TENSOR PRODUCTS OF BANACH ALGEBRAS

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Introduction. In [3] Gelbaum defined the tensor product $A \otimes_C B$ of three commutative Banach algebras, $A$, $B$ and $C$ and established some of its properties. Various examples are given and the particular case where $A$, $B$ and $C$ are group algebras of L.C.A. groups $G$, $H$ and $K$ respectively, is discussed there. It is shown there that if $K$ is compact $L_1(G) \otimes_{L_1(K)} L_1(H)$ is isomorphic to $L_1(\hat{K})$ where $\hat{S}$ is L.C.A.

If and only if $L_1(G) \otimes_{L_1(K)} L_1(H)$ is semisimple.

It is the purpose of this paper to extend these results to the case where $K$ is L.C.A. group and to point out the connection between the tensor product and spectral synthesis.

This paper is divided into three sections: section 1 is a collection of definitions and theorems which appear in [3]; section 2 deals with group algebras as a topological module; and in section 3 we discuss the case of the tensor product of group algebras.

1. Preliminaries. Let $A$, $B$ and $C$ be the commutative Banach algebras where $A$ and $B$ are $C$ modules: $\|ac\| \leq \|a\| \cdot \|c\|$, $\|bc\| \leq \|b\| \cdot \|c\|$ for $a \in A$, $b \in B$ and $c \in C$.

We construct the commutative algebra

$$F_{C}(A, B) = \{f: f \in C^{A \times B}, f(a, 0) = f(0, b) = 0, \gamma_1(f)$$

$$= \Sigma \|f(a, b)\| \cdot \|a\| \cdot \|b\| < \infty \}$$

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where addition and multiplication by scalars are defined as usual and where multiplication of two elements \( f_1, f_2 \in F(A, B) \) is defined by

\[
f_1 \ast f_2(a, b) = \begin{cases} \{ \Sigma f_1(a_1, b_1)f_2(a_2, b_2) : a_1a_2 = a, b_1b_2 = b \} & \text{if } \|a\| \cdot \|b\| > 0 \\ 0 & \text{otherwise} \end{cases}
\]

In \( F(A, B) \) we consider the closed ideal \( I \) (with respect to the semi-norm \( \gamma_1 \)) generated by the functions of the following type:

1. \( f(a_1 + a_2, b_1 + b_2) = -f(a_1, b_1) = -f(a_2, b_1) \)
   
   \( f(a, b) = 0 \) otherwise

2. \( f(a_1, b_1 + b_2) = -f(a_1, b_1) = -f(a_1, b_2) \)
   
   \( f(a, b) = 0 \) otherwise

3. \( f(a_1 \phi_1, b_1) = -f(a_1, b_1 \phi_1) \)
   
   \( f(a, b) = 0 \) otherwise

4. \( f(a_1 \phi_1, b_1 \phi_1) = -f(a_1, b_1) \)
   
   \( f(a, b) = 0 \) otherwise

where \( \phi_1 \) represents either a scalar or an element of \( C \).

With the above notations the tensor product \( D = A \otimes_C B \) is defined to be \( F(A, B)/I \): \( D \) is then a commutative Banach algebra with \( \gamma_1 \) as a (quotient) norm. If \( C \) is the complex numbers we obtain the usual tensor product \( A \otimes B \) endowed with the "greatest cross norm" - the "projective tensor product".

As is customary we denote by \( \mathfrak{m}_A, \mathfrak{m}_B, \mathfrak{m}_C \) and \( \mathfrak{m}_D \) the maximal spaces of \( A, B, C \) and \( D \) respectively.

In order to simplify our next theorems we add the following assumption: For every \( (M_1, M_2) \in \mathfrak{m}_A \times \mathfrak{m}_B \) there exist \( a \in A, b \in B, c_1, c_2 \in C \) such that \( ac_1(M_1)b, c_2(M_2) \neq 0 \).

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In the general case the one point compactification of $\mathbb{m}_A$ (resp. $\mathbb{m}_B$) or equivalently the adjunction of module identity is needed.

