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AN EXTENSION OF THE FEJÉR-JACKSON INEQUALITY

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Abstract

Best-possible results are established for positivity of the partial sums of $\sum \sin k\theta (k+\alpha)^{-1}$. In fact odd sums are positive for $-1 \le \alpha \le \alpha_0 = 2.1\ldots$, while sums with $2k$ terms are positive on the subinterval $]0, \pi - 2\mu_0\pi(4k+1)^{-1}[$, $\mu_0 = 0.8128\ldots$. This is analagous to the Gasper extension of the Szegö-Rogosinski-Young inequality for cosine sums.

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1. Introduction

It is, of course, well-known that, for every positive integer n,

$$\sum_{k=1}^{n} \frac{\sin k\theta}{k} > 0, \qquad 0 < \theta < \pi. \tag{1}$$

This inequality was first mentioned by Fejér in 1910 (see [3]), was proved by Jackson, and subsequently revisited by many mathematicians who have offered different proofs. In parallel, for cosine series, Young [6] proved the following inequality:

$$1 + \sum_{k=1}^{n} \frac{\cos k\theta}{k} \ge 0, \qquad 0 \le \theta \le \pi. \tag{2}$$

Rogosinski and Szegö [5] extended Young's inequality to

$$\frac{1}{1+\alpha} + \sum_{k=1}^{n} \frac{\cos k\theta}{k+\alpha} \ge 0, \qquad 0 \le \theta \le \pi, \quad -1 < \alpha \le 1. \tag{3}$$

The case $\alpha = 1$ is interesting but Gasper showed in [4] that the result admits considerable improvement. In fact he showed that (3) holds for $-1 < \alpha \le \tilde{\alpha} = 4.567\ldots$, and this $\tilde{\alpha}$ is best possible.

In the present paper, we will extend the Fejér-Jackson inequality in a similar way. Thus we shall be concerned with the partial sums

$$T_n^\alpha(\theta) = \sum_{k=1}^{n} \frac{\sin k\theta}{k+\alpha} \qquad (\alpha > -1,\ n \in \mathbb{N}). \tag{4}$$

The odd partial sums are positive for $-1 < \alpha \le \alpha_0$, where $\alpha_0 = 2.1\ldots$ is best possible. The result for even partial sums holds only on a subinterval $]0, \pi - 2\mu_0\pi(4n+1)^{-1}[$, where $\mu_0 = 0.8128252\ldots$ is best possible.

To make matters precise we must define three constants $\lambda_0, \mu_0, \alpha_0$. The first of these is the solution of the equation

$$(1+\lambda)\pi = \tan(\lambda\pi) \qquad 0 < \lambda < 1/2,$$

and it is easy to see that $\lambda_0 = 0.4302967\ldots$ is the point at which the function $(\sin\lambda\pi)/(1+\lambda)$ attains its maximum for $0 < \lambda < 1/2$. We define μ_0 to be the solution of

$$\frac{\sin\mu\pi}{\mu\pi} = \frac{\sin\lambda_0\pi}{(1+\lambda_0)\pi},$$

and α_0 to be the solution of the equation

$$\sum_{k=1}^{\infty} \frac{2k}{(2k-1+\alpha)(2k+\alpha)(2k+1+\alpha)} = \frac{\sin\lambda_0\pi}{2(1+\lambda_0)\pi}.$$

The main results can now be stated.

THEOREM 1. *If* $-1 < \alpha \le \alpha_0$ *then*

$$T_{2n-1}^\alpha(\theta) > 0, \qquad 0 < \theta < \pi, \quad n \in \mathbb{N}.$$

THEOREM 2. *If* $-1 < \alpha \le \alpha_0$ *then*

$$T_{2n}^\alpha(\theta) > 0, \quad 0 < \theta \le \pi - \mu_0\pi/(2n+0.5).$$

THEOREM 3. *If* $\alpha > \alpha_0$ *then there exists an infinite subset* $N \subset \mathbb{N}$ *such that*

$$T_{2n-1}^\alpha\left(\pi - \frac{(1+\lambda_0)\pi}{2n-\frac{1}{2}}\right) < 0, \qquad n \in N.$$

THEOREM 4. *If* $0 < \gamma < \mu_0$ *then there exists an* α *near to but strictly smaller than* α_0 *such that*

$$T_{2n}^{\alpha}(\pi - \gamma\pi/(2n + 0.5)) < 0$$

for an infinite number of n.

