_n(x)$: for Lebesgue almost every $x \in \mathbb{I}$,

$$\lim_{n\to\infty} \frac{\widetilde{S}_n(x) - \max_{1\leq k\leq n}\{a_k(x)a_{k+1}(x)\}}{n\log^2 n} = \frac{1}{2\log 2}.$$

These two results imply that for Lebesgue almost every $x \in \mathbb{I}$,

$$\liminf_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \frac{1}{2\log 2}. \tag{4.1}$$

For its limit superior, we introduce the zero-one law for products of consecutive partial quotients established by Kleinbock and Wadleigh [8]. For any $\Phi : \mathbb{N} \to [1, \infty)$ with $\Phi(n) \to \infty$ as $n \to \infty$, let

$$\mathcal{K}(\Phi) := \{x \in \mathbb{I} : a_n(x)a_{n+1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

Then the Lebesgue measure of $\mathcal{K}(\Phi)$ is zero or one according to whether the series $\sum_{n=1}^{\infty} \frac{\log\Phi(n)}{\Phi(n)}$ converges or diverges. As a result, for $\Phi(n) = n(\log^2 n)(\log\log n)$, we see that the Lebesgue measure of $\mathcal{K}(\Phi)$ is one. Hence, for Lebesgue almost every $x \in \mathbb{I}$,

$$\limsup_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \infty. \tag{4.2}$$

In light of (4.1) and (4.2), we have the following theorem.

Theorem 4.1 (Kleinbock-Wadleigh-Hu-Hussain-Yu's Theorem)**.** *For Lebesgue almost every $x \in \mathbb{I}$,*

$$\liminf_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \frac{1}{2\log 2} \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \infty.$$

In what follows, we study the Baire category and Hausdorff dimension of $\widetilde{S}_n(x)$. For any $0 \leq \alpha \leq \beta \leq \infty$, let

$$\widetilde{S}(\alpha,\beta) := \left\{x \in \mathbb{I} : \liminf_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \alpha,\ \limsup_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \beta\right\}.$$

Using a method similar to that employed in Theorem 1.2, we obtain the necessary and sufficient conditions for $\widetilde{S}(\alpha,\beta)$ to be residual.

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Theorem 4.2. *The set* $\widetilde{S}(\alpha,\beta)$ *is residual if and only if* $\alpha=0$ *and* $\beta=\infty$.

We end this section with the Hausdorff dimension of $\widetilde{S}(\alpha,\beta)$.

Theorem 4.3. *For any* $0\le\alpha\le\beta\le\infty$, *we have* $\dim_{\mathrm{H}}\widetilde{S}(\alpha,\beta)=1$.

To prove Theorem 4.3, we first give a result analogous to Lemma 3.1.

Let $\{\omega_k\}_{k\ge1}$, $\{b_k\}_{k\ge1}$, and $\{c_k\}_{k\ge1}$ be sequences of positive integers satisfying $\omega_k/k\to\infty$, $b_k\to\infty$, and $c_k\to\infty$ as $k\to\infty$. For each $M\in\mathbb{N}$, define

$$E_M(\{\omega_k\},\{b_k\},\{c_k\}):=\Big\{x\in\mathbb{I}:a_{\omega_k}(x)=b_k;\ a_{\omega_k+1}(x)=c_k;\\ 1\le a_j\le M\,(j\ne\omega_k,\omega_k+1)\,,\forall k\in\mathbb{N}\Big\}.$$

The Hausdorff dimension of $E_M(\{\omega_k\},\{b_k\},\{c_k\})$ is as follows.

Lemma 4.4. *Let* $E_M(\{\omega_k\},\{b_k\},\{c_k\})$ *be defined as above. Assume that*

$$\lim_{k\to\infty}\frac{1}{\omega_k}\sum_{j=1}^{k}\log\left(b_jc_j\right)=0. \tag{4.3}$$

Then

$$\lim_{M\to\infty}\dim_{\mathrm{H}}E_M(\{\omega_k\},\{b_k\},\{c_k\})=1.$$

Proof. This result can be regarded as a consequence of Lemma 3.1. To see this, define $\{n_k\}_{k\ge1}$ and $\{u_k\}_{k\ge1}$ as follows: for any $k\in\mathbb{N}$,

$$\begin{cases} n_{2k-1}:=\omega_k,\\ n_{2k}:=\omega_k+1,\end{cases}\quad\text{and}\quad\begin{cases} u_{2k-1}:=b_k,\\ u_{2k}:=c_k.\end{cases}$$

Then $E_M(\{\omega_k\},\{b_k\},\{c_k\})=E_M(\{n_k\},\{u_k\})$. Since $\omega_k/k\to\infty$ as $k\to\infty$, it follows that $n_k/k\to\infty$ as $k\to\infty$. Moreover,

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{n_{2k}}\sum_{j=1}^{2k}\log u_j&=\lim_{k\to\infty}\frac{1}{\omega_k+1}\sum_{j=1}^{k}\left(\log b_j+\log c_j\right)\\&=\lim_{k\to\infty}\frac{\omega_k}{\omega_k+1}\cdot\frac{1}{\omega_k}\sum_{j=1}^{k}\log\left(b_jc_j\right)\\&=0,\end{aligned}$$

and

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{n_{2k-1}}\sum_{j=1}^{2k-1}\log u_j&=\lim_{k\to\infty}\frac{1}{\omega_k}\left(\sum_{j=1}^{k-1}\left(\log b_j+\log c_j\right)+\log b_j\right)\\&\le\lim_{k\to\infty}\frac{1}{\omega_k}\sum_{j=1}^{k}\left(\log b_j+\log c_j\right)\\&=0.\end{aligned}$$

Thus, $\{n_k\}_{k\ge1}$ and $\{u_k\}_{k\ge1}$ satisfy condition (3.2). By Lemma 3.1, we derive that

$$\lim_{M\to\infty}\dim_{\mathrm{H}}E_M(\{\omega_k\},\{b_k\},\{c_k\})=\lim_{M\to\infty}E_M(\{n_k\},\{u_k\})=1.$$

The proof is completed. □

We are now in a position to show that $\dim_{\mathrm{H}}\widetilde{S}(\alpha,\beta)=1$ by using Lemma 4.4.