**Theorem 1.** (i) There are continuous mappings $\mu : \mathbb{m}_A \rightarrow \mathbb{m}_C$, $\nu : \mathbb{m}_B \rightarrow \mathbb{m}_C$ such that, for $(a, b, c) \in A \times B \times C$ and $(M_A, M_B) \in \mathbb{m}_A \times \mathbb{m}_B$,

$$\hat{a}c(M_A) = \hat{a}(M_A)\hat{c}(\mu(M_A)), \quad b\hat{c}(M_B) = \hat{b}(M_B)\hat{c}(\nu(M_B)).$$

(ii) Let $\rho = \mu \times \nu : \mathbb{m}_A \times \mathbb{m}_B \rightarrow \mathbb{m}_C \times \mathbb{m}_C$ and let $\Delta$ be the diagonal of $\mathbb{m}_C \times \mathbb{m}_C$. Then there exists a homomorphism $\tau : \mathbb{m}_D \rightarrow \rho^{-1}(\Delta)$, a locally compact subset of $\mathbb{m}_A \times \mathbb{m}_B$. If $\tau(M_D) = (M_A, M_B)$ and if

$$\mu(M_A) = M_C = \nu(M_B) \text{ then for every } f = f/1 \in D, \text{ with } f(a_n, b_n) = c_n \quad n = 1 \ldots$$

and $f(a, b) = 0$ otherwise,

$$\hat{f}(M_D) = \sum_n \hat{c}_n(M_C)\hat{a}_n(M_A)\hat{b}_n(M_B).$$

**Theorem 2.** Let $\{c\}$ be an approximate identity for $C$. Then $\{c\}$ is also an approximate identity for $A$ if and only if each $a \in A$ is of the form $a_1 c_1$ where $a_1 \in A$ and $c_1 \in C$. Moreover, for $\epsilon > 0$, $a_1$ and $c_1$ can be chosen to satisfy $\|c_1\| = 1$ and $\|a_1 - a\| < \epsilon$.

For the proofs of these theorems as well as several other applications we refer the reader to [1], [3] and [5].

In the next sections we shall denote by $\sum_n c(a_n, b_n)$ and $\sum_n c(\Theta b_n)$ elements of $F(A, B)$ and $D$ respectively.

2. **Group Algebras.** In this section we shall focus our attention upon group algebras with an additional module property. Although some of the results of this section hold for a larger class of multipliers [5] we shall restrict ourselves to the following particular case [3]:

Let $G$ and $K$ be two L.C.A. groups with dual groups $\hat{G}$ and $\hat{K}$ respectively. Let $\theta : K \rightarrow G$ be a (topological) homomorphism of $K$ into $G$ (so that $\theta(K)$ is locally compact and hence $\theta(k)$ is closed [6] and let $\theta^* : \hat{G} \rightarrow \hat{K}$ be the (induced) dual mapping defined by [12]:

$$(\theta(k), a) = (k, \theta^*(a)) \text{ where } a \in \hat{G}.$$

With the above notations we define the "module action" as

$$ac(g) = \int_K a(g - \theta(k))c(k)dk \text{ for } a \in L_1(G), \ c \in L_1(K).$$
Under this definition $L_1(G)$ is an $L_1(K)$ module with $\|ac\| \leq \|a\| \cdot \|c\|$ and $(a_1c)a_2 = (a_1a_2)c$ etc. Indeed, the usual proofs hold here with the obvious modifications.

We now prove several propositions which will be used in the sequel.

**Lemma 3.** Let $a \in \hat{G}$. Then $\hat{a} \hat{c}(a) = \hat{a}(\theta^*(a)) = \hat{a}(\theta^*(a))$ for $a \in L_1(G)$, $c \in L_1(K)$.

**Proof.**

$$\hat{a} \hat{c}(a) = \int \hat{a}(g) \overline{c(g,a)} dg = \int \int a(g - 0(k)) \overline{c(k)(g,a)} dk dg$$

$$= \int \int a(g) \overline{c(k)(g,a)} \theta(k) dk dg$$

$$= \hat{a}(\theta^*(a)) .$$

**Lemma 4.** Let $A^I = \{ \sum_{i=1}^{n} a_i c_i ; a_i \in L_1(G), c_i \in L_1(K) \}$. Then $A^I = L_1(G)$.

Since $A^I$ is a closed ideal in $L_1(G)$ it suffices to show that $A^I$ is not contained in $a$ for every $a \in \hat{G}$. [7, p. 148].

