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EIGENVALUES OF SMOOTH POSITIVE DEFINITE KERNELS

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For positive definite C^1 kernels on a finite real interval the eigenvalues λ_n are known to be $o(1/n^2)$. In this paper this result is shown to be best possible in the best possible sense, namely that, given any decreasing sequence λ_n which is $o(1/n^2)$, there exist positive definite C^1 kernels whose eigenvalues are λ_n.

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1. Introduction

We showed in [4] that for any real positive sequence λ_n for which $n^2\lambda_n$ decreases and tends to zero there exist positive definite C^1 kernels whose eigenvalues are λ_n. We now prove a slightly stronger result namely that if λ_n *itself* decreases and if $n^2\lambda_n \to 0$ then there exist positive definite C^1 kernels whose eigenvalues are λ_n. The proof is based on an idea of Chaundy and Jolliffe originally given in [1] and later reproduced in [5].

2. Trigonometric series

We need some results on trigonometric sine series. The following estimates on partial sums are crucial.

Lemma 1. *For all integers* $N \geqq 1$ *and for all real* t *in the interval* $0 < t < 2\pi$

(i) $\left|\sum_1^N \sin nt\right| \leqq \dfrac{1}{\sin t/2}$,

(ii) $\sum_1^N n \sin nt = O\left(\dfrac{N}{\sin t/2}\right)$.

Proof. Let us write

$$s_N(t) = \sum_1^N \sin nt.$$

Then we have

$$|s_N(t)| = \left|\sum_1^N \sin nt\right|$$

$$\leqq \left|\sum_1^N e^{int}\right|$$

$$= \left|\frac{e^{it} - e^{i(N+1)t}}{1 - e^{it}}\right|$$

$$\leqq \frac{1}{\sin t/2}.$$

Also

$$\sum_1^N n \sin nt = s_1(t) + \sum_2^N n(s_n(t) - s_{n-1}(t))$$

$$= -\sum_1^{N-1} s_n(t) + N s_N(t).$$

Hence

$$\left|\sum_1^N n \sin nt\right| \leqq \frac{2N-1}{\sin t/2}$$

$$= O\left(\frac{N}{\sin t/2}\right).$$

Lemma 2. *If λ_n is a real positive decreasing sequence such that $n^2\lambda_n \to 0$ then*

$$\sum_1^\infty \lambda_n \cos nt$$

is C^1 for all real t.

Proof. The series clearly converges (absolutely) for all t so it is sufficient to show the differentiated series

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$$-\sum_{1}^{\infty} n\lambda_n \sin nt$$

is uniformly convergent over all t.

Suppose $\varepsilon > 0$ is given. Let N be chosen such that $n^2\lambda_n < \varepsilon$ for all $n > N$. Consider the sum

$$\sum_{p}^{q} n\lambda_n \sin nt$$

where p, q are both $> N$ and $0 < t < \pi$. We split this sum into two parts

$$\sum_{p}^{q} = \sum_{p}^{k} + \sum_{k+1}^{q}$$

where k (depending on t) is chosen such that

$$kt \leqq \pi < (k+1)t.$$

Either of these smaller sums may be empty of course.

The first sum can be estimated as follows. Using the inequality $\sin\theta < \theta$ for $\theta > 0$ we have

$$\begin{aligned}\sum_{p}^{k} n\lambda_n \sin nt &< \sum_{p}^{k} \varepsilon t \\ &\leqq \sum_{p}^{k} \varepsilon\pi/k \\ &\leqq \varepsilon\pi.\end{aligned}$$

Estimating the second sum is not quite so easy. If we write

$$\sigma_N(t) = \sum_{1}^{N} n \sin nt$$

then, using the inequality $\sin\theta > 2\theta/\pi$ for $0 < \theta < \pi/2$, we have from Lemma 1

$$\begin{aligned}\sigma_N(t) &= O\left(\frac{N}{\sin t/2}\right) \\ &= O(N\pi/t)\end{aligned}$$

$$=O(N(k+1))$$

$$=O(Nk).$$

Also

$$\sum_{k+1}^{q} n\lambda_n \sin nt = \sum_{k+1}^{q} \lambda_n(\sigma_n(t)-\sigma_{n-1}(t))$$

$$=\lambda_{k+1}\sigma_k(t)+\sum_{k+1}^{q-1}(\lambda_n-\lambda_{n+1})\sigma_n(t)+\lambda_q\sigma_q(t).$$

Therefore

$$\sum_{k+1}^{q} n\lambda_n \sin nt = O\left(\lambda_{k+1}k^2+\sum_{k+1}^{q-1}(\lambda_n-\lambda_{n+1})nk+\lambda_q qk\right)$$

$$=O\left(\lambda_{k+1}(2k+1)k+\sum_{k+2}^{q}\lambda_n k\right)$$

$$=O\left(\varepsilon+\sum_{k+2}^{q}\varepsilon k/n^2\right)$$

$$=O(\varepsilon)$$

since

$$\sum_{k}^{\infty} 1/n^2 = O(1/k).$$

3. The kernels

Lemma 2 above enables us to construct smooth kernels with prescribed eigenvalues in the following way.

Theorem 1. *If λ_n is real positive decreasing and $n^2\lambda_n \to 0$ then the kernel*

$$K(x,t)=\sum_{1}^{\infty}\lambda_n \cos n\pi x \cos n\pi t$$

$$=\frac{1}{2}\sum_{1}^{\infty}\lambda_n(\cos n\pi(x+t)+\cos n\pi(x-t))$$

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is C^1 *and the operator*

$$Tf(x)=\int_{-1}^{1} K(x,t)f(t)\,dt$$

on $L^2[-1,1]$ *has eigenvalues* λ_n.

The same idea can be used to construct positive definite C^p kernels (p odd) with eigenvalues any given λ_n positive decreasing such that $n^{p+1}\lambda_n \to 0$. For p even one has to have

$$\sum_1^\infty n^p \lambda_n < \infty$$

(see [4]) which makes the proof that the corresponding kernel is C^p relatively trivial.

Regrettably this idea cannot be used to construct a symmetric C^1 kernel with eigenvalues any given λ_n decreasing in modulus such that $n^{3/2}\lambda_n \to 0$ (see [3]) since no symmetric C^1 kernel can have eigenvalues e.g. $|\lambda_n| = 1/n^{3/2}(\log n)^{1/2}$ on account of the fact that

$$\sum_1^\infty n^2 \lambda_n^2 < \infty$$

for such kernels (see [2]).

REFERENCES

1. T. W. Chaundy and A. E. Jolliffe, The uniform convergence of a certain class of trigonometrical series, *Proc. London Math. Soc.* (2), **15** (1916), 214–216.

2. J. B. Reade, Eigenvalues of smooth kernels, *Math. Proc. Cambridge Philos. Soc.* **95** (1984), 135–140.

3. J. B. Reade, On the sharpness of Weyl's estimate for the eigenvalues of smooth kernels, *SIAM J. Math. Anal.* **16** (1985), 548–550.

4. J. B. Reade, Positive definite C^p kernels, *SIAM J. Math. Anal.* **17** (1986), 420–421.

5. E. C. Titchmarsh, *Theory of Functions* (Oxford University Press, 1932).

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