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ON THE ORDER OF THE ERROR FUNCTION OF THE (k, r)-INTEGERS—II

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ABSTRACT. In the first part of this note we give a very simple and elegant proof of the theorem on the order of the error function of the (k, r)-integers, which the authors proved earlier using elaborate calculations. We also obtain an improvement of an earlier result on the order of the same error function on the basis of the Riemann hypothesis.

1. **Introduction.** Following the terminology and notation of our paper [4], for given integers k and r with $1<r<k$, we define a (k, r)-integer as a positive integer n of the form $a^k b$ where a and b are integers and b is r-free. Thus n is a (k, r)-integer if and only if its kth power free part is r-free. Also, $Q_{k,r}(x)$ denotes the number of such integers that do not exceed x. $\zeta(r)$ denotes the Riemann zeta function and $\mu(n)$ is the well known Möbius arithmetic function. Let

$$\Delta_{k,r}(x)=Q_{k,r}(x)-x\frac{\zeta(k)}{\zeta(r)}.$$

In [4], we proved the following theorems:

$$\Delta_{k,r}(x)=0(x^{1/r}\delta_r(x)). \tag{1.1}$$

where the 0-constant is independent of x and k, but might depend on r and

$$\delta_r(x)=\exp\{-B_r \log^{3/5} x(\log\log x)^{-(1/5)}\}, \tag{1.2}$$

B_r being a positive constant depending only on r.

On the basis of the Riemann hypothesis,

$$\Delta_{k,r}(x)=0(x^{\beta(k,r)}\omega(x)), \tag{1.3}$$

where again the 0-constant is independent of x and k, but might depend on r and

$$\omega(x)=\exp\{A \log x(\log\log x)^{-1}\}, \tag{1.4}$$

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A being a positive absolute constant and

$$\beta(k, r) = \{2+3\nu(k)\}/\{2r+1+2r\nu(k)\},$$

and $\nu(k)$ is a function of k satisfying

$$\frac{1}{2k+2} \le \nu(k) \le \frac{2}{2k+5} \quad \text{(cf. [4], eq. (2.11)).}$$

The proof of (1.1) given in [4] is long and complicated.

In this note, we first give a very simple and elegant proof of (1.1), showing simultaneously that the 0-constant in (1.1) is not only independent of x and k, but also of r. Next, we improve result (1.3) by proving that

$$\Delta_{k,r}(x) = 0(x^{2/(2r+1)}\omega(x)), \tag{1.5}$$

where the 0-constant is not only independent of x and k, but anlso of r. It is clear that (1.5) is a better result than (1.3) since

$$\frac{2}{2r+1} \le \frac{2+3\nu(k)}{2r+1+2r\nu(k)} \quad \text{for every } k.$$

2. **Proof of (1.1).** Let $Q_r(x)$ denote the number of r-free integers not exceeding x. We use the following estimate due to Walfisz [5, satz 1, p. 192]:

$$Q_r(x) = \frac{x}{\zeta(r)} + 0(x^{1/r}\delta_r(x)), \tag{2.1}$$

where the 0-constant is not only independent of x and k, but also of r. It is Since we have (cf. [4], (2.13))

$$Q_{k,r}(x) = \sum_{n \le \sqrt[k]{x}} Q_r\left(\frac{x}{n^k}\right),$$

it follows that

$$\begin{aligned} Q_{k,r}(x) &= \sum_{n \le \sqrt[k]{x}} \left\{\frac{x}{n^k} \cdot \frac{1}{\zeta(r)} + 0\left(\left[\frac{x}{n^k}\right]^{1/r} \delta_r\left[\frac{x}{{}_nk}\right]\right)\right\} \\ &= x\frac{\zeta(k)}{\zeta(r)} + 0(x^{1/k}) + 0\left(\sum_{n \le \sqrt[k]{x}} \left[\frac{x}{{}_nk}\right]^{1/r-\varepsilon} \left[\frac{x}{{}_nk}\right]^{\varepsilon} \delta_r\left[\frac{x}{{}_nk}\right]\right), \end{aligned}$$

where ε is any number such that $o < \varepsilon < 1/r - 1/k$. It is clear from the expression (1.2) for $\delta_r(x)$ that $x^{\varepsilon}\delta_r(x)$ is a monotonic increasing function of x, and hence $[x/{}_nk]^{\varepsilon}\delta_r[x/{}_nk] \le x^{\varepsilon}\delta_r(x)$. It follows that

$$\begin{aligned} Q_{k,r}(x) &= x\frac{\zeta(k)}{\zeta(r)} + 0(x^{1/k}) + 0\left(x^{\varepsilon}\delta_r(x) \sum_{n \le \sqrt[k]{x}} \left[\frac{x}{{}_nk}\right]^{1/r-\varepsilon}\right) \\ &= x\frac{\zeta(k)}{\zeta(r)} + 0(x^{1/k}) + 0\left(x^{1/r}\,\delta_r(x) \sum_{n \le \sqrt[k]{x}} n^{-k(1/r-\varepsilon)}\right). \end{aligned}$$

Now, since $1 < r < k$, the first 0-term is $0[x^{1/r}\,\delta_r(x)]$ and since $0 < \varepsilon < 1/r - 1/k$, the second 0-term is also $0(x^{1/r}\,\delta_r(x))$. Hence we have

$$Q_{k,r}(x)=x\frac{\zeta(k)}{\zeta(r)}+0(x^{1/r}\,\delta_r(x)).$$

Thus (1.1) is proved, with the 0-constant independent of x, k and r.

3. **Proof of (1.4).** Axer (cf. [1], §7) proved that, under the Riemann hypothesis, we have

$$\Delta_r(x)=0[x^{2/(2r+1)+\varepsilon}],\quad \text{where}\quad \Delta_r(x)=Q_r(x)-x/\zeta(r),$$

for every $\varepsilon>0$. A simple proof of this result using the estimate

$$M(x)=\sum_{n\le x}\mu(n)=0[x^{1/2+\varepsilon}]$$

is given in [2]. Instead of this estimate for $M(x)$, using the better estimate $M(x)=0[x^{1/2}\omega(x)]$, where $\omega(x)$ is given by (1.4), it has been shown (cf. [3], corollary 3.2.1, $n=1$) that

$$\Delta_r(x)=0[x^{2/(2r+1)}\omega(x)],$$

valid under the Riemann hypothesis, where the 0-constant is independent of x and r.

Now, since

$$Q_{k,r}(x)=\sum_{n\le\sqrt[k]{x}}Q_r\left[\frac{x}{n^k}\right],$$

we have

$$\begin{aligned}Q_{k,r}(x)&=\sum_{n\le\sqrt[k]{x}}\left\{\frac{x}{n^k}\cdot\frac{1}{\zeta(r)}+0\left(\left[\frac{x}{n^k}\right]^{2/(2r+1)}\omega\left[\frac{x}{n^k}\right]\right)\right\}\\&=x\frac{\zeta(k)}{\zeta(r)}+0(x^{1/k})+0\left(x^{2/(2r+1)}\omega(x)\sum_{n\le\sqrt[k]{x}}n^{-2k/(2r+1)}\right),\end{aligned}$$

since $\omega(x)$ is a monotonic increasing function of x. Now, since $1<r<k$, we have $2k\ge 2r+1$, so that the first and second 0-terms in the above are $0(x^{2/(2r+1)}\omega(x))$. Thus (1.5) is proved.

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