From Theorem 3 we see that α_0 is best possible in Theorem 1. Theorem 4 shows that μ_0 is best possible in Theorem 2.

The particular case $\alpha = 1$ has been considered by Brown and Wilson [2]. They obtained the following conclusion:

$$T_{2n-1}^{1}(\theta) > 0, \quad 0 < \theta < \pi; \qquad T_{2n}^{1}(\theta) > 0, \quad 0 < \theta < \pi - \pi/2n.$$

2. Basic lemmas

LEMMA 1. *For* $n \geq 3$, $\alpha > -1$, *double partial summation gives*

$$T_n^{\alpha}(\theta) = \sum_{k=1}^{n-2} a_k(\alpha)\sigma_k(\theta) + \frac{\sigma_{n-1}(\theta)}{(n-1+\alpha)(n+\alpha)} + \frac{S_n(\theta)}{n+\alpha} \tag{5}$$

where

$$a_k(\alpha) = \frac{2}{(k+\alpha)(k+1+\alpha)(k+2+\alpha)}, \quad S_k(\theta) = \sum_{j=1}^{k} \sin j\theta, \quad \sigma_k(\theta) = \sum_{J=1}^{k} S_j(\theta).$$

LEMMA 2. *For* $0 < \theta < \pi$,

$$\sigma_k(\theta) > 0. \tag{6}$$

LEMMA 3. *Let* $0 < \delta < 2$, $t_n = \delta\pi/(n + \frac{1}{2})$ *and* $\theta_n = \pi - t_n$. *Then*

$$\lim_{n\to\infty} \frac{\sigma_k(\theta_n)}{\sin\theta_n} = \begin{cases} (k+1)/2 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even,} \end{cases} \tag{7}$$

and

$$0 < \frac{\sigma_k(\theta_n)}{\sin\theta_n} < k + 1, \qquad 1 \leq k \leq n, \quad n \geq 7. \tag{8}$$

All these lemmas admit simple direct proofs. We learned of (6) from [1, (1.9) p.8]. The inequality was found by Lukacs and published in Fejér's paper [3].

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3. Proof of Theorems 3 and 4

PROOF OF THEOREM 3. Here we write $\theta_n = \pi - (1+\lambda_0)\pi/(2n-\frac{1}{2})$. Suppose $T^{\alpha}_{2n-1}(\theta_n) > 0$ for all sufficiently large $n \in \mathbb{N}$. Then by (5) we get

(9)
$$\sum_{k=1}^{2n-3} a_k(\alpha)\frac{\sigma_k(\theta_n)}{\sin\theta_n} + \frac{1}{(2n-2+\alpha)(2n-1+\alpha)}\frac{\sigma_{2n-2}(\theta_n)}{\sin\theta_n} > -\frac{1}{2n-1+\alpha}\frac{S_{2n-1}(\theta_n)}{\sin\theta_n},$$

and the right hand side of (9) tends to $(\sin\lambda_0\pi)/2(1+\lambda_0)\pi$ as n tends to infinity.

By Lemma 3 and using the Dominated Convergence Theorem we deduce from (9) that

$$\sum_{k=1}^{\infty} k a_{2k-1}(\alpha) \geq \frac{\sin\lambda_0\pi}{2(1+\lambda_0)\pi}. \tag{10}$$

Noticing that $k a_{2k-1}(\alpha)$ is decreasing when α is increasing, we conclude that (10) is equivalent to $\alpha \leq \alpha_0$. This completes the proof.

