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Pure point spectrum for dynamical systems and mean, Besicovitch and Weyl almost periodicity

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Abstract. We consider metrizable ergodic topological dynamical systems over locally compact, σ-compact abelian groups. We study pure point spectrum via suitable notions of almost periodicity for the points of the dynamical system. More specifically, we characterize pure point spectrum via mean almost periodicity of generic points. We then go on and show how Besicovitch almost periodic points determine both eigenfunctions and the measure in this case. After this, we characterize those systems arising from Weyl almost periodic points and use this to characterize weak and Bohr almost periodic systems. Finally, we consider applications to aperiodic order.

Key words: discrete spectrum, aperiodic order, almost periodicity properties of generic points
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Contents

1. *Introduction*

This article is concerned with dynamical systems with pure point spectrum. The dynamical systems in question consist of a continuous action of a locally compact abelian group on a compact metric space together with an invariant probability measure on the space. Pure point spectrum means that there exists an orthonormal basis of eigenfunctions. In a sense, such systems are the simplest possible dynamical systems. Their study is a most basic ingredient in the conceptual theory of dynamical systems as witnessed by such fundamental results as the Halmos–von Neumann theorem or the Furstenberg structure theorem. Accordingly, a variety of characterizations for pure point spectrum has been established over the decades.

Recent years have brought two new lines of interest in such systems. One line is given by a series of works which analyze such systems via (weak) notions of equicontinuity [**11**, **16**–**19**, **22**]. The main thrust of these works is to characterize pure point spectrum as well as various strengthened versions thereof (see also [**15**, **23**, **40**, **41**] for related work). The other line comes from the investigation of aperiodic order. Aperiodic order, also known as mathematical theory of quasicrystals, has emerged as fruitful field of (not only) mathematics over the last three decades; see, e.g., [**2**] for a recent monograph and [**5**, **24**] for recent collections of surveys. A key feature of aperiodic order is the occurrence of (pure) point diffraction. A central result in the mathematical treatment of aperiodic order gives that pure point diffraction can be understood as pure point spectrum of suitable associated dynamical systems. In fact, this result is the outcome of a cumulative effort of various people over the last decades [**4**, **12**, **21**, **26**, **30**, **32**, **37**].

A common feature of all these works is that their considerations share a flavor of almost periodicity. On an intuitive level this is not surprising. After all, pure point spectrum means that all spectral measures are pure point measures. This, in turn, is equivalent to the Fourier transforms of these measures being almost periodic [**20**, **35**]. In this way, almost periodicity properties of functions and their averages come into play in a very natural way.

However, what is lacking is a description of pure point spectrum via almost periodicity properties of the *points* of the dynamical system. The goal of this article it to provide such a characterization and to study some of its consequences.

In order to do so, we introduce for points in dynamical systems the concepts of mean almost periodicity, Besicovitch almost periodicity, Weyl almost periodicity, weak almost periodicity and Bohr almost periodicity, respectively.

After the discussion of the necessary background in §2, our first achievement is Theorem 3.8 in §3. This theorem says that a system has pure point spectrum if and only if every (or even just one) generic point is mean almost periodic. If the dynamical system is ergodic this is then equivalent to almost all points being mean almost periodic.

As for mean almost periodicity of points, in turn various characterizations are discussed in §3. One of them proceeds via averages of distance between the orbit of the point and a shifted orbit. Another characterizes mean almost periodicity of a point via almost periodicity properties of the sampling of continuous functions along the orbit of this point. In this way, we have a rather complete and clear picture of the meaning of mean almost periodic points for general dynamical systems.

This picture ties in with various earlier results. In the case where the group is just the integers, a related characterization via sampling of bounded measurable function is given in [**7**] for measurable dynamical systems. In the more specific situation of subshifts over a finite alphabet, there is also a characterization of pure point spectrum via the so-called Besicovitch–Hamming almost periodicity of almost all configurations [**40**, Lemma 5]. For constant length substitutions, the equivalence between mean almost periodicity and pure point spectrum was proved in [**36**, Lemma VI.25]. In the particular case of point processes in $\mathbb{R}^d$, these results have been established in [**21**, Theorem 4.4]. Moreover, we point out a companion article [**31**] dealing with fundamental issues in aperiodic order via almost periodicity of measures. As an application, it treats the situation of a special dynamical system, namely translation bounded measures dynamical systems. These systems are particularly relevant to aperiodic order. In these systems, the points are measures and this allows one to work with almost periodicity properties of measures. Of course, this approach it not available for general dynamical systems.

In terms of methods it should be emphasized that our proof of Theorem 3.8 is completely different from those given in [**7**, **31**, **40**]. It relies on the characterization of pure point spectrum by Bohr almost periodicity of an averaged metric obtained recently in [**29**].

In §4, we then introduce Besicovitch almost periodic points. Although the condition of Besicovitch almost periodicity is strictly stronger than mean almost periodicity, we can still show that an ergodic dynamical system has pure point spectrum if and only if almost all points are Besicovitch almost periodic and this holds if and only if there exists one generic Besicovitch almost periodic point (Theorem 4.7). In this respect the difference between mean and Besicovitch almost periodicity is not too large. The advantage of Besicovitch almost periodic points is that they allow for averaging with characters. In particular, it is possible to compute the eigenfunctions via the Besicovitch almost periodic points (Theorem 4.7). In fact, the complete spectral theory of the dynamical system can be directly computed from any single typical Besicovitch almost periodic point (Theorem 4.4).

In §5, we introduce Weyl almost periodic points. Weyl almost periodicity of a point is substantially stronger than Besicovitch almost periodicity. In fact, a point is Weyl almost periodic if and only if its orbit closure is uniquely ergodic and has pure point spectrum with continuous eigenfunctions (Theorem 5.5). In this case, all points in the orbit closure are Weyl almost periodic (Lemma 5.3). To put these results in perspective, we note that

[11, 16] showed that a dynamical system is mean equicontinuous if and only if it is uniquely ergodic with pure point spectrum and continuous eigenvalues. Hence, a point is Weyl almost periodic if and only if its orbit closure is mean equicontinuous. Thus, our results can be understood to provide a natural pointwise counterpart to the results of [16].

In §6, we investigate two special classes of Weyl almost periodic points, namely Bohr and weakly almost periodic points, respectively. This allows us to characterize Bohr and weakly almost periodic dynamical systems. In particular, we reprove a main result of [33]. An application of our results to aperiodic order is given in §7. This includes an alternative proof for the main results on measure dynamical systems contained in [31].

Our article gives a comprehensive treatment of most relevant concepts of almost periodicity in the context of pure point spectrum. Some parts of our considerations allow for simple abstractions. As this may be of value for further investigations, we include a brief discussion of some basic results in §8.

It seems that Besicovitch almost periodicity is not well known (for groups other than $\mathbb{R}$) and there is also some ambiguity in the way it is defined. For this reason, we include some appendices discussing the almost periodicity properties needed in this article as well as basic properties of the class of continuous functions with these almost periodicity properties.

2. *Background on dynamical systems, pure point spectrum and the upper mean*

In this section, we review the necessary concepts from dynamical systems and introduce the upper mean $\overline{M}$, which is crucial for our subsequent considerations.

Throughout the paper, we denote the set of continuous complex-valued functions on the topological space Y by $C(Y)$.

We consider a compact metric space X equipped with a continuous action

$$\alpha : G \times X \longrightarrow X, \quad (t, x) \mapsto \alpha_t(x),$$

of a locally compact, σ-compact, abelian group G. We then call (X, G) a *dynamical system (over the space X)*. Often, we will also be given a probability measure m on X, which is invariant under the action of G. We then call (X, G, m) a *dynamical system* as well. We write tx instead of $\alpha_t(x)$ for $t \in G$ and $x \in X$. The composition on G itself is written additively and the neutral element of G is denoted as e. We fix a Haar measure on G, which is unique up to multiplication by a positive constant. The Haar measure of a measurable subset $A \subset G$ is denoted by $|A|$ and the integral of an integrable function f on G by $\int f(t)\, dt$.

Whenever a dynamical system (X, G) is given, we furthermore make use of:

- a metric d on X which generates the topology;
- a Følner sequence (B_n) in G, that is, each B_n is an open relatively compact subset of G and

$$\frac{|(B_n \setminus (t + B_n)) \cup ((t + B_n) \setminus B_n)|}{|B_n|} \to 0, \quad n \to \infty,$$

  for all $t \in G$.

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Note that a Følner sequence exists in a locally compact abelian group G if an only if G is σ-compact [39, Proposition B6]. For this reason, we always assume that G is σ-compact.

The *orbit* of x is given by $Gx := \{tx : t \in G\}$ and the *orbit closure* $\overline{Gx}$ is the closure of the orbit. If the orbit closure of $x \in X$ agrees with X, the element x is called *transitive*. If every $x \in X$ is transitive the dynamical system is called *minimal*. The dynamical system (X, G) is called *uniquely ergodic* if there exists only one invariant probability measure on X. We then denote this measure by m and call (X, G, m) uniquely ergodic as well.

The dynamical system (X, G, m) is *ergodic* if any invariant measurable subset A of G satisfies $m(A) = 1$ or $m(A) = 0$. If (X, G, m) is ergodic, any Følner sequence has a subsequence (B_n) for which *Birkhoff's ergodic theorem holds*, that is,

$$\lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(tx)\, dt = \int_X f\, dm$$

is valid for almost every $x \in X$ whenever $f : X \longrightarrow \mathbb{C}$ integrable [34].

Whenever a dynamical system (X, G, m) and a Følner sequence (B_n) is given, a point $y \in X$ is called *m-generic* with respect to the Følner sequence if

$$\lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(ty)\, dt = \int_X f(x)\, dm(x)$$

holds for any continuous $f : X \longrightarrow \mathbb{C}$. If the measure m and the sequence (B_n) are clear from the context, we just speak about generic points. Generic points play a key role in our subsequent considerations as they determine the measure and, in this sense, the whole dynamical system. As is well known (and not hard to see), the set of generic points is measurable and invariant under the group action. Moreover, the set of generic points has full measure if (X, G, m) is ergodic and Birkhoff's ergodic theorem holds along the underlying Følner sequence. Although we do not need it here, it is instructive for our subsequent considerations to note that a converse of sorts holds as well: if m is an invariant probability measure such that the set of m-generic points with respect to some Følner sequence has full measure, then (X, G, m) is ergodic. Thus, ergodicity is a necessary and sufficient condition for having an ample supply of generic points at ones disposal. This is ultimately the reason that most (but not all) of our theorems in the following deal with ergodic systems.

A dynamical system (X, G, m) is said to have *pure point spectrum* if there exists an orthonormal basis of $L^2(X, m)$ consisting of eigenfunctions. Here, an $f \in L^2(X, m)$ with $f \neq 0$ is called an *eigenfunction* if for any $t \in G$ there exist a $\xi(t) \in \mathbb{C}$ with

$$f(t\cdot) = \xi(t) f$$

in the sense of $L^2(X, m)$ functions. In this case, each $\xi(t)$ belongs to the group

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

and the map

$$\xi : G \longrightarrow \mathbb{T}, \quad t \mapsto \xi(t),$$

can easily be seen to be a continuous group homomorphism. It is called *eigenvalue*. Clearly, eigenfunctions to different eigenvalues are orthogonal. As X is compact and metrizable,

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$L^2(X, m)$ is separable. Hence, the set of eigenvalues is (at most) countable. We denote by P_ξ the projection onto the eigenspace of ξ if ξ is an eigenvalue and set $P_\xi = 0$ if ξ is not an eigenvalue. The set of eigenvalues of (X, G, m) will be denoted by $\operatorname{Eig}(X, G, m)$. As is well known the set of eigenvalues is a group if (X, G, m) is ergodic.

For our further discussion, we also rely on some concepts defined purely with respect to G (that is, they do not need the dynamical system). Let (B_n) be a Følner sequence, and let $B(G)$ denote the set of bounded measurable real-valued functions on G. Then, we define the associated *upper mean* via

$$\overline{M}_{(B_n)} : B(G) \longrightarrow [0, \infty), \quad \overline{M}_{(B_n)}(h) := \limsup_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} h(s)\, ds.$$

If the Følner sequence is clear from the context we drop the subscript (B_n). Clearly, $\overline{M}$ gives rise to seminorm on the space of bounded measurable functions on G via $f \mapsto \overline{M}(|f|)$.

A subset A of G is called *relatively dense* if there exists a compact set $K \subset G$ with

$$G = \bigcup_{a\in A} (a + K).$$

A continuous bounded $f : G \longrightarrow \mathbb{C}$ is called *Bohr almost periodic* if for any $\varepsilon > 0$ the set of $t \in G$ with

$$\|f - f(\cdot - t)\|_\infty < \varepsilon$$

is relatively dense. Here, the supremum norm $\|\cdot\|_\infty$ for bounded complex-valued functions on G is defined via $\|f\|_\infty = \sup_{s\in G} |f(s)|$.

3. *Mean almost periodic points and pure point spectrum*

In this section, we introduce and study mean almost periodic points. The main result of this section then provides a characterization of pure point spectrum via mean almost periodicity of points.

Whenever (X, G) is a dynamical system with metric d and (B_n) is a Følner sequence, we define

$$D = D_d^{(B_n)} : X \times X \longrightarrow [0, \infty), \qquad D(x, y) := \overline{M}(s \mapsto d(sx, sy)).$$

Clearly, D is a pseudometric. Moreover, the Følner condition on (B_n) easily gives that D is invariant, that is, satisfies $D(tx, ty) = D(x, y)$ for all $x, y \in X$ and $t \in G$. Furthermore, for each $x \in X$ the function $G \to [0, \infty)$, $t \mapsto D(x, tx)$, is uniformly continuous. Indeed, continuity at $t = 0$ is easily seen from the definition. Moreover, due to the invariance and the pseudometric properties we infer

$$|D(x, tx) - D(x, sx)| \le D(tx, sx) = D(x, (s - t)x). \tag{$\clubsuit$}$$

When combined with continuity at $t = 0$, this gives uniform continuity. We refer to D as the *averaged metric on X associated to d and* (B_n).

Definition 3.1. (Mean almost periodic points) Let (X, G) be a dynamical system, let d be a metric on X generating the topology, let (B_n) be a Følner sequence and let D be

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the associated averaged metric. Then, a point $x \in X$ is called mean almost periodic with respect to d and (B_n) if for every $\varepsilon > 0$ the set

$$\{t \in G : D(x, tx) < \varepsilon\}$$

is relatively dense.

Remark. Almost periodicity properties with respect to $\overline{M}$ are often connected with the name of *Besicovitch*. We use this for a strengthened version to be introduced in the following. Here we stick to the term 'mean' as this seems to be the common term within the study of equicontinuity properties in recent years (see, e.g., [**11**, **16**, **19**] as well as the discussion in Appendix C).

By definition, mean almost periodicity depends on the chosen Følner sequence. In our subsequent discussion of mean almost periodic points, however, we often refrain from explicitly referring to the Følner sequence (B_n) if it is clear from the context which sequence is involved.

LEMMA 3.2. *Let (X, G) be a dynamical system and let d be a metric on X generating the topology. An $x \in X$ is mean almost periodic if and only if the function $G \ni t \mapsto D(x, tx) \in \mathbb{R}$ is Bohr almost periodic.*

Proof. Define f on G via $f(t) := D(x, tx)$. We have already noted that the function f is uniformly continuous. Clearly, Bohr almost periodicity of f implies that x is mean almost periodic (as $f(0) = 0$). Conversely, (♣) gives $|f(t + s) - f(s)| \le f(t)$ for all $s, t \in G$ and mean almost periodicity of x implies Bohr almost periodicity of f. □

Our next aim is to discuss independence of this definition from the metric and to provide an alternative way of defining mean almost periodicity via density of superlevel sets. The proofs of the corresponding statements are not difficult and rather close to each other. They rely on some simple facts stated in the next proposition. We denote the characteristic function of a set $A \subset G$ by 1_A (that is, $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ for $x \notin A$).