This clearly is the case since $d(a,c) = d(a,c) = 0$.

**Proposition 5.** Let $\{ u \}$ be an approximate identity for $L_1(K)$. Then $a u \to a$ for every $a \in L_1(G)$.

**Proof.** Let $\epsilon > 0$. Choose $a_i \in L_1(G), c_i \in L_1(K), i = 1, \ldots, n$ and $u \in \{ u \}$ such that

$$\| a - \sum_{i=1}^{n} a_i c_i \| < \epsilon / 3 \quad \text{and} \quad \| c_i - c_i u \| < \epsilon / 3 \eta .$$

Then,

$$(\| u \| < 1, \eta > \max \| a_j \| ) 1 \leq j \leq n$$

$$\| a - au \| \leq \| a - \sum_{i=1}^{n} a_i c_i \| + \| \sum_{i=1}^{n} a_i c_i - \sum_{i=1}^{n} a_i c_i u \| + \| \sum_{i=1}^{n} a_i c_i u \| < \epsilon / 3 + \epsilon / 3 + \epsilon / 3 = \epsilon .$$

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PROPOSITION 6. (i) Let \( a \in L_1(G), \ c \in L_1(K) \). Suppose \( c = 1 \) on \( \text{supp}(\hat{a}) \). Then \( a = ac \).

(ii) Let \( a \in L_1(G) \), and let \( \varepsilon > 0 \). Then there exist \( a_1 \in L_1(G), \ c_1 \in L_1(K) \) such that \( \| c_1 \| = 1, \| a - a_1 \| < \varepsilon \) and \( a = a_1c_1 \).

Proof. (i) \( a \hat{c}(\omega) = a\hat{c}(\omega) = \hat{a}(\omega) \hat{c}(\omega) \) by Lemma 3 and our assumption. Hence, since \( L_1(G) \) is semisimple \( ac = a \).

(ii) Theorem 2.

3. Tensor Products by Group Algebras. We now turn our attention to a particular example by a tensor product over a Banach algebra - the case where the algebras involved are group algebras of a locally compact abelian group and where the module action is in accordance with the previous section.

One purpose in the development will be the realization of \( D \) as a group algebra \( L_1(S) \) of a L.C.A., \( S \) constructible from \( G, H \) and \( K \). Another equally important problem will be the semisimplicity of \( D \). It is a well-known open question whether the tensor product of semisimple Banach algebras is again semisimple. In the case of group algebras this is true since \( L_1(G) \otimes L_1(H) = L_1(G \times H) \) [2], [4], and [11]. Yet in the general case this is known to be true if the condition of "monomorphy" (\( A \otimes B \rightarrow A \otimes B \) is 1-1) holds, which is the case if either one of the algebras satisfies the Grothendieck condition of approximation [2], [4], [10], and [11].

To focus our ideas let \( G, H, \) and \( K \) be three L.C.A. groups with dual groups \( \hat{G}, \hat{H}, \) and \( \hat{K} \) respectively. Let \( \theta : K \rightarrow G \) and \( \psi : K \rightarrow H \) be homomorphisms of \( K \) into \( G \) and \( H \) respectively. With the previous definitions the tensor product \( D = L_1(G) \otimes L_1(K) \), \( L_1(H) \) is a well-defined Banach algebra. We first characterize its maximal ideal space - which turns out to be a L.C.A. group \( S \) - and then define a linear mapping \( T : F(S) \rightarrow L_1(S) \) which turns out to be an isomorphism of \( D \) onto \( L_1(S) \) provided \( D \) is semisimple. Several rather powerful theorems are used in this development. Besides Cohen's factorization theorem (Proposition 6) [1], [5] we need Grothendieck's characterization of the tensor product [4] (these are used in showing that \( T \) is surjective) and Calderon's result in spectral synthesis [8].

Some of the ideas involved in this discussion appear in [3]. However, our proof of the semisimplicity of \( D \) is entirely different.

To make our discussion complete we indicate the proofs of several
propositions which appear already in [3].