PROOF OF THEOREM 4. Write $u_n = \gamma\pi/(2n+\frac{1}{2})$, $0 < \gamma < \mu_0$. For $n \geq 2$ we have

$$\frac{1}{\sin u_n} T^{\alpha}_{2n}(\pi - u_n) = \sum_{k=1}^{2n-2} a_k(\alpha)\frac{\sigma_k(\pi-u_n)}{\sin u_n} + \frac{1}{(2n-1+\alpha)}\frac{\sigma_{2n-1}(\pi-u_n)}{\sin u_n} + \frac{1}{2n+\alpha}\frac{S_{2n}(\pi-u_n)}{\sin u_n}.$$

Let us write $F(\alpha) = \sum_{k=1}^{\infty} k a_{2k-1}(\alpha)$. Then by Lemma 3 and the Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \frac{1}{\sin u_n} T^{\alpha}_{2n}(\pi - u_n) = F(\alpha) - \frac{\sin\gamma\pi}{2\gamma\pi}. \tag{11}$$

As we chose $F(\alpha_0) = (\sin\mu_0\pi)/2\mu_0\pi$, it follows that $T^{\alpha}_{2n}(\pi - u_n)$ is negative for α near to but strictly less than α_0 and n big enough.

4. Proof of Theorem 1

We achieve the proof by developing a sequence of lemmas.

LEMMA 4. *Let*

$$-\frac{d}{d\alpha}T_n^\alpha(\theta) = U_n^\alpha(\theta) = \sum_{k=1}^{n}\frac{\sin k\theta}{(k+\alpha)^2}, \qquad \alpha > -1, \quad n \in \mathbb{N}.$$

If $-1 < \alpha \le 2.5$ *then*

$$U_{2n-1}^\alpha(\theta) > 0, \qquad 0 < \theta < \pi, \quad n \in \mathbb{N}.$$

PROOF. Simple direct estimates for the other cases allow us to assume $n \ge 3$ and $1 < \alpha \le 2.5$. By using partial summation twice we get

$$U_{2n-1}^\alpha(\theta) = \sum_{k=1}^{2n-3} b_k(\alpha)\sigma_k(\theta) + (c_{2n-2}(\alpha) - c_{2n-1}(\alpha))\sigma_{2n-2}(\theta) + c_{2n-1}(\alpha)S_{2n-1}(\theta),$$

where $b_k(\alpha) = c_k(\alpha) - 2c_{k+1}(\alpha) + c_{k+2}(\alpha)$, $c_k(\alpha) = (k+\alpha)^{-2}$.

Noticing

$$S_{2n-1}(\theta) = \frac{\cos\theta/2 - \cos\left(2n - \frac{1}{2}\right)\theta}{2\sin\theta/2} \ge -\frac{1}{2}\tan\theta/4,$$

we get

$$2\cot\frac{\theta}{4}\cdot U_{2n-1}^\alpha(\theta) > (b_1(\alpha) + b_3(\alpha))8\cos^2\frac{\theta}{4}\cos\frac{\theta}{2} - \frac{1}{(2n-1+\alpha)^2} \qquad (n \ge 3).$$

If $0 < \theta \le 0.85\pi$ and $1 \le \alpha \le 2$ then

$$2\cot\frac{\theta}{4}U_{2n-1}^\alpha(\theta) > (b_1(2) + b_3(2)) \times 1.15177 - \frac{1}{36} > 0 \quad (n \ge 3).$$

When $2 < \alpha \le 2.5$ and $0 < \theta \le 0.85\pi$, since

$$b_1(\alpha) + b_3(\alpha) \ge b_1(2.5) + b_3(2.5) = 0.01942$$

we have

$$2\cot\frac{\theta}{4}U_{2n-1}^\alpha(\theta) > 0.01942 \times 1.15177 - \frac{1}{49} > 0 \quad (n \ge 3).$$

On the other hand, for $0.85\pi < \theta < \pi$, we set $t = \pi - \theta$. Then

$$U_{2n-1}^\alpha(\theta) \ge (b_1(\alpha) + b_3(\alpha))\sin t + \frac{1}{(2n-1+\alpha)^2}\frac{\sin t/2 + \sin\left(2n - \frac{1}{2}\right)t}{2\cos t/2}.$$

We see, if $t \le \pi/\left(2n - \frac{1}{2}\right)$, the above sum is strictly positive. Assume $\pi/\left(2n - \frac{1}{2}\right) < t < 0.15\pi$. This occurs only when $n \ge 4$. Then we have

$$\frac{1}{t}U_{2n-1}^\alpha(\pi - t) > (b_1(\alpha) + b_3(\alpha))\frac{\sin 0.15\pi}{0.15\pi} - \frac{0.51421\left(2n - \frac{1}{2}\right)}{(2n-1+\alpha)^2\pi}$$

and we estimate for $1 \le \alpha \le 2$, $2 \le \alpha \le 2.5$ as before.