PROPOSITION 3.3. *Let $f : G \longrightarrow [0, 1]$ be given. Then, for any $\delta > 0$ the following estimates hold for the set $A(f, \delta) := \{s \in G : f(s) \ge \delta\}$:*

(a) $\overline{M}(1_{A(f,\delta)}) \le 1/\delta\, \overline{M}(f)$;
(b) $\overline{M}(f) \le M(1_{A(f,\delta)}) + \delta$.

Proof

(a) We clearly have $1_{\{s\in G: f(s)\ge\delta\}} \le 1/\delta f$. This easily gives (a).
(b) We compute

$$\overline{M}(f) \le \overline{M}(f \cdot 1_{\{s\in G: f(s)\ge\delta\}}) + \overline{M}(f \cdot 1_{\{s\in G: f(s)<\delta\}}) \le \overline{M}(1_{\{s\in G: f(s)\ge\delta\}}) + \delta.$$

This finishes the proof. □

LEMMA 3.4. (Independence of the metric) *Let (X, G) be a dynamical system with metrizable X and let (B_n) be a Følner sequence on G. Then, mean almost periodicity of an $x \in X$ does not depend on the chosen metric (provided it generates the topology).*

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Proof. Let e, d be two metrics on X, which generate the topology. For $x \in X$ and $t \in G$ we define the functions $d_{t,x}$ and $e_{t,x}$ on G via $d_{t,x}(s) = d(sx, tsx)$ and $e_{t,x}(s) = e(sx, tsx)$, respectively. We show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $t \in G$ and $x \in X$ we have

$$\overline{M}(d_{t,x}) < \varepsilon$$

whenever $\overline{M}(e_{t,x}) < \delta$ holds. The statement with the roles of e and d reversed can be shown analogously and taken together these two statements prove the lemma.

Without loss of generality we assume $d, e \leq 1$.

Let $\varepsilon > 0$ be given. Choose $\delta' > 0$ with $d(z, y) < \varepsilon/2$ whenever $e(z, y) < \delta'$. Set

$$\delta := \delta' \cdot \frac{\varepsilon}{2}.$$

If $\overline{M}(e_{t,x}) < \delta$, then Proposition 3.3(a) gives

$$\overline{M}(1_{\{s:e_{t,x}(s)\geq\delta'\}}) \leq \frac{1}{\delta'}\overline{M}(e_{t,x}) < \frac{\delta}{\delta'} = \frac{\varepsilon}{2}.$$

Furthermore, we note that, by the definition of δ', we have

$$\overline{M}(d_{t,x} 1_{\{s:e_{t,x}(s)<\delta'\}}) \leq \frac{\varepsilon}{2}.$$

Given this we can now estimate

$$\overline{M}(d_{t,x}) \leq \overline{M}(d_{t,x} 1_{\{s:e_{t,x}(s)\geq\delta'\}})) + \overline{M}(d_{t,x}(s) 1_{\{s:e_{t,x}<\delta'\}}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This is the desired statement. □

By the previous lemma mean almost periodicity of a point is independent of the underlying metric. Hence, we can (and will) from now on refrain from specifying a metric when talking about mean almost periodicity.

We define the *upper density* of a subset $A \subset G$ via

$$\text{Dens}(A) := \overline{M}(1_A).$$

LEMMA 3.5. (Mean almost periodicity via density of superlevel sets) *Let (X, G) be a dynamical system with metrizable X and let (B_n) be a Følner sequence on G. Then, the following assertions for $x \in X$ are equivalent:*

(i) *the point x is mean almost periodic;*

(ii) *for any $\delta > 0$ the set of $t \in G$ with*

$$\text{Dens}(\{s \in G : d(sx, tsx) \geq \delta\}) < \varepsilon$$

is relatively dense for any $\varepsilon > 0$.

Proof. We use the notation of the proof of the preceding lemma.

(i)$\Longrightarrow$(ii): Let $\delta > 0$ and $\varepsilon > 0$ be arbitrary. By assertion (i), the set of $t \in G$ with $\overline{M}(d_{t,x}) < \delta\varepsilon$ is relatively dense. Now, for any such $t \in G$ we obtain from Proposition 3.3(a)

$$\overline{M}(1_{\{s:d_{t,x}(s)\geq\delta\}}) < \varepsilon.$$

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(ii)$\Longrightarrow$(i): Let $\varepsilon > 0$ be given. Assume without loss of generality that $d \leq 1$. Set $\delta = \varepsilon/2$. By assertion (ii), the set of $t \in G$ with $\mathrm{Dens}(\{s \in G : d(sx, tsx) \geq \delta\}) < \varepsilon/2$ is relatively dense. Choose such a $t \in G$. Then, Proposition 3.3(b) gives

$$\overline{M}(d_{t,x}) \leq \overline{M}(1_{\{s:d_{t,x}(s)\geq\delta\}}) + \delta = \mathrm{Dens}(\{s \in G : d(sx, tsx) \geq \delta\}) + \delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof. □

We finish this section with the discussion of a further characterization of mean almost periodicity via suitable functions. To state this characterization (and similar characterizations in subsequent sections) it is useful to define for $x \in X$ and $f \in C(X)$ the function

$$f_x : G \longrightarrow \mathbb{C}, \quad f_x(t) = f(tx).$$

Moreover, we set

$$\mathcal{A}_x := \{f_x : f \in C(X)\}.$$

Clearly, $\mathcal{A}_x$ is an algebra.

PROPOSITION 3.6. (Completeness of $\mathcal{A}_x$) *The algebra $\mathcal{A}_x$ is complete with respect to $\|\cdot\|_\infty$.*

Proof. Consider a sequence $(f^{(n)})$ in $C(X)$ such that $(f_x^{(n)})$ is a Cauchy sequence with respect to $\|\cdot\|_\infty$. Then, a direct $\varepsilon/3$ argument shows that the restrictions of $f^{(n)}$ to the orbit closure of x converge uniformly to a continuous function on the orbit closure. Now, the desired statement follows from Tietze's extension theorem. □

A bounded measurable function $f : G \longrightarrow \mathbb{C}$ is *mean almost periodic* with respect to (B_n) if, for every $\varepsilon > 0$, the set

$$\{t \in G : \overline{M}(|f(\cdot) - f(\cdot - t)|) < \varepsilon\}$$

is relatively dense. The set of uniformly continuous mean almost periodic functions is an algebra and closed under complex conjugation and uniform convergence (see Appendix B).

As a consequence of the previous considerations, we can now characterize mean almost periodicity via functions.

LEMMA 3.7. (Mean almost periodicity via functions) *Let (X, G) be a dynamical system and let (B_n) be a Følner sequence on G. For $x \in X$ the following assertions are equivalent.*

(i) *The point x is mean almost periodic.*

(ii) *Every element from $\mathcal{A}_x$ is mean almost periodic.*

(iii) *The set $\{f \in C(X) : f_x$ is mean almost periodic$\}$ separates the points of $\overline{Gx}$.*

(iv) *For any $s \in G$ the function $d_x^{(s)}$ is mean almost periodic, where $d^{(s)} \in C(X)$ is defined via $d^{(s)}(y) := d(sx, y)$.*

Remark. As the proof shows, condition (iii) could equivalently be formulated with $\overline{Gx}$ replaced by the whole space X.

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Proof. (iv)$\Longrightarrow$(iii): This follows as the $d^{(s)}$, $s \in G$, clearly separate the points of $\overline{Gx}$.

(iii)$\Longrightarrow$(ii): Invoking the corresponding properties of mean almost periodic functions, we can easily see that $\{f \in C(X) : f_x \text{ is mean almost periodic}\}$ is an algebra, which is closed under complex conjugation and uniform convergence. This algebra clearly contains the constant functions. Moreover, by assumption (iii), it separates the points of $\overline{Gx}$. Furthermore this algebra contains every function $f \in C(X)$, which vanishes on $\overline{Gx}$ (as $f_x = 0$ for any such functions). Thus, this algebra even separates the points of X. Now, assumption (ii) follows from Stone–Weierstraß' theorem.

(ii)$\Longrightarrow$(i): Choose a countable set $\mathcal{C} \subset C(X)$ such that any $f \in \mathcal{C}$ satisfies $\|f\|_\infty \leq 1$ and such that the elements of $\mathcal{C}$ separate the points of X. Let $c_f > 0$, $f \in \mathcal{C}$, with $\sum_{f\in\mathcal{C}} c_f < \infty$ be given. Then,

$$e(z, y) := \sum_{f\in\mathcal{C}} c_f \, |f(x) - f(y)|$$

is a metric on X, which generates the topology. Moreover, by assumption (ii), the function f_x is mean almost periodic for any $f \in C(X)$ and, hence, any $f \in \mathcal{C}$. This easily gives that the set of $t \in G$ with $\overline{M}(s \mapsto e(sx, (t + s)x)) < \varepsilon$ is relatively dense (compare with Proposition B.4). Thus, x is mean almost periodic with respect to the metric e. As mean almost periodicity does not depend on the metric, we conclude the proof of assumption (i).

(i)$\Longrightarrow$(iv): Let $z \in X$ be arbitrary and define $d^{(z)} \in C(X)$ by $d^{(z)}(y) := d(z, y)$. The triangle inequality for d gives

$$\begin{aligned}\overline{M}(|d^{(z_x)}(\cdot - t) - d_x^{(z)}|) &= \overline{M}(s \mapsto |d(z, (s - t)x) - d(z, sx)|)\\ &\leq \overline{M}(s \mapsto d(sx, (s - t)x)) = D(x, -tx)\end{aligned}$$

for any $t \in G$ and $z \in X$. Now, mean almost periodicity of $d_x^z(\cdot)$ follows (for any $z \in X$) from assumption (i). □

We now come to the main result of this section, which provides a characterization of pure point spectrum via mean almost periodic points.

THEOREM 3.8. *Let (X, G, m) be a dynamical system and let (B_n) be a Følner sequence on G. Assume that there exists a generic point for (X, G, m). Then, the following assertions are equivalent.*

(i) *The dynamical system (X, G, m) has pure point spectrum.*

(ii) *Every generic point of X is mean almost periodic.*

(iii) *One generic point of X is mean almost periodic.*

If (X, G, m) is ergodic and the Birkhoff theorem holds along (B_n), these statements are also equivalent to the following statement.

(iv) *Almost every $x \in X$ is mean almost periodic.*

Remark

(a) In the particular case of point processes in $\mathbb{R}^d$, this result has been proven in [21, Theorem 4.4].

(b) A related result for subshifts over a finite alphabet can be found in [40, Lemma 5]. There, pure point spectrum (i) is characterized via a variant of statement (iv) given

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by a mean almost periodicity condition on points defined via a metric (close in spirit to what is discussed in Lemma 3.5).

It is worth noting here that in [40] the author works over a finite alphabet and uses a version of the mean defined with lim inf. Under the settings of [40], assuming ergodicity, one can show that the limit exists almost surely, and hence the choice of lim inf or lim sup does not matter. In particular, for ergodic systems, one can deduce [40, Lemma 5] from our result. We prefer to work with lim sup because it defines a pseudo-metric, whereas lim inf only gives rise to a partial semi-metric.

(c) The equivalence between mean almost periodicity of an individual element and pure point diffraction has been proven for constant length substitutions in [36, Lemma VI.25], for Meyer sets in [6, 21] and in general in [31].

Proof. In the ergodic case almost every point is generic. Hence, (ii)$\Longrightarrow$(iv) and (iv)$\Longrightarrow$(iii) follow. Thus, we now turn to showing equivalence between assertions (i), (ii) and (iii) in the general case. We clearly have (ii)$\Longrightarrow$(iii). To show (i)$\Longrightarrow$(ii) and (iii)$\Longrightarrow$(i) we define

$$\underline{d} : G \longrightarrow [0, \infty), \quad \underline{d}(t) = \int_X d(x, tx)\, dm(x).$$

The main result of [29] says that assertion (i) is equivalent to $\underline{d}$ being Bohr almost periodic. Thus, it remains to show that:

- Bohr almost periodicity of $\underline{d}$ implies (ii);
- (iii) implies Bohr almost periodicity of $\underline{d}$.

Now, for $t \in G$ we can consider $f_t : X \longrightarrow [0, \infty)$, $f_t(x) = d(x, tx)$. Then, f_t is clearly continuous. Thus, whenever $y \in X$ is generic, we find

$$\underline{d}(t) = \int_X d(x, tx)\, dm(x) = \int_X f_t(x)\, dm(x) = \overline{M}(s \mapsto f_t(sy)) = D(y, ty)$$

for every $t \in G$. Moreover, the triangle inequality gives that $\underline{d}$ is Bohr almost periodic if and only if the set

$$\{t \in G : \underline{d}(t) < \varepsilon\}$$

is relatively dense in G for all $\varepsilon > 0$. Putting this together we easily obtain that $\underline{d}$ is Bohr almost periodic if and only if one (every) generic $y \in X$ is mean almost periodic. □

We emphasize that the first part of the preceding theorem does not need an ergodicity assumption and illustrate this using the following example.

Example: pure point spectrum in non ergodic case. Consider $\{0, 1\}$ with discrete topology and $X = \{0, 1\}^{\mathbb{Z}}$ with product topology. Equip X with the shift action of $\mathbb{Z}$ given by $\alpha_n(x) = x(\cdot - n)$ for $n \in \mathbb{Z}$. Let $\underline{1}$ and $\underline{0}$ be the elements of X which are constant equal to 1 and 0, respectively. Then, clearly each of these elements is invariant under the shift action and so are then the sets $\{\underline{0}\}$ and $\{\underline{1}\}$. Thus,

$$m := \tfrac{1}{2}(\delta_{\underline{0}} + \delta_{\underline{1}})$$

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is an invariant probability measure (where δ_p denotes the unit point mass at p). Obviously, m is not ergodic. The space $L^2(X, m)$ is two-dimensional and $\sqrt{2} \cdot 1_{\{\underline{0}\}}, \sqrt{2} \cdot 1_{\{\underline{1}\}}$ is an orthogonal basis consisting of eigenfunctions (to the eigenvalue 1). In particular, $(X, \mathbb{Z}, m)$ has pure point spectrum. Now, consider the point $x \in X$ with $x(-k)$ arbitrary for $k \geq 0$ and $x(k) = 1$ if $k \in \{2^n, \ldots, 2^n + 2^{n-1} - 1\}$ for some $n \in \mathbb{N}$ and $x(k) = 0$ otherwise. Then, it is not hard to see that x is generic for m with respect to the Følner sequence $B_n = \{1, \ldots, 2^n\}$. Thus, the theorem gives that x is mean almost periodic. In this example neither $\underline{0}$ nor $\underline{1}$ are generic. Hence, m does not give mass to generic points and the set of generic points has measure zero. Note that the construction of x could easily be modified to yield a transitive generic point (by including suitable finite words of slowly increasing length between the blocks of ones and zeros in x).

Combining the previous result, Theorem 3.8, with the characterization of mean almost periodicity via functions in Lemma 3.7, we obtain the following.

COROLLARY 3.9. *Let (X, G, m) be an ergodic dynamical system with metrizable X and assume that Birkhoff's ergodic theorem holds along (B_n). Then, the following assertions are equivalent.*

(i) *The dynamical system (X, G, m) has pure point spectrum.*

(ii) *For almost every $x \in X$ the set $\{f \in C(X) : f_x$ is mean almost periodic$\}$ separates the points of X.*

Remark. A variant of this statement (with assertion (ii) replaced by the stronger condition that $\mathcal{A}_x$ consists only of mean almost periodic functions) is shown in [**28**] based on an earlier version of [**31**]. Our proof is different. Note also that for ergodic systems over $G = \mathbb{Z}$, it is known that pure point spectrum is equivalent to $\mathbb{Z} \ni n \mapsto f(nx)$ belonging to the Besicovitch class for almost every $x \in X$ whenever f is a bounded measurable function on X, see Theorem 3.22 in [**7**]. The condition of Besicovitch class is stronger than mean almost periodicity (see also the next section).