THEOREM 7. (i) The maps \( \mu : \hat{G} \to \hat{K} \) and \( \nu : \hat{H} \to \hat{K} \) are the duals \( \theta^* \) and \( \psi^* \) of the maps \( \theta \) and \( \psi \).

(ii) \( \tau (\mathcal{M}_D) = \{(\alpha, \beta) : \alpha \in \hat{G}, \beta = \hat{H}, \theta^*(\alpha) = \psi^*(\beta)\} = (\theta^* \times \psi^*)^{-1} \Delta \)

where \( \Delta = \text{diagonal of } \hat{K} \times \hat{K} \). Hence \( \tau (\mathcal{M}_D) \) is a closed subgroup of \( \hat{G} \times \hat{H} \).

(iii) \( \tau (\mathcal{M}_D) = G \times H / \tau (\mathcal{M}_D)^+ \) where + is the annihilator.

(iv) \( \tau (\mathcal{M}_D)^+ = (\theta \times -\psi) \Gamma = Q \) where \( \Gamma = \text{diagonal of } K \times K \).

(v) \( \tau (\mathcal{M}_D) = G \times H / Q \).

(vi) If \( \tilde{z} \in \mathcal{D} \) and \( \tilde{z} = \sum c_m (a_m \otimes b_m) \) then for \( M_D \in \mathcal{D} \), \( \tau (\mathcal{M}_D) = (\alpha, \beta) \) and \( \theta^*(\alpha) = \psi^*(\beta) = \gamma \).

Proof. (i), (ii) and (vi) follow from Theorem 1, (iii) follows from (ii) by duality and (v) follows from (iv) by duality. To prove (v) we first note that \( Q \) is closed since its locally compact group in the relative topology (the mappings are open) [6].

Next we show that \( Q \subset \tau (\mathcal{M}_D)^+ \). Indeed, for \( g = \theta(k), h = -\psi(k), k \in K \) and \( (\alpha, \beta) \in \tau (\mathcal{M}_D) \), we have

\[
(g, \alpha)(h, \beta) = (\theta(k), \alpha)(-\psi(k), \beta) = (k, \theta^*(\alpha))(k, \psi^*(\beta)) = (k, \gamma)(k, \gamma) = 1
\]

by (ii) and (iii).

Finally, let \( (\alpha^o, \beta^o) \in Q^+ \) then \( 1 = (\theta(k), \alpha^o)(-\psi(k), \beta^o) \); hence \( \theta^*(\alpha^o) = \psi^*(\beta^o) \). Hence \( (\alpha^o, \beta^o) \in \tau (\mathcal{M}_D) \) (by (ii)) and this completes the proof since \( Q \) is closed.

Let \( T \) be the linear operator defined on the functions in \( F L_1(K) \) with finite support and values in \( L_1(G \times H / Q) \);

\( T \) is defined by

\[
Tc(a, b)(g, h) = \int_{\mathcal{M}_D} a(g - \epsilon(k_1, k_2))c(k_1)b(h + \psi(k_2))dk_1dq_2
\]

= \( \int_{\mathcal{D}} ac(g - \epsilon(k_2))b(h + \psi(k_2))dq_2 \).
where $dq$ represents the Haar measure on $Q = (\mathfrak{c} x - \psi) \text{ diag}(K \times K)$ and $(g, h)$ represents the coset $(g, h) + Q$. By a proper choice of the Haar measures we have that the mapping $F \mapsto \int_Q F((g, h) + q)dq$ is surjective and

$$\int_{G \times H} \int_Q F((g, h) + q)dq \, dg \, dh = \int_{G \times H} F(g, h) \, dg \, dh,$$

[7], [12].

**PROPOSITION 8.** (i) $T$ is bounded, $\|Tf\| \leq \gamma_1(f)$.

(ii) $T[1] = 0$.

(iii) $T$ is multiplicative on $D$ where $T$ denotes the induced mapping by (ii).

(iv) $T: D \to L_1(G \times H/Q)$ is surjective.

(v) $T$ is isomorphic.