LEMMA 5. *Let*

$$f(\theta) = a_1(2.11)\sigma_1(\theta) + a_2(2.11)\sigma_2(\theta) + a_3(2.11)\sigma_3(\theta)$$

$$g_n(\theta) = \frac{\sigma_{2n-2}(\theta)}{2n+0.11} + S_{2n-1}(\theta) \quad (n \geq 3).$$

If $0 < \theta \leq 0.75\pi$ *and* $n \geq 3$ *then*

$$T_{2n-1}^{\alpha_0}(\theta) > f(\theta) + \frac{1}{2n+1.11} g_n(\theta) > 0.$$

PROOF. By Lemma 4 we have $T_{2n-1}^{\alpha_0}(\theta) > T_{2n-1}^{2.11}(\theta)$. Then using Lemma 1 we get for $n \geq 3$

$$T_{2n-1}^{\alpha_0}(\theta) > T_{2n-1}^{2.11}(\theta) \geq f(\theta) + \frac{1}{2n+1.11} g_n(\theta).$$

It is easy to establish that $0.0503 \sin\theta$ is a lower bound for $f(\theta)$, for $0 < \theta \leq 0.75\pi$. For $0 < \theta < 0.5\pi$, this can be combined with the lower bound $-0.5 \tan\theta/4$ for $g_n(\theta)$ to establish the result. For $0.5\pi < \theta \leq 0.75\pi$, we use the estimate

$$g_n(\theta) > \left[\frac{2n-1-\operatorname{cosec}\theta}{(2n+0.11)4\sin^2\theta/2} - \frac{\sin^2\theta/4}{\sin\theta/2 - \sin\theta}\right]\sin\theta > -0.3006\sin\theta.$$

The next two lemmas are straightforward estimates based on Lemma 1.

LEMMA 6. *Let* $t = (1+\delta)\pi/5.5$, $0 < \delta \leq 0.375$. *Then* $T_5^{\alpha_0}(\pi - t) > 0$.

LEMMA 7. *Let* $t = (2\ell - 1 + \delta)\pi/(2n - 0.5)$, $\ell \in \mathbb{N}$, $0 < \delta < 1$, $n \geq 4$. *Define*

$$U_n(t) = \frac{\cos^2 t/2 \cdot T_{2n-1}^{\alpha_0}(\pi - t)}{\sin t},$$

$$D_n(t) = \sum_{k=1}^{\infty} k\, a_{2k-1}(\alpha_0) - \frac{\sin\delta\pi}{2(2\ell-1+\delta)\pi} u_n(t) - \frac{1}{2}\sum_{k=1}^{n-2} a_{2k+1} \frac{\sin^2\left(k+\frac{3}{4}\right)t}{\cos t/2} - v_n(t)$$

with

$$u_n(t) = \left(1 - \frac{\alpha_0 - 0.5}{2n-1+\alpha_0}\right)\frac{t}{2\sin t/2}, \quad v_n(t) = \frac{(\alpha_0 - 1)\sin t + \sin(2n-1)t}{(2n-2+\alpha_0)(2n-1+\alpha_0)4\sin t},$$

and

$$\Delta_n(t) = \frac{\alpha_0}{2}\sum_{k=n}^{\infty} a_{2k-1} + \frac{\sin^2 t/4}{2\cos t/2}\sum_{k=1}^{n-2} a_{2k+1} - \sin^2\frac{t}{2} a_1,$$

where $a_k = a_k(\alpha_0)$. *Then*

$$U_n(t) > \Delta_n(t) + D_n(t).$$

LEMMA 8. *If* $n \geq 4$, $0 < \delta \leq 0.55$ *and* $t = (1+\delta)\pi/(2n-0.5)$ *then*

$$D_n(t) > 0.$$

PROOF. By definition of D_n and the fact that $F(\alpha_0) \geq (\sin\delta\pi)/2(1+\delta)\pi$ we get

$$D_n(t)\cos\frac{t}{2} > F(\alpha_0)(1-u_n(t))\cos\frac{t}{2} - v_n(t) - \frac{1}{2}\sum_{k=1}^{n-2} a_{2k+1}\sin^2\left(k+\frac{3}{4}\right)t.$$