If the system (X, G, m) is uniquely ergodic, then every $x \in X$ is generic irrespective of the underlying Følner sequence (Oxtoby's theorem). Thus, from the previous theorem we obtain immediately the following corollary.

COROLLARY 3.10. *Let (X, G, m) be a dynamical system and let (B_n) be a Følner sequence on G. Assume that (X, G, m) is uniquely ergodic. Then, the following assertions are equivalent.*

(i) *The dynamical system (X, G, m) has pure point spectrum.*

(ii) *Every $x \in X$ is mean almost periodic.*

(iii) *One $x \in X$ is mean almost periodic.*

Remark. The concept of mean almost periodicity depends on the chosen Følner sequence. To see this, consider $X := \{0, 1\}^{\mathbb{Z}}$ with product topology and the shift action of $\mathbb{Z}$ and the Bernoulli measure m (product measure of the measures giving equal weights $1/2$ to $\{0\}$ and $\{1\}$). This system is ergodic and m almost every $x \in X$ contains arbitrary long stretches of zeros. For each of those x we can then choose a Følner sequence (B_n) with

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$\overline{M}(s \mapsto d(sx, (t+s)x) = 0$ for all $t \in \mathbb{Z}$ (by each B_n being chosen 'within' a long stretch of zeros with distance to the boundary of these stretches increasing in n). Hence, each of these x is mean almost periodic. On the other hand, as the system does not have pure point spectrum, we obtain from Theorem 3.8 that not every of these x will be almost periodic with respect to the standard Følner sequence $B_n = \{0, \ldots, n\}$ along which Birkhoff's ergodic theorem holds.

4. *Besicovitch almost periodic points and eigenfunctions*

In this section, we discuss a strengthened version of mean almost periodicity, namely Besicovitch almost periodicity. We show that pure point spectrum can also be characterized via this strengthened version. In fact, our results can be understood as saying that in a dynamical system with pure point spectrum both eigenfunctions and eigenvalues can be read of from any of its (generic) Besicovitch almost periodic points.

We consider a σ-compact, locally compact abelian group together with a Følner sequence (B_n). As usual, the set of all continuous group homomorphisms $\xi : G \longrightarrow \mathbb{T}$ is denoted as $\widehat{G}$ and called the *dual group* of G. We say that a bounded function $f : G \longrightarrow \mathbb{C}$ is *Besicovitch almost periodic* if for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in \widehat{G}$ and $c_1, \ldots, c_k \in \mathbb{C}$ with

$$\overline{M}\bigg(\bigg|f - \sum_{j=1}^{k} c_j \xi_j\bigg|\bigg) < \varepsilon.$$

A discussion of basic properties of uniformly continuous Besicovitch almost periodic functions is given in Appendix C. This shows, in particular, that any uniformly continuous Besicovitch almost periodic function is also mean almost periodic and admits an average (see also the following). The discussion also shows that the set of these functions forms an algebra and is closed under uniform convergence.

Definition 4.1. (Besicovitch almost periodic points) Let (X, G) be a metrizable dynamical system and let (B_n) be a Følner sequence. Then, a point $x \in X$ is called Besicovitch almost periodic with respect to (B_n) if $\mathcal{A}_x$ consists only of Besicovitch almost periodic functions.

As in the definition of mean almost periodicity, Besicovitch almost periodicity also depends on the chosen Følner sequence. In our subsequent discussion, however, we often refrain from explicitly referring to the Følner sequence (B_n) if it is clear from the context which sequence is involved.

Remark

(a) To set this definition in perspective, we refer to Lemma 3.7. This lemma shows that a point is mean almost periodic if and only if $\mathcal{A}_x$ consists of mean almost periodic functions only.

(b) Note also that the statements of Lemma 3.7 remain true (with essentially the same proof) after 'mean almost periodic' is replaced with 'Besicovitch almost periodic'.

(c) As Besicovitch almost periodic functions are mean almost periodic, any Besicovitch almost periodic point is mean almost periodic. The converse is not true. To see this consider $X := \{0, 1\}^{\mathbb{Z}}$ with product topology and the shift action of $\mathbb{Z}$. Set $B_n := \{0, \ldots, n\}$ for n even and $B_n = \{-n, \ldots, -1\}$ for n odd. Consider now $y \in X$ with $y(k) = 1$ for $k \geq 0$ and $y(k) = 0$ otherwise. Then, it is not hard to see that $D(y, ny) = 0$ for all $n \in \mathbb{Z}$. Hence, y is mean almost periodic. On the other hand, consider $f : X \longrightarrow \{0, 1\}$ with $f(x) = x(0)$. Clearly f is continuous. Moreover,

$$a_n := \frac{1}{|B_n|} \sum_{k \in B_n} f(ky)$$

does not converge (as $a_{2n} = 1$ and $a_{2n+1} = 0$ for all $n \in \mathbb{N}$). By the discussion in Appendix C (see also Proposition 4.2), this shows that f_y is not Besicovitch almost periodic. Hence, y is not Besicovitch almost periodic.

We now give a characterization of Besicovitch almost periodic points via existence of means. To state it, we first introduce some notation. For a bounded measurable function $h : G \longrightarrow \mathbb{C}$, we define the *mean* or *average* of h with respect to (B_n) by

$$\mathrm{A}(h) := \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} h(t)\, dt$$

whenever the limit exists.

PROPOSITION 4.2. (Averaging along orbits) *Let (X, G) be a dynamical system and let (B_n) be a Følner sequence. Then, the following assertions are equivalent for $x \in X$.*

(i) *The point x is Besicovitch almost periodic.*

(ii) *For any $f \in C(X)$ there exists a countable set $F_f \subset \widehat{G}$ such that the limits*

$$\mathrm{A}(|f_x|^2) = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} |f(tx)|^2\, dt \quad \text{and} \quad \mathrm{A}(f_x\overline{\xi}) = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(tx)\, \overline{\xi(t)}\, dt$$

exist for all $\xi \in F_f$ and

$$\mathrm{A}(|f_x|^2) = \sum_{\xi \in F_f} |\mathrm{A}(f_x\overline{\xi})|^2$$

holds.

Moreover, in case that assertions (i) and (ii) hold, $A(f\overline{\xi})$ exists and equals zero for $f \in C(X)$ and $\xi \in \widehat{G} \setminus F_f$.

Proof. This is a direct consequence of the definition of Besicovitch almost periodicity of a point and Proposition C.4 in Appendix C. □

Definition 4.3. (Frequency) Let (X, G) be a dynamical system and let $x \in X$ be a Besicovitch almost periodic point. Then, every $\xi \in \widehat{G}$ with $\mathrm{A}(f_x\overline{\xi}) \neq 0$ for some $f \in C(X)$ is called a frequency of x. The set of all frequencies of x is denoted by $\mathrm{Freq}(x)$.

Here is the first main result of this section. It shows that any Besicovitch almost periodic point completely determines a dynamical system with pure point spectrum.

THEOREM 4.4. *Let (X, G) be a dynamical system, let (B_n) be a Følner sequence and let $p \in X$ be a Besicovitch almost periodic point. Then, there exists a (unique) ergodic* *probability measure m on X such that p is generic with respect to m. The dynamical system (X, G, m) has pure point spectrum and $\operatorname{Eig}(X, G, m) = \operatorname{Freq}(p)$ holds. To each eigenvalue $\xi \in \operatorname{Eig}(X, G, m)$ there exists a (unique) eigenfunction $e_\xi \in L^2(X, m)$ with*

$$\int_X f\, \overline{e_\xi}\, dm = \mathrm{A}(f_p \overline{\xi})$$

for all $f \in C(X)$.

Proof. Obviously, the map

$$\Phi : C(X) \longrightarrow \mathbb{C}, \quad f \mapsto \mathrm{A}(f_p),$$

is linear and positive (that is, $\mathrm{A}(f_p) \geq 0$ for $f \geq 0$). Hence, there exists a unique measure m on X with $\Phi(f) = \int_X f\, dm$. Clearly, $m(X) = \int_X 1\, dm = \mathrm{A}(1) = 1$. By the Følner property of (B_n), the mean A is invariant and, thus, so is Φ. This easily gives that m is invariant. Thus, m is an invariant probability measure.

We now turn to the construction of the eigenfunctions. For $\xi \in \widehat{G}$ consider the map

$$\Phi_\xi : C(X) \longrightarrow \mathbb{C}, \quad f \mapsto \mathrm{A}(f_p \overline{\xi}).$$

This map is obviously linear and defined on a dense subspace of $L^2(X, m)$. By the Cauchy–Schwarz inequality, $\mathrm{A}(1) = 1$ and the construction of m we find

$$|\Phi_\xi(f)|^2 = |\mathrm{A}(f_p \overline{\xi})|^2 \leq \mathrm{A}(|f_p|^2)\mathrm{A}(1) = \int_X |f|^2\, dm.$$

Hence, Φ_ξ can be extended to a linear continuous map, again denoted by Φ_ξ, on the whole $L^2(X, m)$. By Riesz's lemma, there exists an $e_\xi \in L^2(X, m)$ with $\|e_\xi\| \leq 1$ and

$$\Phi_\xi(f) = \int_X f\, \overline{e_\xi}\, dm$$

for all $f \in C(X)$. Define

$$E := \{\xi \in \widehat{G} : e_\xi \neq 0\}.$$

By construction, $\xi \in \widehat{G}$ belongs to E if and only if there exists an $f \in C(X)$ with $\mathrm{A}(f_p \overline{\xi}) \neq 0$. Hence, $E = \operatorname{Freq}(p)$ holds. In particular, we have $\mathrm{A}(f_p \overline{\varrho}) = 0$ for all $f \in C(X)$ and $\varrho \in \widehat{G} \setminus E$. A short computation shows for $t \in G$,

$$\begin{aligned} \xi(-t) \int_X f\, \overline{e_\xi}\, dm &= \xi(-t)\mathrm{A}(f_p \overline{\xi}) \\ &= \mathrm{A}(f_p \overline{\xi(t + \cdot)}) \\ (\mathrm{A} \text{ invariant}) &= \mathrm{A}(f_p(-t \cdot)\overline{\xi}) \\ (\text{construction of } m) &= \int_X f(-t \cdot)\, \overline{e_\xi}\, dm \\ (m \text{ invariant}) &= \int_X f\, \overline{e_\xi(t \cdot)}\, dm \end{aligned}$$

for all $f \in C(X)$. As these f are dense in $L^2(X, m)$ this gives

$$e_\xi(t \cdot) = \xi(t) e_\xi$$

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for all $t \in G$. This shows that e_ξ is an eigenfunction (to ξ) for each $\xi \in E$. Clearly, eigenfunctions to different eigenvalues are orthogonal.

Next, we show that the e_ξ, $\xi \in E$, are normalized and form a basis. This gives that $E = \mathrm{Eig}(X, G, m)$ and together with the already shown $E = \mathrm{Freq}(p)$, this will then also imply $\mathrm{Eig}(X, G, m) = \mathrm{Freq}(p)$.

By Parseval's inequality, the definition of m and Proposition 4.2, we have the following:

$$\begin{aligned}\sum_{\xi \in E} \left| \int_X f\, \overline{e_\xi}\, dm \right|^2 &\le \int_X |f|^2\, dm \\ &= \mathrm{A}(|f_p|^2) \\ \text{(Proposition 4.2)} &= \sum_{\xi \in \widehat{G}} |\mathrm{A}(f_p \overline{\xi})|^2 \\ \text{(construction of } E) &= \sum_{\xi \in E} |\mathrm{A}(f_p \overline{\xi})|^2 \\ \text{(construction of } e_\xi) &= \sum_{\xi \in E} \left| \int_X f\, \overline{e_\xi}\, dm \right|^2.\end{aligned}$$

This shows

$$\sum_{\xi \in E} \left| \int_X f\, \overline{e_\xi}\, dm \right|^2 = \int_X |f|^2\, dm$$

for all $f \in C(X)$. This is only possible if $\|e_\xi\| = 1$ for all $\xi \in E$ and (e_ξ) form an orthonormal basis of $L^2(X, m)$.

It remains to show ergodicity: for each eigenvalue $\xi \in E$ we have constructed an eigenfunction e_ξ and we have shown that these form a complete set (that is, e_ξ, $\xi \in E$, is an orthonormal basis). Hence (as each of these eigenfunctions belong to different eigenspaces), each eigenspace is one dimensional. In particular. the eigenspace to the eigenvalue 1 is one dimensional and the system is ergodic. □

Remark. Note that the proof relies (and only relies) on the characterizing properties of Besicovitch almost periodic points given in Proposition 4.2.

The previous theorem shows that any Besicovitch almost periodic point is generic with respect to a (uniquely determined) measure. It may well be that different Besicovitch almost periodic points are generic with respect to different measures. Consider, for example, the full shift $X = \{0, 1\}^{\mathbb{Z}}$ with $Tx(n) = x(n + 1)$. Then, any periodic element of X is Besicovitch almost periodic. Clearly, elements with different periods will not be generic with respect to the same measure. This motivates the following definition.

Definition 4.5. (Generic Besicovitch almost periodic points) Let (X, G) be a dynamical system and let (B_n) be a Følner sequence on G. Then, for any invariant probability measure m on (X, G), we denote by $\mathrm{Bap}(X, G, m)$ the set of those Besicovitch almost periodic points which are generic with respect to m.

We have the following consequence of the preceding theorem.

PROPOSITION 4.6. *Let (X, G, m) be a dynamical system and let (B_n) be a Følner sequence. Then,* $\mathrm{Bap}(X, G, m)$ *is a measurable invariant set.*

Proof. Clearly, $\mathrm{Bap}(X, G, m)$ is invariant as both the set of generic points and the set of Besicovitch almost periodic points are invariant. If $\mathrm{Bap}(X, G, m)$ is empty, there is nothing left to show. Thus, consider the case $\mathrm{Bap}(X, G, m) \neq \varnothing$. By Theorem 4.4, the dynamical system (X, G, m) then has pure point spectrum and the set of its eigenvalues $\mathrm{Eig}(X, G, m)$ equals $\mathrm{Freq}(p)$ for any $p \in \mathrm{Bap}(X, G, m)$. Set $E := \mathrm{Eig}(X, G, m)$.

CLAIM. *Let D be a dense subset of $C(X)$. Then, we have $p \in \mathrm{Bap}(X, G, m)$ if and only if the following three points hold:*

- *$A(f_p)$ exists and equals $\int_X f\, dm$ for all $f \in D$;*
- *$A(f_p\overline{\xi})$ exists for all $\xi \in E$ and $f \in D$;*
- *$A(|f_p|^2) = \sum_{\xi\in E} |A(f_p\overline{\xi})|^2$ for all $f \in D$.*

Proof of claim. Consider $p \in \mathrm{Bap}(X, G, m)$. Then, p is generic and the first point holds (even for all $f \in C(X)$). In particular, $A(|f_p|^2)$ exists. Now, the second and third point follow from Proposition 4.2 as p is Besicovitch almost periodic with set of frequencies given by E.

Consider now a $p \in X$ satisfying the three points in the claim. By density of D in $C(X)$, we then easily infer that $A(f_p) = \int_X f\, dm$ holds for all $f \in C(X)$ and $A(f_p\overline{\xi})$ exists for all $f \in C(X)$ and $\xi \in E$. In particular, we have $A(|f_p|^2) = \int |f|^2\, dm$ for all $f \in C(X)$. Given this, we can now follow the proof of Theorem 4.4 to conclude the existence of (pairwise orthogonal) eigenfunctions e_ξ to $\xi \in \mathrm{Eig}(X, G, m)$ with $\|e_\xi\| \leq 1$ and

$$\sum_{\xi\in E} \left| \int_X f\, \overline{e_\xi}\, dm \right|^2 = \int_X |f|^2\, dm$$

for all $f \in D$. As D is dense, this is only possible if $\|e_\xi\| = 1$ holds for all $\xi \in E$ and the $e_\xi, \xi \in E$, are an orthonormal basis. This finishes the proof of the claim. □

Given the claim, the desired measurability follows easily: by compactness and metrizability of X we can choose a countable dense subset D of $C(X)$. Then, the claim gives that $p \in X$ belongs to $\mathrm{Bap}(X, G, m)$ if countably many conditions are satisfied. Clearly, each of these conditions gives a measurable set. □

The following theorem can be seen as both a converse to Theorem 4.4 and an analog to Theorem 3.8.

