PART I

STABILITY, N- AND 3-BODY PROBLEMS, VARIABLE MASS

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STABILITY OF THE SOLAR SYSTEM

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ABSTRACT

This paper reviews the present status of research on the problem of stability of satellite and planetary systems in general. In addition new results concerning the stability of the solar system are described. Hill's method is generalized and related to bifurcation (or catastrophe) theory. The general and the restricted problems of three bodies are used as dynamical models. A quantitative measure of stability is introduced by establishing the differences between the actual behavior of the dynamical system as given today and its critical state. The marginal stability of the lunar orbit is discussed as well as the behavior of the Sun-Jupiter-Saturn system. Numerical values representing the measure of stability of several components of the solar system are given, indicating in the majority of cases bounded behavior.

INTRODUCTION

No more appropriate subject than the unsolved problem of the stability of the solar system may be contributed to a volume honoring Professor Y. Hagihara. His fundamental contributions to celestial mechanics underline not only the importance of this subject but point out the basic difficulties encountered such as the non-uniform convergence of series solutions and the non-existence of sufficient number of uniform integrals. Indeed, the semi-convergent series as shown by Poincaré (1892-1899), may be truncated and give solutions for certain intervals of time with excellent accuracy, but these series are of no use when solutions in the domain $-\infty \leq t \leq \infty$ are to be studied. Numerical integrations give meaningful results for limited periods of time, but, once again, their applicability to stability problems raises serious questions. Modern, powerful topological theories by Kolmogorov, Arnold and Moser, while representing significant advances in the theory of dynamical systems are not applicable to the stability of the solar system because the actual perturbations are above the limits set by the theory.

R. L. Duncombe (ed.), Dynamics of the Solar System, 7–15.

With Hagihara (1970-1976), we may state that if the trigonometric series used in celestial mechanics are not uniformly convergent then they do not represent the solution of the differential equations describing the solar system. On the other hand, if these series were convergent and would contain no secular terms, the solar system's stability would be established by the classical methods of general perturbation theory.

A complete solution would be available if sufficient number of uniform integrals would exist. Once again, the non-existence of such integrals was shown by Poincaré (1892-1899) and by Bruns (1887). The use of existing integrals allows the reduction of the order of the problem as well as allows the establishment of certain topological properties of the manifold of the motion. This approach originally proposed by Hill (1878) is one of the few methods which, with all its disadvantages, may be used in a precise, mathematically well formulated and exact manner to establish - if not the stability - at least certain boundaries of the motion.

The basic problem of the stability of the solar system may be stated in a variety of ways. Hagihara's (1957) unambiguous and clear mathematical formulation is: "What is the interval of time at the end of which the solar system deviates from the present configuration by a previously assigned small amount?"

Professor Hagihara's basic question has only partial answers today. An introductory, modern, non-mathematical treatment of the subject of the stability of the solar system is available elsewhere (Jefferys and Szebehely, 1978).

THREE APPROACHES TO THE PROBLEM OF STABILITY

In this section three fundamental approaches to stability are shortly described and their advantages and disadvantages are listed.

(A) The method applicable to the solar system and offering mathematically precise results is the previously mentioned method of Hill (1878). It uses integrals of the motion and was employed by Hill to study the stability of the lunar orbit. The method is also applicable to the model of the restricted problem (Szebehely, 1967) and it was generalized to other dynamical systems such as the general problem of three bodies. Between 1968 and 1977, a series of papers appeared in which the topology of the general problem of three bodies are discussed by Golubev (1968), Smale (1970), Marchal and Saari (1975), Bozis (1976), Zare (1977) and Szebehely (1977).

The ideal in its simplest form may be expressed as:

$$V^2 = V(x,y)-C \ ,$$

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which for a two-degrees of freedom dynamical system represents the energy integral with V(x,y) the potential, v the velocity and C the constant of integration, determined by the initial conditions. For a given value of C motion is possible in those regions of the plane (x,y) for which V(x,y) > C. In this way boundaries of possible motions may be established by a study of the topology of the curves C = V(x,y). Values of C associated with topological changes are bifurcation of critical values.

The greatest advantage of the method is that it is an exact, nonlinear analysis. Consequently, it's free of the criticisms raised against the method of small disturbances, periodic or quasi-periodic motions have no importance and the fact that it is also devoid of any reference to commensurabilities may be considered an advantage because of the ambiguous role of resonances in celestial mechanics. Another great advantage of the method is its applicability to non-integrable dynamical systems, such as the gravitational problem of n bodies. On the other hand, its greatest disadvantage is that it gives partial results only and at best it establishes upper or lower boundaries of the motion. The importance of these limitations calls for an explanation by two simple examples. The "stability" of the lunar orbit proved by Hill may be formulated precisely as follows: if the dynamical model adopted by Hill is accepted to describe the motion of the Moon, then the Moon may not leave the Earth and may not become an (independent) planet of the Sun. Note that this result does not mean that the Moon cannot collide with the Earth and that all Hill's assumptions must be dynamically correct. In this example Hill meant by "stability" the existence of an upper bound of the Moon's distance from the Earth.