**Proof.** (i), (ii) and (iii) are straightforward.

(iv) is a consequence of Propositions 6 and the isometric isomorphism between $L_1(G) \otimes L_1(H)$ and $L_1(G \times H)$. Indeed, let $\Sigma a_n b_n \in L_1(G \times H)$; write $a_n = a c_n$ and consider $\Sigma c_n (a_n \otimes b_n) \in D$ by proper choice of $a_n$ and $c_n$. Then

$$T \Sigma c_n (a_n \otimes b_n) = \int_Q a_n c_n (g^{-0}(k_2)b_n (h + \psi(k_2)) dq_2$$

which is surjective.

(v) One half is obvious. The second follows directly from the identity $\tilde{z}(M_D) = \tilde{z}(\alpha, \beta)$ where $(\alpha, \beta) \leftrightarrow M_D$ (Theorem 7).

We include a detailed proof of this identity since the involved computations are typical. To this end let $d\sigma = dq \, dh$; then
\[
\sum_{\alpha} a_\alpha = \sum_{\alpha} a_{\alpha} \quad \text{and} \quad \sum_{\beta} b_\beta = \sum_{\beta} b_{\beta}
\]

In order to simplify some of the statement, we introduce the following definitions:

Let

\[S = \{ (\alpha, \beta, \gamma) ; \alpha \in \hat{G}, \beta \in \hat{H}, \gamma \in K, \theta^*(\alpha) = \psi^*(\beta) = \gamma \} .\]

A cube is a set of the form \(E = E_\alpha \times E_\beta \times E_\gamma\) where \(E_\alpha, E_\beta,\) and \(E_\gamma\) are subsets of \(\hat{G}, \hat{H}\) and \(K\) respectively and \(E \cap S = \emptyset\). An element \(X = \sum_{i=1}^m a_i b_i c_i\) of \(L_4(G \times H \times K)\) where \(a_i \in L_1(G), b_i \in L_1(H),\) \(c_i \in L_1(K), i = 1, \ldots, n\) will be called a generator. A term \(abc\) will be a component of the generator.

**Lemma 9.** (i) \(S\) is a closed subgroup of \(\hat{G} \times \hat{H} \times \hat{K}\).

(ii) If \(E = E_\alpha \times E_\beta \times E_\gamma\) is a cube then \(\theta^*(E_\alpha) \cap \psi^*(E_\beta) \cap E_\gamma = \emptyset\).

**Proof.** (i) If \(\lambda = (\alpha, \beta, \gamma)\) does not belong to \(S\) then for \(\theta^*(\alpha) = \gamma_1 \neq \gamma, \psi^*(\beta) = \gamma_2 \neq \gamma\) we choose two disjoint neighbourhoods (in \(K\)) \(V_\gamma\) of \(\gamma\) and \(V_{\gamma_1}\) of \(\gamma_1\). Then \(\theta^{-1}_g(V_\gamma) \times \hat{H} \times V_{\gamma_1}\) is a cube neighbourhood of \(\lambda\).

(ii) (By contradiction) If \(\lambda \in \theta^*(E_\alpha) \cap \psi^*(E_\beta) \cap E_\gamma\) then \(\gamma = \theta^*(\alpha) = \psi^*(\beta)\) and \((\alpha, \beta, \gamma) \in S \cap E\).

**Lemma 10.** Let \(f \in L_4(G \times H \times K)\) with \(f = 0\) on \(S\). Then for arbitrary \(\epsilon > 0\) there exists a generator \(z\) with components \(z_i, i = 1, \ldots, L = L(\epsilon, s, t)\) such that
(i) \( \{ \text{supp} \hat{z}_i \} \) are compact cubes;

(ii) \( \| z-f \| < \epsilon \).

Proof. Without loss of generality we may assume that the support of \( \hat{f} \) is a compact set disjoint from \( S \). For, by Calderon's Theorem, [1], \( S \) is a spectral set and the usual triangle inequality completes the argument.

Let \( \text{supp} \hat{f} = \Lambda \) be a compact set disjoint from \( S \). Choose generators \( x \) and \( y \) such that \( \hat{x} = 1 \) on \( \Lambda \) and the \( \text{supp} \hat{x}_i \) are compact cubes where \( \hat{x}_i \) are the components for \( x \) for \( i = 1, \ldots, n \) and \( \| y-f \| < \epsilon / \| x \| \). Then \( z = x \ast y \) satisfies the lemma.