THEOREM 4.7. (Discrete spectrum via Besicovitch almost periodic points) *Let (X, G, m) be an ergodic dynamical system and let (B_n) be a Følner sequence along which Birkhoff's ergodic theorem holds. Then, the following assertions are equivalent.*

(i) *The dynamical system (X, G, m) has pure point spectrum.*
(ii) $m(\mathrm{Bap}(X, G, m)) = 1$.
(iii) $\mathrm{Bap}(X, G, m) \neq \varnothing$.

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If one of the equivalent conditions (i), (ii) and (iii) holds, then $\mathrm{Eig}(X, G, m) = \mathrm{Freq}(x)$ *for every* $x \in \mathrm{Bap}(X, G, m)$*. Moreover, in this case, for any* $f \in C(X)$ *and* $\xi \in \widehat{G}$ *the function*

$$e_{f,\xi} : X \longrightarrow \mathbb{C}, \quad e_{f,\xi}(x) := \begin{cases} \mathrm{A}(f_x\overline{\xi}), & x \in \mathrm{Bap}(X, G, m), \\ 0, & \textit{otherwise}, \end{cases}$$

satisfies $P_\xi f = e_{f,\xi}$ *(in* $L^2(X, m)$*),* $e_{f,\xi}(tx) = \xi(t)e_{f,\xi}(x)$ *for all* $t \in G$ *and* $x \in X$ *and has constant modulus on* $\mathrm{Bap}(X, G, m)$*.*

Remark. By Theorem 4.4, the existence of a generic Besicovitch almost periodic point entails the ergodicity of m. For this reason, the ergodicity assumption in the above theorem cannot be dropped.

Proof. The implication (ii)$\Longrightarrow$(iii) is clear. The implication (iii)$\Longrightarrow$(i) follows from Theorem 4.4. We now show (i)$\Longrightarrow$(ii). As the set of generic points has full measure, it suffices to show that almost every $x \in X$ is Besicovitch almost periodic. To do so, we denote the inner product on $L^2(X, m)$ by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|_2$. Let $\xi_1, \xi_2, \xi_3, \ldots,$ be an enumeration of $\mathrm{Eig}(X, G, m)$. Choose for any $\xi \in E$ a normalized eigenfunction $e_\xi : X \longrightarrow \mathbb{C}$. Without loss of generality, we can assume

$$e_\xi(sx) = \xi(s)\, e_\xi(x)$$

for all $s \in G$ and $x \in X$. (Otherwise, we could replace e_ξ by $\widetilde{e}_\xi$ defined by

$$\widetilde{e}_\xi(x) := \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} e(sx)\, \overline{\xi}(s)\, ds$$

if the limit exists and $\widetilde{e}_\xi(x) = 0$ otherwise.) Moreover, for $\xi = 1$ we choose the constant function 1.

Consider now an arbitrary $g \in C(X)$. By Birkhoff's ergodic theorem, we can then find a subset X_g of X of full measure such that

$$\int_X \bigg| g - \sum_{j=1}^{k} \langle g, e_{\xi_j} \rangle e_{\xi_j} \bigg|\, dm(x) = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} \bigg| g(sx) - \sum_{j=1}^{k} \langle g, e_{\xi_j} \rangle e_{\xi_j}(sx) \bigg|\, ds$$

for all $x \in X_g$ and $k \in \mathbb{N}$. Let $D \subset C(X)$ be a countable dense subset. Define

$$X' := \bigcap_{g\in D} X_g.$$

Then, X' has full measure and a short computation gives

$$\int_X \bigg| f - \sum_{j=1}^{k} \langle f, e_{\xi_j} \rangle e_{\xi_j} \bigg|\, dm(x) = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} \bigg| f(sx) - \sum_{j=1}^{k} \langle f, e_{\xi_j} \rangle e_{\xi_j}(sx) \bigg|\, ds$$

for all $f \in C(X)$, $x \in X'$ and $k \in \mathbb{N}$. This, in turn, implies that any $x \in X'$ is Besicovitch almost periodic: indeed, a short calculation invoking Birkhoff's ergodic theorem and the

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Cauchy–Schwarz inequality shows

$$\begin{aligned}\overline{M}\bigg(\bigg|f(\cdot x)-\sum_{j=1}^{k}\langle f,e_{\xi_j}\rangle\, e_{\xi_j}(x)\,\xi_j(\cdot)\bigg|\bigg) &= \overline{M}\bigg(\bigg|f(\cdot x)-\sum_{j=1}^{k}\langle f,e_{\xi_j}\rangle\, e_{\xi_j}(\cdot x)\bigg|\bigg)\\ \text{(Birkhoff's ergodic theorem)} &= \lim_{n\to\infty}\frac{1}{|B_n|}\int_{B_n}\bigg|f(sx)-\sum_{j=1}^{k}\langle f,e_{\xi_j}\rangle e_{\xi_j}(sx)\bigg|\,ds\\ &= \int_X\bigg|f-\sum_{j=1}^{k}\langle f,e_{\xi_j}\rangle e_{\xi_j}\bigg|\,dm(x)\\ \text{(Cauchy–Schwarz inequality)} &\le \bigg\|f-\sum_{j=1}^{k}\langle f,e_{\xi_j}\rangle e_{\xi_j}\bigg\|_2\\ &\to 0,\ n\to\infty.\end{aligned}$$

This gives the desired claim.

We now turn to the remaining statements. The equality $\mathrm{Freq}(x)=\mathrm{Eig}(X,G,m)$ for an element $x\in\mathrm{Bap}(X,G,m)$ directly follows from Theorem 4.4. As for $e_{f,\xi}$, we note that it is well defined and invariant (as $\mathrm{Bap}(X,G,m)$ is invariant). The equality $\mathrm{Freq}(x)=\mathrm{Eig}(X,G,m)$ for $x\in\mathrm{Bap}(X,G,m)$ rather directly gives that $e_{f,\xi}$ vanishes identically for $\xi\in\widehat{G}\setminus\mathrm{Eig}(X,G,m)$. In particular, it has constant modulus on $\mathrm{Bap}(X,G,m)$. Now, by Theorem 4.4, for each $\xi\in E$ and $x\in\mathrm{Bap}(X,G,m)$ there exists a normalized eigenfunction $e_\xi^{(x)}$ with

$$e_{f,\xi}(x)=\mathrm{A}(f_x\overline{\xi})=\langle f,e_\xi^{(x)}\rangle.$$

As each eigenspace is one dimensional, the $e_\xi^{(x)}$ arising for different $x\in\mathrm{Bap}(X,G,m)$ will only differ by a factor of modulus one. This gives the statement on constancy of the modulus.

That $e_{f,\xi}$ is the projection onto the eigenspace of ξ follows from standard theory; see, e.g., [27] for a recent discussion. □

5. *Weyl almost periodic points, unique ergodicity and continuity of eigenfunctions*

In this section, we consider a strengthening of Besicovitch almost periodicity, namely Weyl almost periodicity. We show that Weyl almost periodicity extends from one point to its orbit closure. This allows us to characterize transitive systems all of whose points are Weyl almost periodic. These are exactly the uniquely ergodic dynamical systems with pure point spectrum and continuous eigenfunctions. This ties in with various recent investigations (see the following for details).

Let (X,G) be a dynamical system. Whenever d is a metric on X generating the topology and (B_n) is a Følner sequence, we define for each $n\in\mathbb{N}$ the map

$$\overline{M}_n: B(G)\longrightarrow\mathbb{R},\quad \overline{M}_n(f):=\sup_{s\in G}\frac{1}{|B_n|}\int_{B_n+s}f(t)\,dt.$$

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This gives then rise to the functions

$$D_n := D_{n,d} : X \times X \longrightarrow [0, \infty), \quad D_n(x, y) := \overline{M}_n(s \mapsto d(sx, sy)),$$

for each $n \in \mathbb{N}$. Then, each D_n can easily be seen to be an invariant metric. Moreover, for each $x \in X$ the function $t \mapsto D_n(x, tx)$ is uniformly continuous (by the argument used in §2 to show uniform continuity of D). The function D_n is referred to as the *averaged metric* on level n.

A bounded measurable function $f : G \longrightarrow \mathbb{C}$ is *Weyl almost periodic* if for each $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in \widehat{G}$ and $c_1, \ldots, c_k \in \mathbb{C}$ with

$$\limsup_{n\to\infty} \overline{M}_n\bigg(\bigg|f - \sum_{j=1}^{k} c_j \xi_j\bigg|\bigg) < \varepsilon.$$

As discussed in Appendix D, an equivalent alternative characterization is that for each $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and a relatively dense set $R \subset G$ with

$$\overline{M}_N(|f - f(\cdot - t)|) < \varepsilon$$

for all $t \in R$. A crucial feature of Weyl almost periodic functions is the existence of the limits

$$\lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n+s_n} f(t)\, \xi(t)\, dt$$

irrespective of (and uniform in) the chosen sequence $(s_n) \in G$ for each $\xi \in \widehat{G}$, see Appendix D.

Definition 5.1. (Weyl almost periodic points) Let (X, G) be a dynamical system, let d be a metric on X generating the topology, let (B_n) be a Følner sequence and let D_n, $n \in \mathbb{N}$, be the associated averaged metrics. Then, a point $x \in X$ is called Weyl almost periodic with respect to d and (B_n) if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$\{t \in G : D_N(x, tx) < \varepsilon\}$$

is relatively dense.

Remark. It follows from Proposition D.1 that an $x \in X$ is Weyl almost periodic if and only if for each $\varepsilon > 0$ there exists a relatively dense set $R \subset G$ and an $N_0 \in \mathbb{N}$ such that $D_N(x, tx) < \varepsilon$ for all $N \geq N_0$ and $t \in R$.

Arguing as in §3 with $\overline{M}$ replaced by $\overline{M}_n$ we see that Weyl almost periodicity is independent of the chosen metric and the following holds.

LEMMA 5.2. *Let (X, G) be a dynamical system, let d be a metric on X generating the topology, let (B_n) be a Følner sequence and let D_n, $n \in \mathbb{N}$, be the associated averaged metrics. Then, the following assertions for $x \in X$ are equivalent.*

(i) *The point x is Weyl almost periodic.*
(ii) *The algebra $\mathcal{A}_x$ consists of Weyl almost periodic functions.*
(iii) *The set $\{f \in C(X) : f_x$ is Weyl almost periodic$\}$ separates the points of X.*

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(iv) *For any $s \in G$ the function $d_x^{(s)}$ is Weyl almost periodic, where $d^{(s)} \in C(X)$ is defined via $d^{(s)}(y) := d(sx, y)$.*

The previous lemma implies, in particular, that any Weyl almost periodic point is Besicovitch almost periodic. It is not hard to see by examples that the converse does not hold.

Weyl almost periodicity has a stability property.

LEMMA 5.3. (Stability of Weyl almost periodicity along orbit closures) *Let (X, G) be a dynamical system. Assume that $x \in X$ is Weyl almost periodic. Then, any element in the orbit closure of x is Weyl almost periodic.*

Proof. The function D_N is lower semi-continuous for each $N \in \mathbb{N}$ as it is a supremum over continuous functions. From this and the invariance of D_N we find

$$D_N(y, ty) \leq \liminf_{n\to\infty} D_N(s_n x, ts_n x) = D_N(x, tx)$$

whenever $s_n x \to y$ for a sequence (s_n) in G. This easily gives the desired statement. □

PROPOSITION 5.4. *Let (X, G) be a dynamical system with transitive element $p \in X$. Let p be Weyl almost periodic. Then (X, G) is uniquely ergodic, has pure point spectrum, all eigenfunctions are continuous and* Freq(x) = Eig(X, G, m) *holds for all $x \in X$. Moreover, for any $f \in C(X)$ and $\xi \in \widehat{G}$, the averages*

$$A_n(f_x\overline{\xi}) := \frac{1}{B_n}\int_{B_n} f(tx)\,\overline{\xi(t)}\,dt$$

converge (uniformly in x) towards the projection of f onto the eigenspace of ξ.

Proof. It is well known that unique ergodicity is equivalent to uniform (in $y \in X$) convergence of the averages

$$\frac{1}{|B_n|}\int_{B_n} f(ty)\,dt$$

for each continuous $f : X \longrightarrow \mathbb{C}$. Now, uniform existence of these averages on the orbit of x is a direct consequence of Weyl almost periodicity. This easily gives uniform existence on the orbit closure. As the orbit closure is X the desired statement on unique ergodicity follows. Denote the unique invariant probability measure by m.

By the previous lemma and the transitivity assumption on p, every $x \in X$ is Weyl almost periodic. In particular, every element is Besicovitch almost periodic. As (X, G) is uniquely ergodic every $x \in X$ is also generic with respect to m. Hence, $X = \text{Bap}(X, G, m)$ follows. By Theorem 4.7, this implies pure point spectrum as well as pointwise convergence of the averages $A_n(f_x\overline{\xi})$ to the projection of f onto the eigenspace of ξ for each $f \in C(X)$ and $\xi \in \widehat{G}$. Now, by Weyl almost periodicity these averages converge uniformly in $x \in X$. Hence, their limit is continuous and continuity of the eigenfunctions follows. □

THEOREM 5.5. *Let (X, G) be a dynamical system with transitive point $p \in X$. Then, the following assertions are equivalent.*

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(i) *The dynamical system* (X, G) *is uniquely ergodic with pure point spectrum and continuous eigenfunctions.*
(ii) *The point* p *is Weyl almost periodic.*
(iii) *Every* $x \in X$ *is Weyl almost periodic.*

In this case, we have $\mathrm{Freq}(x) = \mathrm{Eig}(X, G, m)$ *for all* $x \in X$.

Proof. The implication (iii)⟹(ii) is obvious whereas (ii)⟹(iii) follows from Lemma 5.3.

The implication (ii)⟹(i) and the last part of the theorem were shown in Proposition 5.4. It remains to show the reverse implication (i)⟹(ii): this follows by a variant of the proof of the corresponding part in Theorem 4.7. We denote the unique invariant measure by m and use the notation introduced in the proof of Theorem 4.7. Thus, we denote the inner product on $L^2(X, m)$ by $\langle\cdot, \cdot\rangle$ and the associated norm by $\|\cdot\|_2$. As the spectrum is pure point, there exists an orthonormal basis e_ξ, $\xi \in \mathrm{Eig}(X, G, m)$, of $L^2(X, m)$ with e_ξ being an eigenfunction to the eigenvalue ξ for each $\xi \in \mathrm{Eig}(X, G, m)$. By assumption, each e_ξ, $\xi \in E$, can be chosen continuous. By unique ergodicity, we then find for any finite subset $A \subset \mathrm{Eig}(X, G, m)$ and any $y \in X$:

$$\begin{aligned}
\limsup_{n\to\infty} \overline{M}_n\bigg(\bigg|f(\cdot y) - \sum_{\xi\in A}\langle e_\xi, f\rangle e_\xi(y)\xi(\cdot)\bigg|\bigg) &= \limsup_{n\to\infty} \overline{M}_n\bigg|f(\cdot y) - \sum_{\xi\in A}\langle e_\xi, f\rangle e_\xi(\cdot y)\bigg| \\
&= \lim_{n\to\infty} \frac{1}{|B_n|}\int_{B_n}\bigg|f(ty) \\
&\quad - \sum_{\xi\in A}\langle e_\xi, f\rangle e_\xi(ty)\bigg|\,dt \\
&= \int_X \bigg|f(x) - \sum_{\xi\in A}\langle e_\xi, f\rangle e_\xi(x)\bigg|\,dm(x) \\
\text{(Cauchy–Schwarz inequality)} &\le \bigg\|f - \sum_{\xi\in A}\langle e_\xi, f\rangle e_\xi\bigg\|_2.
\end{aligned}$$

As f_ξ, $\xi \in E$, is a basis of $L^2(X, m)$, the last term becomes arbitrarily small for suitable $A \subset E$. This shows that $t \mapsto f(ty)$ is Weyl almost periodic for any $y \in X$. □

Remark. Recently, systems satisfying the equivalent conditions of the theorem have attracted substantial interest.