Simple power series estimates yield

$$u_n(t) < 1 - \frac{1.6}{2n+1.1} + \frac{0.9880}{(2n-0.05)^2-0.9880}, \tag{12}$$

$$v_n(t) < \frac{0.4053}{(2n+0.1)(2n+1.1)}, \qquad 0<\delta<1, \tag{13}$$

$$\cos\frac{t}{2} \geq 1 - \frac{2.9640}{(2n-0.5)^2}. \tag{14}$$

Combining these inequalities with (crude) direct estimates of the upper bound of $0.5\sum_{k=1}^{n-2} a_{2k+1}\sin^2(k+3/4)t$ for each n from 4 to 12 allows us to verify that $D_n(t) > 0$ for $4 \leq n \leq 12$.

Now we assume $n \geq 13$ and observe that

$$\begin{aligned}
&(2n-0.5)^2 D_n(t)\cos\frac{t}{2}\\
&\quad > F(\alpha_0)\left(3.2n - 4.3640 - \frac{0.5950}{2n+1.11} - \frac{0.9761}{(2n-0.5)^2-0.9880}\right)\\
&\qquad - \frac{(1.55\pi)^2}{8}\sum_{k=1}^{n-2}\frac{1}{k+4.65} - 0.4053\\
&\quad > 0.3476n - 0.8821 - 2.9640\log((n+2.65)/4.65).
\end{aligned}$$

If we write

$$\phi(n) = 0.3476n - 0.8821 - 2.9640\log((n+2.65)/4.65)$$

then we find

$$\phi(13) = 0.0396 > 0, \qquad d\phi/dn > 0 \text{ for } n \geq 13.$$

Hence we conclude $D_n(t) > 0$ for $n \geq 13$ and complete the proof.

LEMMA 9. *If* $n \geq 4$, $0.55 < \delta < 1$ *and* $t = (1+\delta)\pi/(2n-0.5)$ *then* $D_n(t) > 0$.

PROOF. It is easy to verify that

$$\sin\left(\delta\pi - \frac{0.5(1+\delta)\pi}{2n-0.5}\right) > (\alpha_0 - 1)\sin\frac{(1+\delta)\pi}{2n-0.5}$$

for $0.55 < \delta \leq 0.8$ and $n \geq 4$. Hence by (13), $-v_n(t) > 0$ $(0.55 < \delta \leq 0.8, n \geq 4)$.

On the other hand,

$$\frac{\sin\delta\pi}{2(1+\delta)\pi} \leq \frac{\sin 0.55\pi}{2(1.55\pi)} = 0.10142 \qquad (\delta > 0.55).$$

Therefore

$$D_n(t) \geq 0.10862 - 0.10142\, u_n(t) - \frac{1}{2\cos t/2}\sum_{k=1}^{n-2} a_{2k+1}\sin^2\left(k+\frac{3}{4}\right)t \tag{15}$$

when $0.55 < \delta \leq 0.8$ and $n \geq 4$. In this case,

$$\begin{aligned} u_n(t) &< \left(1 - \frac{1.6}{2n+1.11}\right)\left(1 + \frac{(1.8\pi)^2}{24(2n-0.5)^2 - (1.8\pi)^2}\right) \\ &< 1 - \frac{1.6}{2n+1.11} + \frac{1.3324}{(2n-0.5)^2 - 1.3324}. \end{aligned} \tag{16}$$

According to (15) and (16), calculating directly we get

$$\begin{aligned} D_4(t) &\geq 0.00720 + \frac{0.16227}{(2n+1.11)} - \frac{0.13513}{(2n-0.5)^2 - 1.3324} \\ &\quad - 0.53776\left(\alpha_3\sin^2\frac{7}{4}t + a_5\right) > 0.01592, \\ D_5(t) &\geq 0.02029 - 0.52999\left(a_3\sin^2\frac{7}{4}t + a_5 + a_7\right) > 0.01374, \\ D_6(t) &\geq 0.01854 - 0.51550\left(a_3\sin^2\left(\frac{7}{4}\frac{1.8\pi}{11.5}\right) + a_5 + a_7 + a_9\right) > 0.01229. \end{aligned}$$

For $n \geq 7$ we have

$$\begin{aligned} &(2n-0.5)^2 D_n(t) \\ &\quad > 0.00720(2n-0.5)^2 + 0.15099(2n-2.11) - 4.08649\log((n+2.65)/4.65). \end{aligned}$$

It is easy to see this is positive by an argument similar to that in the proof of Lemma 8.