The second example is the application of the method to the stability of Saturn in the Sun-Jupiter-Saturn system. The meaning of "stability" now becomes quite different. If the assumptions of the (circular) restricted problem are accepted then Saturn can not penetrate the region occupied by the Sun and by Jupiter, i.e., it can not become an interior planet or it can not form a binary system with Jupiter. But, Saturn may escape the system. Therefore, by "stability" we mean a lower bound for Saturn's distance from the Sun. Both examples lack two bounds. The lunar distance has no lower bound and the orbit of Saturn has no upper bound. Both examples ignore other influences such as tidal effects, resonance conditions and perturbations by other planets. Nevertheless, the limited results of these examples, accepting the validity of the dynamical models are exact and no mathematical approximations are involved.

As the degree of freedom of the dynamical system increases the results remain exact but their significance becomes more limited. If the model of the restricted problem is not applicable and the general three-body problem must be used, the topology of the boundary surfaces becomes rather complicated and in few cases may we offer statements concerning boundedness or stability of general validity. For instance, with our present day knowledge regarding the existence of uniform integrals of the three-body problem we find that <u>all</u> boundary surfaces are open to infinity, making escape always possible. Direct exchange, on the other hand, between the roles played by the three bodies, is not always possible.

A distinct advantage of Hill's method is that it may be used to define a measure of stability via bifurcation theory. When the controlling parameters of the system are such that in its present state the system is far removed from the nearest bifurcation then a strong stability exists. On the other hand, when slight changes in the dynamic parameters describing the system throw it into bifurcation, its stability is low. A measure of stability $S = (s_{ac}-s_{cr})/s_{cr}$ was introduced by Szebehely (1977) indicating the differences between the present (actual) system and its nearest bifurcation value. This parameter is the Jacobian constant s=C for dynamical systems described by the restricted problem and it is the quantity $s = -(c^2h)/G^2m^5$ for the general three-body problem. (Here c is the angular momentum, h the total energy, G the constant of gravity and m the average mass). The subscripts "ac" and "cr" refer to the actual or present value and to the critical or bifurcation value respectively. It may be shown for the restricted problem that $C_{cr} \geq 3$ and that for the general problem $s_{cr} \neq 0$ as long as two of the three bodies have non-zero mass. The normalization is quite arbitrary as is the definition used for C and h. Nevertheless, accepting the same definition for all examples offers meaningful comparisons regarding the measures of stability. Note that if $S > 0$ the dynamical system is removed from its nearest possible bifurcation. When $S < 0$ the possibility of changing to a different type of motion exists but will not necessarily occur.

(B) Stability investigations by numerical methods is the second basic approach. A great variety of possibilities exist along these lines and with present day high speed electronic computers, interesting experimental results became available in the past few years. The stability of numerically established periodic and quasi-periodic planetary and satellite type orbits give indication of the long-range behavior of the system. Because of the numerical nature of these investigations, they are approximate and their precise mathematical validity is always questionable. Using the restricted problem Hénon (1970) studied the linear stability of periodic and quasi-periodic orbits as well as their global, non-linear behavior using the method of the surfaces of section (Hénon and Heiles, 1964). These results reveal the existence of stable retrograde orbits, far removed from the primaries. Similar numerical investigations for planetary type three and many-body periodic orbits, performed by Hadjidemetriou (1976), indicate linear stability of the solar system using the model of the general problem of three bodies.

It is known that when the participating masses in the general problem of three bodies are of the same order of magnitude, random initial conditions result, most of the time, in escapes even for negative total energy (Szebehely, 1971). Such unstable motions contradict the generally found stable behavior of planetary type motions when the mass of one

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body dominates. Kuiper (1973), therefore, recommended that increased masses for the planets be used in numerical experiments and an artificial instability be created. With increased masses a shorter numerical integration time is expected to determine the behavior of the system. Numerical integrations have shown (Nacozy, 1976) no secular trends in the motion of the Sun-Jupiter-Saturn system until the masses of Jupiter and Saturn were increased 25 times of their actual value. The sudden appearance of instability with increased masses is an indication of the stability of the system with its present (actual) masses. This method is known as the Kuiper-Nacozy-Szebehely (K-N-S) theory. Note that Hill's method gives a factor of 15 instead of 25 for the onset of possible instability, (Szebehely and McKenzie, 1977).

Other long-time numerical integrations seem to show stability for over 10^7 years. These integrations do not include the inside planets, and do not offer analytical proof of stability inspite of their unquestionably ingenious techniques and reliable results (Birn (1973), Cohen, Hubbard and Oesterwinter (1968)).