(a) For $G = \mathbb{Z}$ various equivalent characterizations of assertion (i) have been investigated in [**11**]. In particular, it is shown there that assertion (i) is equivalent to the topological isomorphy of the system to its maximal equicontinuous factor. The case of general amenable groups G has been studied in [**16**]. In particular, it has been shown there that assertion (i) is equivalent to the continuity of the averaged metric D on $X \times X$. This continuity is known as *mean equicontinuity* of the system. In this context our preceding result is remarkable as it does not assume control of D on the whole of $X \times X$ but just on $\{(x, tx) : t \in G\}$. In this sense, we have found a

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pointwise characterization of mean equicontinuity. Note, however, that we require a rather uniform control on the orbit of this one point.

Note that [11, 16] also study Besicovitch and Weyl-type averages and metrics defined by them. It then turns out that for the continuity of some maps with respect to these metrics (mean equicontinuity) it is not relevant whether the Besicovitch or the Weyl-type average is considered. This is very different from our previous considerations. The reason is that the continuity condition in [11, 16] involves all points of the dynamical system and this then implies unique ergodicity, which ensures uniform behavior of all points.

(b) A large class of examples satisfying the conditions of the theorem are weakly almost periodic systems. A recent study of such systems is carried out in [33] to which we refer for a precise definition and further references, see also §6.

(c) A most important class of examples for the theorem are dynamical systems arising from regular cut and project schemes. Such systems are at the core of the study of aperiodic order (see [2]). They belong to the special class of dynamical systems known as translation-bounded measure dynamical systems (TMDSs), see §7 for details. In fact, a huge bulk of material in the theory of aperiodic order deals with TMDSs satisfying assertion (i) of the theorem. A characterization of such systems via an almost periodicity property of its points had been missing for a long time. It was only given recently in [31]. The preceding theorem is a generalization of the corresponding result of [31] in that it is not restricted to TMDSs but rather applies to general dynamical systems.

If a system is minimal every point is transitive and we can note the following immediate consequence of the preceding theorem.

COROLLARY 5.6. *Let (X, G) be a minimal dynamical system. Then, the following assertions are equivalent.*

(i) *Every point in X is Weyl almost periodic.*
(ii) *There exists a Weyl almost periodic point in X.*
(iii) *The dynamical system (X, G) is uniquely ergodic with pure point spectrum and continuous eigenfunctions.*

Remark. A system can well be Weyl almost periodic without being minimal. Consider, for example, the orbit closure in the subshift $(\{0, 1\}^{\mathbb{Z}}, \mathbb{Z})$ of the element ω defined with $\omega_0 = 1$ and $\omega_n = 0$ for $n \neq 0$.

6. *Weakly and Bohr almost periodic dynamical systems*

In the preceding section, we have met Weyl almost periodic points. In this section, we introduce two special classes of Weyl almost periodic points, namely Bohr almost periodic points and weakly almost periodic points. This allows us to reanalyze and characterize weakly almost periodic dynamical system and Bohr almost periodic dynamical system via the new approach in this paper. Specifically, the main result of this section shows that a (transitive) dynamical system is weakly almost periodic (Bohr almost periodic) if and only if every of its points is weakly almost periodic (Bohr almost periodic). We refer to [33]

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for a recent discussion of Bohr and weakly almost periodic systems including relevance, background and further references.

Let G be a σ-compact locally compact abelian group and denote the set of uniformly continuous and bounded functions on G by $C_u(G)$. This space is a Banach space when equipped with the supremum norm $\|\cdot\|_\infty$. An $f \in C_u(G)$ is called *weakly almost periodic* if the set $\{f(\cdot - t) : t \in G\}$ is relatively compact in $C_u(G)$ with respect to the weak topology of the Banach space $(C_u(G), \|\cdot\|_\infty)$. Clearly, any Bohr almost periodic $f \in C_u(G)$ is weakly almost periodic (as Bohr almost periodicity means, by definition, that the set $\{f(\cdot - t) : t \in G\}$ is relatively compact in the original topology). In fact, a main result on weakly almost periodic functions (see, e.g., [1] or [13, 33]) gives that any weakly almost periodic f can be (uniquely) decomposed into $f = g + h$ with $g \in C_u(G)$ Bohr almost periodic and $h \in C_u(G)$ satisfying

$$\lim_{n\to\infty} \sup_{s\in G} \frac{1}{|B_n|} \int_{B_n} |h(s+t)|\, dt = 0$$

for any Følner sequence (B_n). The existence of this decomposition is [35, Theorem 4.7.11], whereas the uniqueness follows immediately from [35, Lemma 4.6.8].

When combined with (♡) in Appendix A this easily gives that any weakly almost periodic function is Weyl almost periodic.

Definition 6.1. (Weakly and Bohr almost periodic points) Let (X, G) be a dynamical system.

(a) A point $x \in X$ is called weakly almost periodic if $\mathcal{A}_x$ consists only of weakly almost periodic functions.

(b) A point $x \in X$ is called Bohr almost periodic if $\mathcal{A}_x$ consists only of Bohr almost periodic functions.

Remark. By the discussion preceding the definition, any Bohr almost periodic point is weakly almost periodic and any weakly almost periodic point is Weyl almost periodic.

In the subsequent discussion, the weakly almost periodic case and the Bohr almost periodic case can be mostly treated in parallel. To facilitate the reading, we then give statements for the weakly almost periodic case and mention the Bohr almost periodic case in brackets only.

With essentially the same proof as Lemma 3.7, we obtain the following statement.

LEMMA 6.2. *Let (X, G) be a dynamical system and let $x \in X$ be given. Then, the following assertions are equivalent.*

(i) *The point x is weakly (Bohr) almost periodic.*

(ii) *The weakly (Bohr) almost periodic functions in $\mathcal{A}_x$ are dense in $(\mathcal{A}_x, \|\cdot\|_\infty)$.*

(iii) *The set $\{f \in C(X) : f_x$ is weakly (Bohr) almost periodic $\}$ separates the points of X.*

(iv) *For any $s \in G$ the function $d_x^{(s)}$ is weakly (Bohr) almost periodic, where $d^{(s)} \in C(X)$ is defined via $d^{(s)}(y) := d(sx, y)$.*

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Whenever (X, G) is a dynamical system, any $f \in C(X)$ gives rise to a function on the product $G \times X$, namely

$$P_f : G \times X \longrightarrow \mathbb{C}, \quad (t, x) \mapsto f(tx).$$

Roughly speaking one can say that the preceding discussion was concerned with almost periodicity properties of the functions $f_x = P_f(\cdot, x)$ for $x \in X$. It is natural to consider almost periodicity properties of restrictions of the P_f to X as well for fixed $t \in G$. This leads to the notion of weakly (Bohr) almost periodic dynamical system. For $f \in C(X)$ and $t \in G$, we define

$$f_t : X \longrightarrow \mathbb{C}, \quad f_t(x) = f(tx) = P_f(t, \cdot).$$

Definition 6.3. (Weakly and Bohr almost periodic dynamical systems) The dynamical system (X, G) is called weakly almost periodic and Bohr almost periodic (in some papers this is called an almost periodic dynamical system or strongly almost periodic dynamical system) if for any $f \in C(X)$ the family $\{f_t : t \in G\}$ has compact closure in the weak topology and the Banach space topology of $(C(X), \|\cdot\|_\infty)$, respectively.

The next lemma relates weakly (Bohr) almost periodic dynamical systems to weakly (Bohr) almost periodic points.

LEMMA 6.4. *Let (X, G) be a dynamical system.*

(a) *If (X, G) is weakly (Bohr) almost periodic, then every $x \in X$ is a weakly (Bohr) almost periodic.*
(b) *If $x \in X$ is transitive and weakly (Bohr) almost periodic, then (X, G) is a weakly (Bohr) almost periodic dynamical system.*

Proof. Fix an arbitrary $x \in X$. Define $F : C(X) \to C_u(G)$ via

$$F(f)(t) = f_x.$$

(a) It is easy to see that F is well defined and $\|F\| \leq 1$. It follows that F is continuous and, hence, also weakly continuous [35, Lemma 4.4.2]. Therefore, the image of a compact (respectively, weak compact) set is compact (respectively, weak compact). As F commutes with the group action, the claim follows.
(b) We know that F is continuous. Moreover, because x has dense orbit, it follows immediately that for all $f \in C(X)$ we have

$$\|F(f)\|_\infty = \|f\|_\infty.$$

Therefore, F is an isometry and, hence, it induces an isomorphic isometry $F : C(X) \to \mathrm{Im}(F)$. In particular, $\mathrm{Im}(F)$ is closed in $C_u(G)$ and the mapping

$$F^{-1} : \mathrm{Im}(F) \to C(X),$$

is a continuous operator and, hence, is also weakly continuous [35, Lemma 4.4.2]. Therefore, F^{-1} maps compact and weakly compact sets into compact and weakly compact sets, respectively. This easily gives the desired statement. □

We obtain the following immediate consequence of the preceding lemma.

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THEOREM 6.5. *Let (X, G) be a dynamical system with transitive $p \in X$. Then, the following assertions are equivalent.*

(i) *(X, G) is weakly almost periodic.*
(ii) *Every $x \in X$ is weakly almost periodic.*
(iii) *The point $p \in X$ is weakly almost periodic.*

We also note the following consequence of the theorem.

COROLLARY 6.6. *Let (X, G) be a dynamical system with transitive $p \in X$. For $s, t \in G$ define $g_{s,t} : X \to \mathbb{C}$ via*

$$g_{s,t}(x) = d(sp, tx).$$

Then, $p \in X$ is weakly almost periodic if and only if for each $s \in G$ the set $\{g_{s,t} : t \in G\}$ has compact closure in the weak topology of $(C(X), \|\cdot\|_\infty)$.

Proof. Indeed, the only if part of the statement is immediate from the definition of weak almost periodicity and the preceding theorem. To show the if part, we denote (in line with [33])

$$\mathrm{WAP}(X) := \{f \in C(X) : \{f_t : t \in G\} \text{ has compact closure in the weak topology}\}.$$

Then, by [33, Propositions 3.3, 3.4] WAP(X) is a closed algebra of $(C(X), \|\cdot\|_\infty)$, which contains the constant function 1. Moreover, we have $g_{t,1} \in \mathrm{WAP}(X)$ for all $t \in G$. We next show that the functions $g_{t,1}$ separate the points of X. Let $x \neq y \in X$ be arbitrary, and let $r = d(x, y)$.

As p is a transitive point, there exists some $t \in G$ such that $d(tp, x) < r/3$. Then,

$$g_{t,1}(x) = d(tp, x) < \frac{r}{3} \quad \text{and} \quad g_{t,1}(y) = d(tp, y) \geq d(x, y) - d(tp, x) > \frac{2r}{3}.$$

This shows that $g_{t,1}(x) \neq g_{t,1}(y)$. Therefore, WAP$(X)$ is a closed algebra of $(C(X), \|\cdot\|_\infty)$ which is separating the points and, hence, by the Stone–Weierstraß theorem, WAP$(X) = C(X)$. This gives the desired statement. □

The preceding theorem allows us to recapture a main result of [33] (compare with [19, Corollary 2.18]).

COROLLARY 6.7. *Let (X, G) be a transitive weakly almost periodic dynamical system. Then, (X, G) is uniquely ergodic with pure point spectrum and continuous eigenfunctions.*

Proof. Let p be a transitive point. Then, by Theorem 6.5, p is weakly almost periodic. Hence, p is Weyl almost periodic as well and the claim follows from Theorem 5.5. □

As should be clear from the preceding discussion, the analog of Theorem 6.5 with 'weakly almost periodic' replaced by 'Bohr almost periodic' holds as well (and, similarly, for Corollary 6.6). In fact, a somewhat stronger statement is true for Bohr almost periodic points. To state this properly, we introduce the following notation when dealing with an dynamical system (X, G) with metric d. For $f \in C(X)$ we consider the mapping

$$\pi_f : X \longrightarrow C_u(G), \quad \pi_f(x) := f_x,$$

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and, when an $x \in X$ is fixed, we set

$$Y(f, x) := \overline{\{f_{tx} : t \in G\}},$$

where the closure is taken in $C_u(G)$ with the (usual) supremum norm. We also define

$$\overline{d} : X \times X \longrightarrow [0, \infty), \quad \overline{d}(x, y) := \sup_{s \in G} d(sx, sy).$$

Clearly, $\overline{d}$ is a metric.

THEOREM 6.8. *Let (X, G) be a dynamical system. For $x \in X$ the following assertions are equivalent.*

(i) *The element $x \in X$ is Bohr almost periodic.*
(ii) *For any $f \in C(X)$, the map $\pi_f : \overline{Gx} \longrightarrow C_u(G)$, $y \mapsto f_y$, is continuous with range given by $Y(f, x)$.*
(iii) *There exists a G-invariant metric on $\overline{Gx}$ generating the topology.*
(iv) *The function $\overline{d}$ is continuous on $\overline{Gx}$.*
(v) *The orbit closure $\overline{Gx}$ admits a structure of a locally compact group such that $G \longrightarrow \overline{Gx}$, $t \mapsto tx$, becomes a continuous group homomorphism.*

In particular, the orbit closure of any Bohr almost periodic point is minimal.

Proof. (i)$\Longrightarrow$(ii): Clearly, $\pi_f(tx) = f_{tx} = f_x(\cdot + t)$ for any $t \in G$. Thus, it suffices to show that f_y belongs indeed to $Y(f, x)$ for any $y \in \overline{Gx}$ and π_f is continuous on $Y(f, x)$. It is enough to show that $\pi_f(y_n)$ converges to $\pi_f(y)$ whenever (y_n) is a sequence in $\overline{Gx}$ converging to $y \in \overline{Gx}$ such that $\pi_f(y_n)$ belongs to $Y(f, x)$. Now, it is not hard to see that the functions $\pi_f(y_n)$ converge pointwise to $\pi_f(y)$. Moreover, by assertion (i), the set $Y(f, x)$ is compact and, hence, $\pi_f(y_n)$ has a uniform convergent subsequence. Now, any such subsequence must converge to $\pi_f(y)$ (as uniform convergence implies pointwise convergence). This gives the desired convergence statement.

(ii)$\Longrightarrow$(iii): By assertion (ii), the map $\overline{d}_f$ with $\overline{d}_f(y, z) := \|\pi_f(y) - \pi_f(z)\|_\infty$ is a continuous pseudometric on $\overline{Gx}$. It is clearly invariant. Now, choose a countable dense subset $D \subset C(X)$ separating the points of $\overline{Gx}$. Assume without loss of generality that any element of D is normalized, and choose for any $f \in D$ a $c_f > 0$ with $\sum_{f \in D} c_f < \infty$. Then, $\sum_{f \in D} c_f \overline{d}_f$ is a continuous invariant metric on $\overline{Gx}$. As $\overline{Gx}$ is compact any continuous metric determines its topology.

(iii)$\Longrightarrow$(iv): Let $\overline{d}'$ be a continuous invariant metric on $\overline{Gx}$. Let $\varepsilon > 0$ be arbitrary. As $\overline{d}'$ is a continuous metric on $\overline{Gx}$ there exists a $\delta > 0$ with $d(y_1, y_2) \leq \varepsilon$ for $y_1, y_2 \in \overline{Gx}$ whenever $\overline{d}'(y_1, y_2) \leq \delta$. As $\overline{d}'$ is invariant, we obtain then $d(ty_1, ty_2) \leq \varepsilon$ for all $t \in G$ and, hence, $\overline{d}(y_1, y_2) \leq \varepsilon$ whenever $\overline{d}'(y_1, y_2) \leq \delta$ for $y_1, y_2 \in \overline{Gx}$. This is the desired statement.