Assume now $0.8 < \delta < 1$. In this case

$$\frac{\sin\delta\pi}{2(1+\delta)\pi} < 0.05197, \qquad u_n < 1, \qquad v_n < 0.00207.$$

Hence

$$D_n(t) > 0.05458 - 5.40181\frac{1}{(2n-0.5)^2}\log\frac{n+2.65}{4.65} > 0.$$

LEMMA 10. *If* $t = (1+\delta)\pi/(2n-0.5)$, $0 < \delta < 1$ *and* $n \geq 4$, *then* $\Delta_n(t) > 0$.

PROOF. Since $2.10 < \alpha_0 < 2.11$, we have

$$\frac{1}{2}\alpha_0 \sum_{k=n}^{\infty} a_{2k-1} > 2.10 \sum_{k=n}^{\infty}(2k+2.11)^{-3}.$$

It turns out that the ensuing estimate

$$\Delta_n(t) > 0.13125(n+1.055)^{-3} + \sin^2\frac{t}{4}\frac{a_3+a_5}{2} - a_2\sin^2\frac{t}{2}$$

leads quickly to the required result.

Lemmas 8-10 treat the case $\ell = 1$. Now we assume $\ell \geq 2$, $t = (2\ell-1+\delta)\pi/(2n-0.5)$, $0 < \delta < 1$. And we keep $t < 0.25\pi$. This restriction requires $n \geq 7$.

LEMMA 11. *If* $t = (2\ell-1+\delta)\pi/(2n-0.5) < 0.25\pi$, $0 < \delta < 1$, *and* $\ell \geq 2$ *then* $T_{2n-1}^{\alpha_0}(\pi - t) > 0$.

PROOF. In this case $u(t) < \pi\,(8\sin\pi/8)^{-1}$, $[\sin\delta\pi/2(2\ell-1+\delta)\pi] < (6\pi)^{-1}$, $v_n(t) < 0.00197$ $(n \geq 7)$ and

$$\frac{1}{2}\sec\frac{t}{2}\sum_{k=1}^{n-2} a_{2k+1}\sin^2\left(k+\frac{3}{4}\right)t < \frac{1}{2}\sec\frac{\pi}{8}\sum_{2}^{\infty} a_{2k-1}.$$

These estimates combined show that $D_n(t) > a_1$, and since $\Delta_n(t) > -a_1$, we achieve the proof.

Lemmas 5-11 now demonstrate that $T_{2n-1}^{\alpha_0}(\theta) > 0$, $0 < \theta < \pi$, for $n \geq 3$. We complete the proof of Theorem 1 by checking the case $n = 2$ directly.

5. Proof of Theorem 2

Because of the many parallels with the proof of Theorem 1 we merely sketch the salient steps. A fuller version of this (and the preceding proofs) is available from the authors on request.

LEMMA 12. *For U_n^α as in Lemma 4, if $-1 < \alpha \le 2.3$ then $U_{2n}^\alpha(\theta) > 0$, $0 < \theta \le \pi - \mu_0\pi/(2n+0.5)$ (recall $\mu \approx 0.8128252$).*

PROOF. We have, in a way similar to the proof of Lemma 4,

$$2\cot\frac{\theta}{4}U_{2n}^\alpha(\theta) > (b_1(\alpha)+b_3(\alpha))8\cos^2\frac{\theta}{4}\cos\frac{\theta}{2} - \frac{1}{(2n+\alpha)^2} \qquad (n \ge 3).$$

So, we get $U_{2n}^\alpha(\theta) > 0$ for $0 < \theta \le 0.85\pi$ and $-1 < \alpha \le 2.3$, $n \ge 3$. On the other hand, when $0.85\pi < \theta \le \pi - \mu_0\pi/(2n+0.5)$ by writing $t = \pi - \theta$ we have

$$U_{2n}^\alpha(\theta) > (b_1(\alpha)+b_3(\alpha))\sin t + \frac{1}{(2n+\alpha)^2}\frac{\sin\frac{t}{2} - \sin(2n+0.5)t}{2\cos\frac{t}{2}},$$

and this leads quickly to a proof. The cases $n = 1, 2$ are handled separately.