(C) The third approach to stability investigations is the classical general perturbation method. Once again no mathematical results exist which would show long-time behavior because of the previously mentioned divergence of the series. Until new, uniformly convergent series solutions become available, the various averaging techniques, truncations, secular perturbations, etc., will furnish results useful for finite time only. Sundman (1912), has shown the existence of the solution of the pertinent differential equations. The radius of convergence of his series is so small that not even a general pattern of dynamical behavior emerges, not to mention stability properties. Message's (1978) recent proof of the non-existence of secular terms to any order in the formal solution for the semi-major axis is an important result considering formal solutions. An analytical approach along these lines is the K-A-M (Kolmogorov (1954), Arnol'd (1963), Moser (1973)) theory which states that if the perturbations to a soluble (two-body) problem are small enough, the commensurabilities are of high enough order and certain continuity conditions are satisfied, then quasi-periodic solutions exist. Unfortunately, for the solar system the first condition is not satisfied and, therefore, the KAM theory is not applicable.

STABILITY MEASURES FOR THE SOLAR SYSTEM

In this section, the results of calculations are given according to the first approach (A). The satellite systems of the solar system are discussed first, followed by remarks on the planetary system itself. The numerical values given are based on the most recent astronomical constant accepted by the International Astronomical Union in 1976. Table I. lists values of S for the satellite systems.

PLANET	SATELLITE	STABILITY	PLANET	SATELLITE	STABILITY
Earth	Moon	0.00015	Saturn	Mimas	0.72023
Mars	Phobos	0.00254		Enceladus	0.56154
	Deimos	0.00098		Tethys	0.45187
Jupiter	V	1.33570		Dione	0.35133
	Io	0.56521		Rhea	0.25030
	Europa	0.35058		Titania	0.10458
	Ganymede	0.21478		Hyperion	0.08531
	Callisto	0.11663		Iapetus	0.03221
	XIII	0.0106		Phoebe	0.00313
	VI	0.00988	Uranus	Miranda	0.33493
	VII	0.00944		Ariel	0.21569
	X	0.00925		Umbriel	0.15462
	XII R	-0.00536		Titania	0.09358
	XI R	-0.00612		Oberon	0.06939
	VIII R	-0.00656	Neptune	Triton	0.21629
	IX R	-0.00666		Nereid	0.01206

Table I. Stability of Satellites in the Solar System

The model of the restricted problem is used with the three bodies being the Sun, the planet and its satellite. The assumptions of the restricted problem are acceptable for this case since the satellites' effects of the primaries are less than approximately 1% when compared to the effects of the primaries on each other. (The only exception is of Neptune's Triton). Attention is directed to the low stability of the Earth's moon. This small positive number ($S = 1.5 \times 10^{-4}$) becomes negative ($S = -2.75 \times 10^{-4}$) if the model of the general problem of three bodies is used (Szebehely and McKenzie, 1977), indicating a borderline case of bifurcation. The four outer retrograde satellites of Jupiter are below the bifurcation value ($S < 0$) suggesting the possibility of capture-origin.

The discussion of satellite stability in the solar system may be closed by mentioning several approaches to the problem of finding the limiting orbital radii of natural or artificial satellites around the planets of the solar system. Such formulae were given among others by Hagihara (1952), Kuiper (1953), and Szebehely (1978).

Regarding planetary stability, the dynamical situation is considerably more complicated. Table II. shows the results of the bifurcation computations, assuming the model of the restricted problem and using the Sun and Jupiter as primaries. This Table needs careful interpretation. The numbers decrease as the planetary orbit approaches Jupiter's orbit and the minimum is shown for Saturn. Uranus, Neptune and Pluto show increasing numbers since they are farther removed from Jupiter's orbit.

MERCURY	VENUS	EARTH	MARS	SATURN	URANUS	NEPTUNE	PLUTO
3.60	1.61	1.00	0.48	0.07	0.35	0.64	0.85

Table II. Stability of Sun-Jupiter-Planet Systems.

Table III. shows more appropriate combinations from a dynamical point of view. Since the principal effect in the secular variation in the mean motion on Mercury is by Venus, on Venus is by the Earth, on the Earth by Jupiter and on Mars by the Earth, Table III. shows also the corresponding stability measures.

PRIMARIES	SUN-MERCURY	SUN-VENUS	SUN-EARTH
PLANET	VENUS	EARTH	MARS
STABILITY	0.110	0.027	0.041

Table III. Planetary Stabilities in the Solar System.

Studies performed regarding the effect of Jupiter's orbital eccentricity e_J on the stability of planetary orbits shows that the increase of e_J has a destabilizing effect. The same is true if the values used for the planetary masses are increased and the general problem is used as the underlying dynamical model.

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DISCUSSION

Kiang: I would like to suggest a new approach to questions of stability, less sophisticated than the topological method. As soon as we write Hill's equation for a closed orbit (in phase space, in general), we can define its stability in terms of the solution of Hill's equation. In this way, I was able to show that a typical Hilda asteroid (at the 3/2 commensurability) is stable while a Hecuba asteroid (at the 2/1 point) with the same eccentricity is unstable. In this particular problem it is possible to write down Hill's equation only after radically departing from the classical treatment of the virtual displacement.

Szebehely: Thank you.