(iv)$\Longrightarrow$(v): Using the invariant metric $\overline{d}$ it is not hard to see that there is a group structure on $\overline{Gx}$ with $tx + sx = (t + s)x$. Here, we show only that this is well defined. The remaining statements then follow easily. Assume $tx = t'x$ and $sx = s'x$. Then triangle inequality and invariance of the metric gives

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$$\overline{d}((t+s)x,(t'+s')x) \le \overline{d}((t+s)x,(t+s')x) + \overline{d}((t+s')x,(t'+s')x)$$
$$\le \overline{d}(sx,s'x) + \overline{d}(tx,t'x) = 0.$$

This shows well definedness.

(v)$\Longrightarrow$(i): This is standard. We include some details for convenience of the reader. Let f be a continuous function on X. We have to show that the set $S := \{f_x(t+\cdot) : t \in G\} = \{f_{tx} : t \in G\}$ has compact closure in $C_u(G)$ with respect to the supremum norm. From assertion (iv), we easily see that $\pi_f : \overline{Gx} \longrightarrow C_u(G)$, $y \mapsto f_y$, is continuous. Hence, $\pi_f(\overline{Gx})$ is compact and, as it clearly contains S, the desired statement follows.

The minimality statement follows directly from assertion (iii). □

The preceding result shows that Bohr almost periodic points give rise to minimal orbit closures. In fact, within the weakly almost periodic points one can even characterize the Bohr almost periodic points by minimality of their orbit closures.

PROPOSITION 6.9. *Let (X, G) be a dynamical system and let $x \in X$ be weakly almost periodic. Then, x is Bohr almost periodic if and only if its orbit closure $\overline{Gx}$ is minimal.*

Proof. We have just seen in Theorem 6.8 that the orbit closure of a Bohr almost periodic point is minimal. Thus, consider now a weakly almost periodic point $x \in X$ with minimal orbit closure $\overline{Gx}$. Then, clearly $(\overline{Gx}, G)$ is weakly almost periodic by Lemma 6.4(b) and it is minimal (by assumption). Now, as is well known (see, e.g., [**33**]) any minimal component of a weakly almost periodic system is Bohr almost periodic. Now, the desired claim follows from Lemma 6.4(a). □

Remark. We note that a Bohr almost periodic dynamical system does not need to be minimal as can easily be seen by considering the 'disjoint union' of two Bohr almost periodic systems.

It is possible to characterize Bohr almost periodic points by almost periodicity properties of the metric $\overline{d}$.

THEOREM 6.10. *Let (X, G) be a dynamical system with metric d. Then the following assertions are equivalent for a point $p \in X$.*

(i) *The point p is Bohr almost periodic.*
(ii) *The function $G \mapsto [0,\infty)$, $t \mapsto \overline{d}(p, tp)$, is Bohr almost periodic.*

Proof. We show that (i) implies (ii): by assertion (i) and Theorem 6.8, the orbit closure of p is minimal and the function $\overline{d}$ is a continuous metric on $\overline{Gp}$. Let $\varepsilon > 0$ be given. As $\overline{d}$ is continuous and every Bohr almost periodic point has a minimal orbit, there exists a relatively dense set $R \subset G$ with $\overline{d}(sp, p) < \varepsilon$ for all $s \in R$. As $\overline{d}$ is invariant, this gives

$$|\overline{d}(p,(t+s)p) - \overline{d}(p,tp)| \le \overline{d}((t+s)p,tp)) \le \overline{d}(sp,p) < \varepsilon$$

for all $t \in G$ and $s \in R$. As $\varepsilon > 0$ was arbitrary this gives assertion (ii).

We now show that assertion (ii) implies that p is Bohr almost periodic: it is not hard to see that assertion (ii) implies that $t \mapsto d(sp,(t+s),p)$ is Bohr almost periodic for any

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$s \in G$. This easily implies that Lemma 6.2(iv) holds and assertion (i) follows from that lemma. □

7. *Application to measure dynamical systems*

In this section, we use our results to shed light on recent investigations of aperiodic order. A fundamental issue in the study of aperiodic order is pure point diffraction; see, e.g., [4] for a recent survey. Indeed, understanding of pure point diffraction has been a driving force in the field; see, e.g., the survey article [25]. Recently, a complete understanding of pure point diffraction via mean almost periodicity has been provided in [31]. That article mostly deals with single measures. However, it also includes results on pure point spectrum of certain dynamical systems, namely measure dynamical systems. Here, we discuss how our results allow one to provide a different approach to these results.

We start with a discussion of TMDSs. Such dynamical systems were brought forward in [3] to provide a systematic framework to study aperiodic order. In our exposition we follow [3] to which we refer for further details, proofs and references.

We denote by $C_c(G)$ the vector space of continuous complex-valued functions on G with compact support. This space is equipped with the inductive limit topology of the injections

$$C_K(G) \longrightarrow C_c(G), \quad \varphi \mapsto \varphi,$$

for $K \subset G$ compact. Here, $C_K(G)$ denotes the subspace of $C_c(G)$ consisting of functions with support in K. The measures on G are the elements of the dual space of $C_c(G)$. The total variation $|\mu|$ of a measure μ is the smallest positive measure with

$$|\mu(\varphi)| \leq |\mu|(\varphi)$$

for all $\varphi \in C_c(G)$ with $\varphi \geq 0$. A measure μ on G is called *translation bounded* if its total variation $|\mu|$ satisfies

$$\|\mu\|_K := \sup |\mu|(t + U) < \infty$$

for one (all) relatively compact open U in G. We denote that set of all translation bounded measures by $M^\infty(G)$ and equip it with the vague topology. Then, G admits a natural action on $M^\infty(G)$ by translations. More specifically, for $t \in G$ and $\mu \in M^\infty(G)$ the measure $t\mu$ is defined by

$$t\mu(\varphi) = \mu(\varphi(\cdot + t))$$

for all $\varphi \in C_c(G)$.

A subset $\Omega \subset M^\infty(G)$ which is invariant under the translation action is compact if and only if it is vaguely closed and there exists a constant C such that [39, Theorem A.8]

$$\|\mu\|_U \leq C \quad \text{for all } \mu \in \Omega.$$

Whenever Ω is a compact subset of $M^\infty(G)$, which is invariant under the translation action to and m is an invariant probability measure on X, we call (X, G, m) a *dynamical system of translation bounded measures* or just TMDS for short. If G is second countable than any TMDS is metrizable. Hence, the theory developed above applies to TMDS

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whenever G is second countable. Consider now an arbitrary TMDS (Ω, G, m) and define for any $\varphi \in C_c(G)$ the function

$$N_\varphi : \Omega \longrightarrow \mathbb{C}, \quad N_\varphi(\omega) = \omega(\varphi).$$

Then, N_φ belongs to $C(\Omega)$. In addition, there exists a unique translation bounded measure $\gamma = \gamma^m$ on (X, G, m) with

$$\gamma(\varphi * \widetilde{\psi}) = \langle N_\varphi, N_\psi \rangle$$

for all $\varphi, \psi \in C_c(G)$ and all $t \in G$. (Note that [3] uses a different sign in the definition of N (called f there) as well as has the inner product linear in the second argument. This results in a different display of the formula for γ, namely $(\gamma * \widetilde{\varphi} * \psi)(0) = \langle f_\varphi, f_\psi \rangle$.) The measure γ is called the *autocorrelation* of the TMDS. This measure allows for a Fourier transform $\widehat{\gamma}$ which is a (positive) measure on $\widehat{G}$. It is known as *diffraction* of the TMDS. Of particular interest in this theory are now those TMDSs whose diffraction is a pure point measure. By a main result of [3] (see references there also for earlier results) the diffraction of a TMDS is pure point if and only if the TMDS has pure point spectrum. Thus, for this reason, TMDSs with pure point spectrum are of utmost relevance in the field of aperiodic order. One particular question is the calculation of the atoms of $\widehat{\gamma}$. Here, the basic idea is that

$$\widehat{\gamma}(\{\xi\}) = \lim_{n\to\infty} \left| \frac{1}{|B_n|} \int_{B_n} \xi(t)\, d\omega(t) \right|^2$$

(with (B_n) being a Følner sequence). Validity of this formula is often discussed under the heading of the Bombieri–Taylor conjecture.

Having provided the framework of TMDSs, we now discuss how the theory developed in the previous section can be used in the study of aperiodic order.

A translation bounded measure ω is called *mean almost periodic* (*Besicovitch almost periodic*, *Weyl almost periodic*, respectively) if for any $\varphi \in C_c(G)$ the function

$$\omega * \varphi : G \longrightarrow \mathbb{C}, \quad (\omega * \varphi)(t) = \int \varphi(t - s)\, d\omega(s),$$

is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic, respectively).

PROPOSITION 7.1. *Let (Ω, G, m) be a TMDS and let (B_n) be a Følner sequence. Then, for $\omega \in \Omega$ the following assertions are equivalent.*

(i) *The measure ω is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic, weak almost periodic, Bohr almost periodic).*

(ii) *The function $t \mapsto N_\varphi(t\omega)$ is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic, weak almost periodic, Bohr almost periodic) for any $\varphi \in C_c(G)$.*

(iii) *The point $\omega \in \Omega$ is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic, weak almost periodic, Bohr almost periodic).*

Proof. We only discuss mean almost periodicity. The remaining statements follow analogously.

(i)$\Longleftrightarrow$(ii): A short computation shows

$$N_\varphi(t\omega) = (\omega * \widetilde{\varphi})(t),$$

where $\widetilde{\varphi} : G \longrightarrow \mathbb{C}, t \mapsto \overline{\varphi}(-t)$. This gives that ω is mean almost periodic if and only if $t \mapsto N_\varphi(t\omega), \varphi \in C_c(G)$, is mean almost periodic and the equivalence between assertions (i) and (ii) follows.

(iii)$\Longrightarrow$(ii): This follows easily as any $N_\varphi, \varphi \in C_c(G)$, is a continuous function on Ω.

(ii)$\Longrightarrow$(iii): It is not hard to see that the set of $N_\varphi, \varphi \in C_c(G)$, separates the points of Ω and is closed under complex conjugation. Hence, the algebra generated by the N_φ is dense in the continuous functions on Ω with respect to the supremum norm (see, e.g., [4] for further discussion of this type of argument). This gives that assertion (ii) implies assertion (iii). □

When dealing with TMDS (Ω, G, m) we can now use the previous proposition to replace the assumption that $\omega \in \Omega$ is mean almost periodic as element of the dynamical system (Ω, G, m) by the assumption that ω is a mean almost periodic measure (and, similarly, with mean almost periodic replaced by Besicovitch almost periodic, Weyl almost periodic, weak almost periodic, Bohr almost periodic). This allows then for reformulations of our main results. We include the most important consequences next.

THEOREM 7.2. *Let (Ω, G, m) be an ergodic TMDS. Assume that G is second countable and that (B_n) is a Følner sequence along which Birkhoff's ergodic theorem holds. Then, the following assertions are equivalent.*

(i) *(Ω, G, m) has pure point spectrum.*
(ii) *Almost every $\omega \in \Omega$ is a mean almost periodic measure.*
(iii) *Almost every $\omega \in \Omega$ is a Besicovitch almost periodic measure.*

Moreover, in this case, for each $\xi \in \widehat{G}$ with $\widehat{\gamma}(\{\xi\}) \neq 0$, the Fourier–Bohr coefficient

$$a_\xi(\omega) = \lim_n \frac{1}{|B_n|} \int_{B_n} \overline{\xi(t)}\, d\omega(t)$$

exists for m-almost all ω and satisfies

$$\widehat{\gamma}(\{\xi\}) = |a_\xi(\omega)|^2.$$

Furthermore, there exists a non-trivial eigenfunction $f_\xi \in L^1(\Omega, m)$ such that $f_\chi(\omega) = a_\chi(\omega)$ for all $\omega \in$ Bap(X, G, m).

Proof. The equivalence follows by combining Theorems 3.8, 4.7 and Proposition 7.1.

Next, let $\xi \in \widehat{G}$ with $\widehat{\gamma}(\{\xi\}) \neq 0$ and let $\varphi \in C_c(G)$ be so that $\widehat{\varphi}(\xi) \neq 0$. Then, by Proposition 4.2 applied to N_φ, the limit

$$A((N_\varphi)_\omega \bar{\xi}) := \lim_n \frac{1}{|B_n|} \int_{B_n} \overline{\xi(t)} \varphi * \omega(t)\, dt$$

exists for m almost all $\omega \in X$. By [31, Corollary 1.1], the Fourier–Bohr coefficients exist for all such ω and satisfy

$$A((N_\varphi)_\omega \bar{\xi}) = a_\xi(\omega) \widehat{\varphi}(\chi).$$

Theorem 4.7 the implies the existence of the eigenfunction.

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Finally, by [4], $|\widehat{\varphi}|^2\widehat{\gamma}$ is the spectral measure for $N_\varphi \in L^2(X, m)$. The last claim follows now from Theorem 4.7 and [27, Theorem 3]. □

By combining Theorem 4.4 with Proposition 7.1 we also immediately obtain the following consequence.

THEOREM 7.3. *Let ω be a Besicovitch almost periodic measure and let (B_n) be a van Hove sequence along which Birkhoff's ergodic theorem holds. Then, there exists a (unique) G-invariant ergodic measure m on $\mathbb{X}(\mu)$ such that μ is generic for m.*

Finally, by combining Theorem 5.5 with Proposition 7.1 we also get the following result.

THEOREM 7.4. *Let ω be a translation bounded measure and let (B_n) be a van Hove sequence along which Birkhoff's ergodic theorem holds. Then, the following are equivalent.*

(i) *μ is Weyl almost periodic.*
(ii) *Every $\omega \in \mathbb{X}(\mu)$ is Weyl almost periodic.*
(iii) *The dynamical system $(\mathbb{X}(\mu), G)$ is uniquely ergodic with pure point spectrum and continuous eigenfunctions.*

8. *Abstract generalizations*

When closing the article, it may be instructive to stop a moment to have a look at the overall theme of this article from a more abstract point of view. The general approach in this article may be described as follows: we consider a dynamical system (X, G) and say that a point $x \in X$ is $(*)$ almost periodic if $\mathcal{A}_x$ consists of $(*)$ almost periodic functions only. Then, the preceding sections have been devoted to a thorough study of consequences of existence of $(*)$ almost periodic points for $(*)$ being replaced by Bohr, weak, Weyl, Besicovitch and mean (and in this order these are increasingly weaker notions of almost periodicity). Now, of course, any other concept of almost periodicity for a function could also be taken as the starting point of the theory. Then, some of our considerations will easily carry over. This is discussed in this section.

Let $C_{\mathsf{u}}(G)$ be the set of uniformly continuous bounded functions on G and consider a dynamical system (X, G). Whenever we are given an $\mathbb{A} \subset C_{\mathsf{u}}(G)$, we can define for $p \in X$

$$\mathbb{A}_p := \{f \in C(X) : f_p \in \mathbb{A}\}.$$

Then, $\mathbb{A}_p$ inherits various properties of $\mathbb{A}$. In particular, if $\mathbb{A}$ is an algebra, then so is $\mathbb{A}_p$ and if $1 \in \mathbb{A}$ then $\mathbb{A}_p$ contains the constant function 1. Moreover, if $\mathbb{A}$ is closed in $(C_{\mathsf{u}}(G), \|\cdot\|_\infty)$, then $\mathbb{A}_p$ is closed in $(C(X), \|\cdot\|_\infty)$. Then, as abstraction of Lemma 3.7 (with the same proof), we obtain the following lemma.