Now to establish Theorem 4 we need only prove

$$(17) \qquad T_{2n}^{\alpha_0}(\theta) > 0, \qquad 0 < \theta \le \left(1 - \frac{\mu_0}{2n+0.5}\right)\pi, \qquad n \in \mathbb{N}.$$

By an argument similar to Lemma 5, we get

LEMMA 13. *If $0 < \theta \le 0.75\pi$ and $n \ge 3$ then $T_{2n}^{\alpha_0}(\theta) > 0$.*

Now we assume $\theta = \pi - t$, $\mu_0\pi/(2n+0.5) \le t < 0.25\pi$ and write $V_n(t) = \cos^2\frac{t}{2}(\sin t)^{-1}T_{2n}^{\alpha_0}(\pi - t)$. By a calculation similar to that in the proof of Lemma 6 we obtain

$$(18) \qquad \begin{aligned} V_n(t) > {} & -a_1\sin^2\frac{t}{2} + \sum_{k=1}^{\infty} ka_{2k-1} - \sum_{k=n+1}^{\infty} ka_{2k-1} \\ & + \frac{1}{2\cos t/2}\sum_{k=1}^{n-1} a_{2k+1}\left(\sin^2\frac{t}{4} - \sin^2\left(k+\frac{3}{4}\right)\right) \\ & + \frac{n}{2(2n-1+\alpha_0)(2n+\alpha_0)}\left(1 - \frac{2}{2n+1+\alpha_0}\right) \\ & + \frac{\sin 2nt}{4(2n-1+\alpha_0)(2n+\alpha_0)\sin t}\left(1 - \frac{2}{2n+1+\alpha_0}\right) \\ & + \frac{1}{4(2n+\alpha_0)} - \frac{\sin(2n+0.5)t}{2(2n+\alpha_0)2\sin t/2}. \end{aligned}$$

We write

$$(19) \qquad \begin{aligned} \delta_n = {} & \frac{n}{2(2n-1+\alpha_0)(2n+\alpha_0)}\left(1 - \frac{2}{2n+1+\alpha_0}\right) \\ & - \frac{1}{2}\frac{1}{2n+1+\alpha_0} + \frac{1}{4(2n+\alpha_0)}, \end{aligned}$$

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and note that

$$\delta_n > \frac{0.2225 - 0.7775(n+0.555)^{-1}}{(2n+\alpha_0)(2n+1+\alpha_0)}. \tag{20}$$

We still use the notation Δ_n defined in Lemma 7, and write also

$$G_n(t) = F(\alpha_0) - \frac{\sin(2n+0.5)t}{2(2n+\alpha_0)2\sin\frac{t}{2}} - \frac{1}{2\cos\frac{t}{2}}\sum_{k=1}^{n-1} a_{2k+1}\sin^2\left(k+\frac{3}{4}\right)t$$
$$+ \delta_n + \left(1 - \frac{2}{2n+1+\alpha_0}\right)\frac{\sin 2nt}{4(2n-1+\alpha_0)(2n+\alpha_0)\sin t}.$$

Then we get

$$V_n(t) > G_n(t) + \Delta_{n+1}(t).$$

LEMMA 14. *If* $\mu_0 t/(2n+0.5) \le t < \pi/(2n+0.5)$, $n \ge 3$ *then* $G_n(t) > 0$ *and* $\Delta_{n+1}(t) > 0$.

LEMMA 15. *If* $t = (2\ell\pi + \delta\pi)/(2n+0.5) < 0.25\pi$, $\ell \ge 1$ *and* $0 < \delta < 1$, *then* $V_n(t) > 0$. *In this case,* $n \ge 4$ *and we have* $G_n(t) > 0.02598$ *and* $\Delta_{n+1}(t) > -0.00452$.

A combination of Lemmas 13-15 together with a direct check of low order cases completes the proof of Theorem 2.

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