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Proof of Theorem 4.3. We onely give the proof for the case where $0 < \alpha \le \beta < \infty$. Let $\omega_1 = b_1 = c_1 := 1$. For all $k \ge 2$, let $\omega_k := k^2$,

$$b_k := \begin{cases} \lceil 8\alpha k \log k \rceil, & \text{if } \lceil \log_2 \log_2 k \rceil \text{ is even;} \\ \lceil 8\beta k \log k \rceil, & \text{if } \lceil \log_2 \log_2 k \rceil \text{ is odd,} \end{cases} \tag{4.4}$$

and $c_k := \lfloor \log k \rfloor$. Then $\{\omega_k\}_{k\ge1}, \{b_k\}_{k\ge1}$ and $\{c_k\}_{k\ge1}$ satisfy condition (4.3). Moreover, we claim that $E_M(\{\omega_k\}, \{b_k\}, \{c_k\})$ is a subset of $\widetilde{S}(\alpha, \beta)$. In fact, for any $x \in E_M(\{\omega_k\}, \{b_k\}, \{c_k\})$, we see that $8\alpha k \log k \le a_{k^2}(x) \le 8\beta k \log k + 1$, $\log k - 1 \le a_{k^2+1}(x) \le \log k$, and $1 \le a_j(x) \le M$ for all $j \notin \{k^2, k^2+1\}$. Hence,

$$\widetilde{S}_n(x) \ge \sum_{k=2}^{t(n)} 8\alpha k \log k (\log k - 1) \ge 8\alpha \int_1^{t(n)} \left(x \log^2 x dx - x \log x \right) dx,$$

and

$$\begin{aligned} \widetilde{S}_n(x) &\le \sum_{k=2}^{t(n)} (8\beta k \log k + 1) \log k + M \sum_{k=2}^{t(n)} (8\beta k \log k + \log k + 1) + nM^2 \\ &\le \int_1^{t(n)+1} \left(8\beta x \log^2 x + 8M\beta x \log x + (M+1) \log x \right) dx + nM(M+1). \end{aligned}$$

Our analysis focuses exclusively on the main term of the integral, since the contributions from the integrals of the remaining terms are negligible, being infinitesimal of the order $n \log^2 n$. Since $\sqrt{n} - 1 \le t(n) \le \sqrt{n}$ and

$$\int x \log^2 x dx = \frac{1}{8} x^2 \left(\log^2 x^2 - 2 \log x^2 + 2 \right) + C, \tag{4.5}$$

we deduce that

$$\alpha \le \liminf_{n\to\infty} \frac{\widetilde{S}_n(x)}{n \log^2 n} \le \limsup_{n\to\infty} \frac{\widetilde{S}_n(x)}{n \log^2 n} \le \beta. \tag{4.6}$$

Note that for all $k \in \mathbb{N}$,

(1) if $\phi(2m-1) < k \le \phi(2m)$, then $\lceil \log_2 \log_2 k \rceil$ is even, and so $a_{k^2}(x) = \lceil 8\alpha k \log k \rceil$ and $a_{k^2+1}(x) = \lfloor \log k \rfloor$;
(2) if $\phi(2m) < k \le \phi(2m+1)$, then $\lceil \log_2 \log_2 k \rceil$ is odd, ans so $a_{k^2}(x) = \lceil 8\beta k \log k \rceil$and $a_{k^2+1}(x) = \lfloor \log k \rfloor$,

and $1 \le a_j(x) \le M$ for all $j \notin \{k^2, k^2+1\}$. Thus,

$$\begin{aligned} \widetilde{S}_{\phi(2m)}(x) &\ge \sum_{k=\phi(2m-2)+1}^{\phi(2m-1)} 8\beta k \log k (\log k - 1) \\ &= 8\beta \int_{\phi(2m-2)}^{\phi(2m-1)} \left(x \log^2 x dx - \log x \right) dx. \end{aligned}$$

By (4.5) and the definition of $\phi(n)$, we obtain

$$\limsup_{m\to\infty} \frac{\widetilde{S}_{\phi(2m)}(x)}{\phi(2m) \log^2 \phi(2m)} \ge \beta. \tag{4.7}$$

Similarly, we can derive that

$$\liminf_{m\to\infty} \frac{\widetilde{S}_{\phi(2m+1)}(x)}{\phi(2m+1) \log^2 \phi(2m+1)} \le \alpha. \tag{4.8}$$

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Combining (4.6), (4.7) and (4.8), we conclude that

$$\liminf_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \alpha \quad \text{and} \quad \limsup_{n\to\infty} \frac{\widetilde{S}_n(x)}{n\log^2 n} = \beta,$$

i.e., $x \in \widetilde{S}(\alpha,\beta)$. So $\dim_{\mathrm{H}} \widetilde{S}(\alpha,\beta) \geq \dim_{\mathrm{H}} E_M(\{\omega_k\},\{b_k\},\{c_k\})$. Letting $M \to \infty$, it follows from Lemma 4.4 that $\dim_{\mathrm{H}} \widetilde{S}(\alpha,\beta) = 1$. □

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