LEMMA 8.1. *Let $\mathbb{A}$ be a closed subalgebra of $C_{\mathsf{u}}(G)$ containing the constant function 1. Then, the following assertions are equivalent for $p \in X$.*

(i) *$\mathbb{A}_p = C(X)$.*
(ii) *$\mathbb{A}_p$ separates the points of $\overline{Gp}$.*
(iii) *For each $s \in G$ the function $d^{(s)} \in C(X)$ with $d^{(s)}(y) := d(sp, y)$ belongs to $\mathbb{A}_p$.*

A particular way to obtain a closed algebra $\mathbb{A}$ is by suitable seminorms. This is discussed next: call a seminorm N on $C_{\mathsf{u}}(G)$ *admissible* if it is G-invariant and satisfies:

- $N(f) \leq N(g)$ whenever $|f| \leq g$;
- $N(1) = 1$.

Note that any admissible seminorm N satisfies $N \leq \|\cdot\|_\infty$ (as $|f| \leq \|f\|_\infty \cdot 1$).

Remark (Examples). It is not hard to see that $\|\cdot\|_\infty$ and $\overline{M} \circ |\cdot|$ as well as

$$N(f) := \limsup_{n\to\infty} \sup_{t\in G} \frac{1}{|B_n|} \int_{t+B_n} |f(t)|\, dt$$

are admissible seminorms on $C_u(G)$.

Definition 8.2

(a) We say that an $f \in C_u(G)$ is *N-almost periodic* if for any $\varepsilon > 0$ the set

$$\{t \in G : N(|f - f(\cdot - t)|) < \varepsilon\}$$

is relatively dense in G.

(b) We say that an $f \in C_u(G)$ is *N-trig almost periodic* if for any $\varepsilon > 0$, there exists some trigonometric polynomial P such that $N(|f - P|) < \varepsilon$.

LEMMA 8.3. *Let N be an admissible seminorm. Then,*

$$\mathbb{A}^N := \{f \in C_u(G) : f \textit{ is } N\textit{-almost periodic}\}$$

and

$$\mathbb{A}^T := \{f \in C_u(G) : f \textit{ is } N\textit{-trig almost periodic}\}$$

are closed subalgebras of $C_{\mathsf{u}}(G)$. *Both subalgebras contain the Bohr almost periodic functions and* $\mathbb{A}^T \subseteq \mathbb{A}^N$ *holds.*

In particular, $1 \in \mathbb{A}^N$.

Proof. The statement on $\mathbb{A}^N$ is an abstraction of Theorem B.3 in our context. It can be shown by replacing $M \circ |\cdot|$ with N in the proof of this theorem. Similarly, the statement on $\mathbb{A}^T$ is an abstraction of Lemma C.2 in our context and can be shown by mimicking the proof of that lemma. This also gives the last statement. □

Definition 8.4. (N-almost periodic points) Let (X, G) be a dynamical system and let N be an admissible seminorm on $C_u(G)$. We say, that an $x \in X$ is *N-almost periodic* if f_x is N-almost periodic for any $f \in C(X)$, that is, if $\mathbb{A}_x^N = C(X)$ holds.

For a continuous metric d on X and $x \in X$ we define

$$d^{N,x} : G \longrightarrow [0, \infty), \quad d^{N,x}(t) := N((s \mapsto d(sx, (t + s)x).$$

Reasoning as in §3, we can infer that for any continuous metric d the function $d^{N,x}$ is uniformly continuous. Having set things up, we can now discuss the following abstraction of the results in §3 (where we include some details for the convenience of the reader).

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Analogously to Lemma 3.2 we find the following.

LEMMA 8.5. *Let (X, G) be a dynamical system. Let N be admissible. Let d be a continuous metric on X. Then, the following assertions for $x \in X$ are equivalent.*

(i) *For any $\varepsilon > 0$ the set*

$$\{t \in G : d^{N,x}(t) < \varepsilon\}$$

is relatively dense.

(ii) *The function $d^{N,x}$ is Bohr almost periodic.*

The following is an abstraction of both the equivalence between assertions (i) and (ii) in Lemma 3.7 (with $N = \overline{M} \circ |\cdot|$) and Theorem 6.10 (with $N = \|\cdot\|_\infty$).

THEOREM 8.6. *Let (X, G) be a dynamical system. Let N be admissible. Then, for $x \in X$ the following assertions are equivalent.*

(i) *The point x is N-almost periodic.*

(ii) *There exists a continuous metric d on X such that $d^{N,x}$ is Bohr almost periodic.*

(iii) *For every continuous metric d on X the function0 $d^{N,x}$ is Bohr almost periodic.*

Remark. As X is a compact metric space, a metric on X is continuous if and only if it generates the topology.

Proof. (iii)$\Longrightarrow$(ii): This is clear.

(ii)$\Longrightarrow$(iii): This is the analog of Lemma 3.4 in our context. It can be shown by a variant of the proof of that lemma. Some extra effort is needed as N is not defined on measurable bounded functions but only on $C_{\mathsf{u}}(G)$. This is tackled by means of Urysohn's lemma. As every locally compact group is a normal space, this lemma allows one to separate arbitrary closed disjoint sets by continuous functions and this is what we use. We now present the details.

Let e be any metric on X such that $e^{N,x}$ is almost periodic and let d be any other metric on X, such that d, e generate the topology. Without loss of generality we can assume that $d, e \leq 1$. Let $\varepsilon > 0$ be arbitrary. By Lemma 8.5, it suffices to show that the set of $t \in G$ with $d^{N,x}(t) \leq \varepsilon$ is relatively dense.

Choose $\delta' > 0$ with $d(y, z) < \varepsilon/2$ whenever $e(y, z) < \delta'$. Set

$$\delta := \frac{\varepsilon}{4\delta'}.$$

Let $t \in G$ be so that $e^{N,x}(t) < \delta$. By Lemma 8.5 the set of such $t \in G$ is relatively dense. Thus, it remains to show $d^{N,x}(t) < \varepsilon$ for any such $t \in G$. Define $e_{t,x}$ and $d_{t,x}$ on G by $e_{t,x}(s) := e(sx, (t + s)x)$ and $d_{t,x}(s) := d(sx, (t + s)x)$. Set

$$A := \left\{s : e_{t,x} \geq \delta'\right\}; \quad B := \left\{s : e_{t,x} \leq \frac{\delta'}{2}\right\}.$$

Then, by Urysohn's lemma, there exists some continuous function $f : G \to [0, 1]$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in B$. Set $g := 1 - f$. Then, $e_{t,x} \geq$

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$\delta'/2f$ and, hence, $\delta'/2N(f) \le N(e_{t,x}) \le \delta$, showing

$$N(f) \le \frac{2\delta}{\delta'} = \frac{\varepsilon}{2}.$$

Moreover, $d_{t,x}(s)g(s) \le \varepsilon/2$. Indeed, if $s \in A$, then $g(s) = 0$ by the definition of g, and if $s \notin A$, then $d_{t,x}(s) \le \varepsilon/2$ by our choice of δ'. This shows

$$N(d_{t,x}) \le \frac{\varepsilon}{2}.$$

Therefore, using $d \le 1$ we obtain

$$d^{N,x}(t) = N(d_{t,x}) \le N(d_{t,x} f) + N(d_{t,x} g) \le N(f) + N(d_{t,x} g) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(i)$\Longrightarrow$(ii): This can be shown as (ii)$\Longrightarrow$(i) in Lemma 3.7.

(ii)$\Longrightarrow$(i): As in the proof of (i)$\Longrightarrow$(iv) of Lemma 3.7 we infer from assertion (ii) that d_x^z is N-almost periodic for any $z \in X$. Now, assertion (i) follows from Lemma 8.1. □

COROLLARY 8.7. *Let (X, G, m) be a dynamical system with metric d. Let N be an admissible seminorm and let $x \in X$ be N-almost periodic. If*

$$\int_X f(y)\, dm(y) \le N(f_x)$$

holds for all $f \in C(X)$, then (X, G, m) has pure point spectrum.

Remark. Note that the assumption holds whenever there exists a F ølner sequence (B_n) along which Birkhoff's ergodic theorem holds and N satisfies $\overline{M} \circ |\cdot| \le N$ and x is generic with respect to m.

Proof. As x is N-almost periodic the preceding theorem (combined with Lemma 8.5) gives that for any $\varepsilon > 0$ the set of $t \in G$ with $d^{N,x}(t) < \varepsilon$ is relatively dense. By using the assumption on N with $f(y) = d(y, ty)$ (for $t \in G$ fixed), we furthermore find for the function

$$\underline{d} : G \longrightarrow [0, \infty), \quad \underline{d}(t) = \int_X d(y, ty)\, dm(y),$$

the inequality

$$\underline{d}(t) \le d^{N,x}(t)$$

for all $t \in G$. Hence, for any $\varepsilon > 0$ the set of $t \in G$ with $\underline{d}(t) < \varepsilon$ is relatively dense as well. This gives us that $\underline{d}$ is Bohr almost periodic and the desired statement on pure point spectrum follows from the main result of [**29**] (compare with the proof of Theorem 3.8 as well). □

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A. *Appendix. Bohr almost periodic functions*

In this section, we briefly present some background on Bohr almost periodic function. This is completely standard and can be found in many places. We include a discussion here in order to be self-contained and to set the perspective on the weaker (and less well-known) notions of almost periodicity underlying our considerations. For more details, we refer the reader to [9, 35].

Let G be a locally compact abelian group. Recall from §2 that a continuous function $f : G \longrightarrow \mathbb{C}$ is called *Bohr almost periodic* if for any $\varepsilon > 0$ the set of $t \in G$ with

$$\|f - f(\cdot - t)\|_\infty < \varepsilon \qquad (\spadesuit)$$

is relatively dense. Any such $t \in G$ is then called an *ε-almost period* of *f*. It turns out that a continuous $f : G \longrightarrow \mathbb{C}$ is Bohr almost periodic if and only if $\overline{\{f(\cdot - t) : t \in G\}}$ is compact, where the closure is taken with respect to the supremum norm. It is not hard to see that any Bohr almost periodic function is bounded and uniformly continuous. Moreover, the Bohr almost periodic functions form a closed subalgebra of the algebra of all uniformly continuous bounded functions on G. The main structural result on Bohr almost periodic functions is that a function *f* is Bohr almost periodic if and only if for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in \widehat{G}$ and $c_1, \ldots, c_k \in \mathbb{C}$ with

$$\left\| f - \sum_{j=1}^{k} c_j \xi_j \right\|_\infty < \varepsilon. \qquad (\heartsuit)$$

The basic idea of the weaker concepts of almost periodicity discussed in the subsequent sections is to replace the supremum norm $\| \cdot \|_\infty$ in ($\spadesuit$) and ($\heartsuit$) by suitable (semi)norms arising by averaging procedures.

B. *Appendix. Mean almost periodic functions*

The main result of this appendix shows that the bounded uniformly continuous mean almost periodic functions form a closed subalgebra of the algebra of bounded uniformly continuous functions on G. This is certainly well known and a proof can be given by standard means. For the convenience of the reader and in order to keep this article self-contained we include a discussion in the following. As our article deals with abelian groups, we assume that the group G in the following is abelian. Note, however, that this is not used in the proofs.

We start with a general result on discrete geometry of groups.

PROPOSITION B.1. *Let G be a locally compact abelian group. Let D and E be relatively dense subsets of G and V a relatively compact open neighborhood of the neutral element. Then, $((D - D) + V) \cap ((E - E) + V)$ is relatively dense.*

Proof. As both D and E are relatively dense, we can choose an open relatively compact set $U \subset G$ with the property that any translate of U intersects both D and E. As addition is continuous on G, we can choose furthermore a relatively compact open neighborhood W with $W = -W$ and $W + W \subset V$. As U is relatively compact, there exist $N \in \mathbb{N}$ and $z_1, \ldots z_N \in G$ with

$$U = \bigcup_{n=1}^{N} (z_n + W) \cap U.$$

Consider now an arbitrary $t \in D$. Then, there exists an $s \in E$ with $t - s \in U$. Hence, for any $t \in D$ we can find $s_t \in E$ and $n_t \in \{1, \ldots, N\}$ with

$$(t - s_t) \in z_{n_t} + W.$$

Fix now for each $n \in \{1, \ldots, N\}$ elements $t_n \in D$ and $s_n \in E$ with

$$(t_n - s_n) \in z_n + W.$$

(If some of the n does not admit elements, we just remove this n from our list.) Then, for any $t \in D$ we can find $s_t \in E$ and $n \in \{1, \ldots, N\}$ such that both $t - s_t$ and $t_n - s_n$ belong to $z_n + W$. Hence, we find

$$t - s_t + w = t_n - s_n + w'$$

for suitable $w, w' \in W$. This gives

$$t - t_n = s_t - s_n + v$$

with $v = w' - w \in W - W \subset V$. Now, we clearly have $t - t_n \in (D - D) + V$ and $(s_t - s_n) + v \in (E - E) + V$. Moreover, the set of all $t - t_n$ is relatively dense as t is an arbitrary element of the relatively dense D and there are only finitely many t_n. This finishes the proof. □

Let G be a σ-compact locally compact abelian group and let (B_n) be a Følner sequence on G. Let f be a uniformly continuous bounded function on G. Let $\varepsilon > 0$ be given. As usual we say that a $t \in G$ is an ε-almost period of f if

$$\overline{M}(|f - f(\cdot - t)|) < \varepsilon.$$

Denote the set of all ε-almost periods of f by $\mathrm{AP}(f, \varepsilon)$. Then, it is not hard to see that

$$\mathrm{AP}(f, \varepsilon) - \mathrm{AP}(f, \varepsilon) \subset \mathrm{AP}(f, 2\varepsilon).$$

A uniformly continuous bounded $f : G \longrightarrow \mathbb{C}$ is mean almost periodic if for any $\varepsilon > 0$ the set $\mathrm{AP}(f, \varepsilon)$ is relatively dense.

LEMMA B.2. *Let G be a σ-compact locally compact abelian group and let (B_n) be a Følner sequence on G. Let a natural number n and uniformly continuous bounded mean almost periodic functions $f_1, \ldots, f_n$ on G be given. Then, the set*

$$\bigcap_{k=1}^{n} \mathrm{AP}(f_k, \varepsilon)$$

is relatively dense in G for any $\varepsilon > 0$.

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Proof. This is shown by induction in n. The case $n = 1$ is clear. Thus, assume now the statement holds for a chosen n. Let $\varepsilon > 0$ and uniformly continuous functions $f_1, \ldots, f_{n+1}$ be given. Then, the set $D := \bigcap_{k=1}^{n} \mathrm{AP}(f_k, \varepsilon/3)$ is relatively dense by assumption. As the functions f_k, $k = 1, \ldots, n+1$, are uniformly continuous we can find an open relatively compact neighborhood V of the neutral element such that

$$\|f_k - f_k(\cdot - s)\|_\infty < \frac{\varepsilon}{3}$$

for all $s \in V$ and $k = 1, \ldots, n+1$. Set $E := \mathrm{AP}(f_{n+1}, \varepsilon/3)$. Then, the previous proposition gives that

$$((D - D) + V) \cap ((E - E) + V)$$

is relatively dense. On the other hand, it is not hard to see that

$$(D - D) + V \subset \mathrm{AP}(f_k, \varepsilon), \quad k = 1, \ldots, n \quad \text{and} \quad (E - E) + V \subset \mathrm{AP}(f_{n+1}, \varepsilon).$$

This finishes the proof. □

THEOREM B.3. *Let G be a σ-compact locally compact abelian group and let (B_n) be a Følner sequence on G. Then, the set of all uniformly continuous bounded mean almost periodic functions is invariant under taking complex conjugates and a closed subalgebra of the uniformly continuous bounded functions on G equipped with $\|\cdot\|_\infty$.*

Proof. We have to show that the set in question is closed under complex conjugation, sums, products, multiplication by scalars and uniform convergence.

It is not hard to see that the set in question is closed under complex conjugation, uniform convergence and multiplication by scalars.

We next show that it is closed under sums: let f, g be mean almost periodic uniformly continuous bounded functions. Then, the previous lemma easily gives that $f + g$ is also mean almost periodic.