LEMMA 11. Let \( \hat{f} = \sum (a \otimes b) \in L^1(G) \otimes L^1(K) \). Then \( f = \sum a \otimes b \in L^1(G \times H \times K) \). If \( \hat{f} \equiv 0 \) then \( f = 0 \) on \( S \).

Proof. \( \| f \| \leq \sum \| a_n \| \| b_n \| \| c_n \| = \gamma \sum c_n (a_n \otimes b_n) < \infty \).

Also, \( f(\alpha, \beta, \gamma') = \sum a_n b_n c_n (\alpha, \beta, \gamma') = \sum a_n b_n c_n (\alpha, \beta) c_n (\gamma') = f(\alpha, \beta, \gamma) \) by Theorem 7.

LEMMA 12. Let \( \hat{f} = c(a \otimes b) \in L^1(G) \otimes L^1(K) \). Let \( \phi^* \) \( (\text{supp} \hat{a}) \), \( \phi^* \) \( (\text{supp} \hat{b}) \), \( \text{supp} \hat{c} \) be compact subsets of \( \hat{K} \). Then, if \( \phi^* \) \( (\text{supp} \hat{a}) \cap \phi^* \) \( (\text{supp} \hat{b}) \cap \text{supp} \hat{c} = \phi \), \( \hat{f} = 0 \).

Proof. Choose \( V_1, V_2, V_3 \) neighbourhoods of \( \phi^* \) \( (\text{supp} \hat{a}) \), \( \psi^* \) \( (\text{supp} \hat{b}) \), \( \phi \) \( \text{supp} \hat{c} \) respectively such that \( V_1 \cap V_2 \cap V_3 = \phi \). Choose local identities \( c_1, c_2, c_3 \in L^1(K) \) such that \( \hat{c}_i = 0 \) outside of \( V_i \), \( i = 1, 2, 3 \) and \( \hat{c}_1 = 1 \) on \( \phi^* \) \( (\text{supp} \hat{a}) \), \( \hat{c}_2 = 1 \) on \( \phi^* \) \( (\text{supp} \hat{b}) \), \( c_3 = 1 \) on \( \text{supp} \hat{c} \). Now \( V_1 \cap V_2 \cap V_3 = \phi \) implies \( c_1 c_2 c_3 = 0 \) whence \( c(a \otimes b) = c_1 c_2 c_3 (a \otimes b) = 0(a \otimes b) = 0 \).

COROLLARY. Let \( z \) be a generator with components \( z_i = a_i b_i c_i \), \( i = 1, \ldots, n \). Let \( \text{supp} \hat{z}_i \) be a compact cube. Then \( z = \sum c_i (a_i \otimes b_i) = 0 \).

Proof. By Lemma 9, \( \phi^* \) \( (\text{supp} \hat{a}_i) \cap \psi^* \) \( (\text{supp} \hat{b}_i) \cap \text{supp} \hat{c}_i = \phi \), \( i = 1, \ldots, n \). Hence, by Lemma 11 and the compactness of \( \phi^* \) \( (\text{supp} \hat{a}_i) \), etc., we get the required result.
THEOREM 13. $\mathbf{D}$ is semisimple.

Proof. Let $\bar{y} = \sum_{n} c_n (a_n \otimes b_n)$ be such that $\bar{y} \neq 0$. Consider, in accordance with Lemma 11, $y = \sum_{n} a_n b_n c_n \in L^1(G \times H \times K)$. Let $\epsilon > 0$.

By Lemma 10 there exists a generator $z$ with components $z_i$, $i = 1, \ldots, L$ such that $\text{supp } z_i$ are compact cubes and $\|z - y\| < \epsilon$. By the previous corollary $\sum c_i (a_i \otimes b_i) = 0$. On the other hand we have that $\gamma_1 (\bar{y} - \sum c_i (a_i \otimes b_i)) \leq \|y - z\| < \epsilon$. Hence $\gamma_1 (\bar{y}) < \epsilon$, and $\mathbf{D}$ is semisimple.

REFERENCES


12. A. Weil, L'intégration sans les groupes topologiques et ses
    application. (Hermann, Paris 1953).

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