Finally, we deal with products: let f, g be mean almost periodic uniformly continuous bounded functions. Then, a short computation gives for any $t \in G$

$$\begin{aligned} |f(s)g(s) - f(s-t)g(s-t)| &\le |f(s)g(s) - f(s)g(s-t)| \\ &\quad + |f(s)g(s-t) - f(s-t)g(s-t)| \\ &\le \|f\|_\infty |g(s) - g(s-t)| + \|g\|_\infty |f(s) - f(s-t)|. \end{aligned}$$

From this we easily obtain

$$\overline{M}(|fg - (fg)(\cdot - t)|) \le \|f\|_\infty \overline{M}(|g - g(\cdot - t)|) + \|g\|_\infty \overline{M}(|f - f(\cdot - t)|).$$

Now, the desired statement follows easily from the preceding lemma.

The last statement is clear. □

For later use we also note the following proposition.

PROPOSITION B.4. *Let (f_n) be a sequence of uniformly continuous bounded mean almost periodic functions on G with $\|f_n\| \le 1$ for all n. Let $c_n > 0$ with $\sum_{n=1}^{\infty} c_n < \infty$ be given.*

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Then, there exists for any $\varepsilon > 0$ *a relatively dense set* D *in* G *with*

$$\overline{M}\left(\sum_{n=1}^{\infty} c_n |f - f(\cdot - t)|\right) < \varepsilon$$

for all $t \in D$.

Proof. Choose n_0 large enough so that $\sum_{k=n_0+1}^{\infty} c_k < \varepsilon/4$. Then, $\sum_{n=n_0+1}^{\infty} c_n \|f - f(\cdot - t)\|_\infty < \varepsilon/2$ by assumption on the f_n and the c_n. By Lemma B.2 there exists a relatively dense set D in G with $\overline{M}(|f_k - f_k(\cdot - t)|) < \varepsilon/2n_0$ for $k = 1, \ldots, n_0$. Now, the statement follows easily. □

Remark. The considerations of this section carry over when $\overline{M} \circ |\cdot|$ is replaced by any invariant seminorm N on the algebra of bounded uniformly continuous functions on G satisfying:

- $N(f) \le N(g)$ whenever $|f| \le g$;
- $N(1) = 1$.

This point is taken up in §8.

C. *Appendix. Besicovitch almost periodic functions and existence of means*

We consider a σ-compact locally compact abelian group G together with a Følner sequence (B_n). Our aim is to study the set of uniformly continuous bounded functions $f : G \longrightarrow \mathbb{C}$, which are Besicovitch almost periodic, that is, they satisfy that for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in \widehat{G}$ and $c_1, \ldots, c_k \in \mathbb{C}$ with

$$\overline{M}\left(\left|f - \sum_{j=1}^{k} c_{\xi_j} \xi_j\right|\right) < \varepsilon.$$

For an in-depth study of these functions, we the reader refer to [31]. Here, we briefly summarize what is needed for the purposes of the present article.

PROPOSITION C.1. *Let* G *be a* σ*-compact locally compact abelian group and let* (B_n) *be a Følner sequence on* G*. Then, any uniformly continuous bounded Besicovitch almost periodic function is mean almost periodic.*

Proof. Let $\varepsilon > 0$ be given. Choose $c_1, \ldots, c_k \in \mathbb{C}$ and $\xi_1, \ldots, \xi_k \in \widehat{G}$ such that $P := \sum_{j=1}^{k} c_j \xi_j$ satisfies

$$\overline{M}(|f - P|) < \varepsilon.$$

As $\overline{M}$ is invariant this inequality will then continue to hold if f is replaced by $f(\cdot - t)$ and P is replaced by $P(\cdot - t)$ for any $t \in G$. Now, clearly P is Bohr almost periodic. Hence, there exists a relatively dense set $R \subset G$ with $\|P - P(\cdot - t)\|_\infty < \varepsilon$ for all $t \in R$. This easily gives

$$\overline{M}(|f - f(\cdot - t)|) \le \overline{M}(|f - P|) + \overline{M}(|P - P(\cdot - t)|) + \overline{M}(|P(\cdot - t) - f(\cdot - t)|) < 3\varepsilon$$

for all $t \in R$. □

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From the definition and simple algebraic manipulations, we infer the following.

LEMMA C.2. *Let G be a σ-compact locally compact abelian group and let (B_n) be a Følner sequence on G. Then, the set of all uniformly continuous bounded Besicovitch almost periodic functions is invariant under taking complex conjugates and a closed subalgebra of the uniformly continuous bounded functions on G equipped with $\|\cdot\|_\infty$. It contains all Bohr almost periodic functions and is contained in the algebra of mean almost periodic functions.*

Proof. The set is clearly closed under taking limits with respect to $\|\cdot\|_\infty$ as well as under addition and taking complex conjugates. To show that it is closed under multiplication consider f, g in this set and let $\epsilon > 0$ be arbitrary. Let P, Q be trigonometric polynomials so that

$$\overline{M}(|f-P|) < \frac{\varepsilon}{2\|g\|_\infty + 4} \quad \text{and} \quad \overline{M}(|g-Q|) < \frac{\varepsilon}{4\|f\|_\infty + 1}.$$

Define

$$Q'(x) := \begin{cases} Q(x) & \text{if } |Q(x)| \le \|g\|_\infty + 1, \\ \dfrac{Q(x)}{|Q(x)|}(\|g\|_\infty + 1) & \text{otherwise.} \end{cases}$$

Then, Q' is a Bohr almost periodic function and $|g - Q'| \le |g - Q|$, which gives $\overline{M}(|g-Q'|) \le \overline{M}(|g-Q|)$.

As Q' is Bohr almost periodic, there exists a trigonometric polynomial R such that $\|Q' - R\|_\infty < \min\{\varepsilon/(4\|f\|_\infty + 1), 1\}$. In particular,

$$\|R\|_\infty \le \|Q'\|_\infty + 1 \le \|g\|_\infty + 2.$$

Then,

$$\begin{aligned} \overline{M}(|fg - PR|) &\le \overline{M}(|fg - fR|) + \overline{M}(|fR - PR|) \\ &\le \|f\|_\infty \overline{M}(|g-R|) + \|R\|_\infty \overline{M}(|f-P|) \\ &\le \|f\|_\infty (\overline{M}(|g-Q'|) + \overline{M}(|Q'-R|)) + \frac{\varepsilon}{2} \\ &\le \|f\|_\infty (\overline{M}(|g-Q'|) + \|Q'-R\|_\infty) + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

This shows that the set in question is closed under multiplication. It clearly contains all Bohr almost periodic functions and is contained in the set of mean almost periodic functions. □

Remark. The functions referred to as Besicovitch almost periodic were introduced by Besicovitch in [8] for $G = \mathbb{R}$. A corresponding class of functions was then studied by Følner for general locally compact abelian groups [14]. This class, however, does not coincide with the Besicovitch class for $G = \mathbb{R}$. Another approach to Besicovitch almost periodic functions is developed by Davis in [10]. An account of these developments with a focus on aperiodic order is given in Lagarias survey [25]. Here, we have taken a 'shortcut':

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we have not defined Besicovitch almost periodic functions by some intrinsic features. Instead we have defined them by what would be a main result in a proper theory starting with an intrinsic definition.

A crucial feature of Besicovitch almost periodic function is existence of means in the following sense.

LEMMA C.3. *Let G be a σ-compact locally compact abelian group and let (B_n) be a Følner sequence on G. Let $f : G \longrightarrow \mathbb{C}$ be a uniformly continuous, bounded Besicovitch almost periodic function. Then, the limit*

$$\mathrm{A}(f) = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(s)\, ds$$

exists.

Proof. The statement is well known for functions of the form $f = \sum_{j=1}^{k} c_j \xi_j$ with $\xi_1, \ldots, \xi_k \in \widehat{G}$, $c_1, \ldots, c_k \in \mathbb{C}$. It then follows by a limiting procedure for Besicovitch almost periodic functions. □

In fact, existence of means together with a Parseval type equality is a characterizing feature of Besicovitch almost periodic functions.

PROPOSITION C.4. *Let G be a σ-compact locally compact abelian group and let (B_n) be a Følner sequence on G. Let $f : G \longrightarrow \mathbb{C}$ be a uniformly continuous and bounded. Then, f is Besicovitch almost periodic if and only if there exists a countable set $F \subset \widehat{G}$ such that the following three statements hold.*

- *The limit $\mathrm{A}(|f|^2) = \lim_{n\to\infty}(1/|B_n|)\int_{B_n} |f(t)|^2\, dt$ exists.*
- *For any $\xi \in F$ the limit*

$$\mathrm{A}(f\overline{\xi}) = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(t)\, \overline{\xi(t)}\, dt$$

  exists.
- *The equality*

$$\mathrm{A}(|f|^2) = \sum_{\xi\in F} |\mathrm{A}(f\overline{\xi})|^2$$

  holds.

In this case $\mathrm{A}(f\overline{\xi}) = 0$ holds for all $\xi \in \widehat{G} \setminus F$.

Proof. Let f be Besicovitch almost periodic. Clearly, any $\xi \in \widehat{G}$ is Bohr almost periodic and, hence, Besicovitch almost periodic. Thus, $f\overline{\xi}$ is Besicovitch almost periodic as a product of Besicovitch almost periodic functions. Hence, the second point holds (even for all $\xi \in \widehat{G}$). Similarly, $|f|^2$ is Besicovitch almost periodic. Given this, the first point follows. The last point is contained in [31].

Now, consider f satisfying the three points. Let $\xi_1, \xi_2, \ldots$ be an enumeration of the $\xi \in F$. A computation involving the Cauchy–Schwarz inequality in the first step, $\overline{M}(1) = 1$

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in the second step and existence of averages A in the third step gives

$$\begin{aligned}
&\overline{M}\left(\left|f-\sum_{j=1}^{N} A(f\overline{\xi_j})\xi_j\right|\right)^2\\
&\leq \overline{M}\left(\left|f-\sum_{j=1}^{N} A(f\overline{\xi_j})\xi_j\right|^2\right)\overline{M}(1)\\
&= \overline{M}\left(|f|^2 - f\overline{\sum_{j=1}^{N} A(f\overline{\xi_j})\xi_j} - \overline{f}\sum_{j=1}^{N} A(f\overline{\xi_j})\xi_j + \sum_{j,k=1}^{N} A(f\overline{\xi_j})\overline{A(f\overline{\xi_k})}\xi_j\overline{\xi_k}\right)\\
&= \mathrm{A}(|f|^2) - \mathrm{A}\left(f\overline{\sum_{j=1}^{N} A(f\overline{\xi_j})\xi_j}\right) - \mathrm{A}\left(\overline{f}\sum_{j=1}^{N} A(f\overline{\xi_j})\xi_j\right) + \mathrm{A}\left(\sum_{j=1,k}^{N} A(f\overline{\xi_j})\overline{A(f\overline{\xi_k})}\xi_j\overline{\xi_k}\right)\\
&= \mathrm{A}(|f|^2) - \sum_{j=1}^{N} |A(f\overline{\xi})|^2\\
&\to 0
\end{aligned}$$

for $N \to \infty$. Here, the penultimate step is a direct computation invoking that A is linear with $A(\eta) = 0$ for $0 \neq \eta \in \widehat{G}$. Indeed, this shows that f can be approximated in mean by a trigonometric polynomial arbitrarily well. This finishes the proof. □

D. *Appendix. Weyl almost periodic functions and uniform means*

In this appendix, we consider a uniform type of mean.

Let (B_n) be a Følner sequence in G and define for bounded $f : G \longrightarrow \mathbb{R}$ and $n \in \mathbb{N}$

$$\overline{M}_n(f) := \sup_{r\in G} \frac{1}{|B_n|}\int_{B_n+r} f(t)\,dt.$$

PROPOSITION D.1. *For a bounded measurable function $f : G \longrightarrow \mathbb{R}$ and $\varepsilon > 0$ the following assertions are equivalent.*

(i) *There exists an $N \in \mathbb{N}$ with $\overline{M}_N(f) < \varepsilon$.*

(ii) *There exists an $N_0 \in \mathbb{N}$ with $\overline{M}_N(f) < \varepsilon$ for all $N \geq N_0$.*

Proof. (ii)⟹(i): This is clear.

(i)⟹(ii): Consider $n \in \mathbb{N}$. Define for $r \in G$

$$C_r := \frac{1}{|B_N|}\int_{B_N}\left(\int \frac{1}{|B_n|}\int_{B_n+r} f(s+t)\,ds\right)dt.$$

Fubini's theorem and the assumption (i) directly give

$$C_r = \frac{1}{|B_n|}\int_{B_n+r}\left(\frac{1}{|B_N|}\int_{B_N} f(s+t)\,dt\right)ds \leq \frac{1}{|B_n|}\int_{B_n+r} \overline{M}_N(f)\,ds = \overline{M}_N(f).$$

On the other hand, we can easily see that (for large n) the additional averaging over B_N does not play a role. More specifically, we can compute as follows:

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$$\begin{aligned}\left|C_r - \frac{1}{|B_n|}\int_{B_n+r} f(s)\,ds\right| &= \left|C_r - \frac{1}{|B_N|}\int_{B_N}\left(\frac{1}{|B_n|}\int_{B_n+r} f(s)\,ds\right)dt\right| \\ &= \left|\frac{1}{|B_N|}\int_{B_N}\left(\frac{1}{|B_n|}\int_{B_n+r}(f(s+t)-f(s))\,ds\right)dt\right| \\ &\leq \frac{1}{|B_N|}\int_{B_N} 2\|f\|_\infty \frac{|(B_n+t)\,\triangle\, B_n|}{|B_n|}\,dt.\end{aligned}$$

Now, due to the Følner condition the integrand in the last term can easily be seen to go pointwise to zero for $n \to \infty$. As B_N is compact, we find convergence to zero of the integral for each fixed N. As there is no dependence on $r \in G$, this convergence is independent of $r \in G$. This easily gives the desired statement. □

A bounded $f : G \longrightarrow \mathbb{C}$ is Weyl almost periodic if for all $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in \widehat{G}$ and $c_1, \ldots, c_k \in \mathbb{C}$ with

$$\limsup_{n\to\infty} \overline{M}_n\left(\left|f - \sum_{j=1}^{k} c_{\xi_j}\xi_j\right|\right) < \varepsilon.$$

By the preceding proposition it is possible to replace this condition by the requirement that for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, $k \in \mathbb{N}$, $\xi_1, \ldots, \xi_k \in \widehat{G}$ and $c_1, \ldots, c_k \in \mathbb{C}$ with

$$\overline{M}_N\left(\left|f - \sum_{j=1}^{k} c_{\xi_j}\xi_j\right|\right) < \varepsilon.$$

As usual it is also possible to characterize this by relative denseness of ε-almost periods. More specifically, as discussed in [38] a measurable bounded $f : G \longrightarrow \mathbb{C}$ is Weyl almost periodic if and only if for each $\varepsilon > 0$ there exists a relatively dense set $D \subset G$ and an $N_0 \in \mathbb{N}$ with

$$\overline{M}_n\left(\left|f - f(\cdot - t)\right|\right) < \varepsilon$$

for all $t \in D$ and $n \geq N_0$. By the preceding proposition, validity for all $n \geq N_0$ can be replaced by validity for one n.

Clearly, the set of Weyl almost periodic functions forms an algebra, is closed under uniform limits and under multiplication with Bohr almost periodic functions. Moreover, a crucial feature of Weyl mean almost periodic functions is uniform existence of means.

LEMMA D.2. *Let G be a locally compact abelian group and let (B_n) be a Følner sequence on G. Let $f : G \longrightarrow \mathbb{C}$ be a bounded Weyl almost periodic function. Then, for any sequence (r_n) in G the limit*

$$\lim_{n\to\infty} \frac{1}{|B_n|}\int_{B_n+r_n} f(s)\,ds$$

exists and is independent of the sequence. In particular, the convergence is uniform in the chosen sequence.

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The proof follows the same lines as the proof of the corresponding statement for Besicovitch almost periodic functions in the preceding section. For this reason, we leave it to the reader.

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