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MONTE CARLO METHODS FOR BACKWARD EQUATIONS IN NONLINEAR FILTERING

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Abstract

We consider Monte Carlo methods for the classical nonlinear filtering problem. The first method is based on a backward pathwise filtering equation and the second method is related to a backward linear stochastic partial differential equation. We study convergence of the proposed numerical algorithms. The considered methods have such advantages as a capability in principle to solve filtering problems of large dimensionality, reliable error control, and recurrency. Their efficiency is achieved due to the numerical procedures which use effective numerical schemes and variance reduction techniques. The results obtained are supported by numerical experiments.

Keywords: Pathwise filtering equation; stochastic partial differential equation; Monte Carlo technique; Kallianpur–Striebel formula; mean-square and weak numerical methods

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space, let $\mathcal{F}_t$, $0 \le t \le T$, be a filtration satisfying the usual hypotheses, and let $(w(t), \mathcal{F}_t)$ and $(v(t), \mathcal{F}_t)$ be d_1-dimensional and r-dimensional independent standard Wiener processes, respectively. We consider the following classical filtering scheme:

$$\mathrm{d}X = \alpha(X)\,\mathrm{d}s + \sigma(X)\,\mathrm{d}w(s), \qquad X(0) = x, \tag{1.1}$$

$$\mathrm{d}y = \beta(X)\,\mathrm{d}s + \mathrm{d}v(s), \qquad y(0) = 0, \tag{1.2}$$

where $X(t) \in \mathbb{R}^d$ is the unobservable signal process, $y(t) \in \mathbb{R}^r$ is the observation process, $\alpha(x)$ and $\beta(x)$ are d-dimensional and r-dimensional vector functions, respectively, and $\sigma(x)$ is a $d \times d_1$ dimensional matrix function. We assume that the functions α, β, and σ are bounded and have bounded derivatives up to some order. The vector $X(0) = x$ in (1.1) can be random, it is independent of both w and v and its density $\varphi(\cdot)$ is supposed to be known.

Let $f(x)$ be a function on $\mathbb{R}^d$ with the same properties as those of α, β, and σ. The filtering problem consists in constructing the estimate $\hat{f}(t) = \hat{f}(X(t))$ based on the observation $y(s)$, $0 \le s \le t$, which is the best in the mean-square sense. Our aim is to give effective numerical procedures for realization of the conditional mean:

$$\hat{f}(t) = \hat{f}(t, y(\cdot)) := \mathrm{E}[f(X(t)) \mid y(s),\ 0 \le s \le t]. \tag{1.3}$$

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To this end, we consider Monte Carlo methods. They exploit the Kallianpur–Striebel formula.

The first method is based on backward pathwise filtering equations [3], [6]–[8],[14], [23], [24], [26]. We emphasize that the pathwise filtering equations are ordinary (not stochastic) partial differential equations. Owing to this fact, nonlinear filtering problems are reduced to comparatively standard problems for linear equations of parabolic type and probabilistic representations of their solutions can be obtained. Then the ideas of the weak-sense numerical integration of stochastic differential equations (SDEs) are used and, finally, the Monte Carlo technique is applied. Among numerous works devoted to the numerical solution of nonlinear filtering problems, very few (see [27] and [28]) use the pathwise approach. Here we revisit this approach and develop it further. In particular, we prove that the Euler method converges with weak order $1-\varepsilon$ for almost every observation $y(s)$, $0 \le s \le t$. The proof is nontrivial since here a part of the SDE system is approximated in the weak sense while the other part (which involves the observation) is simulated pathwisely.

The second method deals with direct Monte Carlo simulation of the numerator and denominator in the Kallianpur–Striebel formula. To the best of the authors' knowledge, the first theoretical study in this direction was done in [25]. This approach is related to backward linear stochastic partial differential equations (SPDEs) of nonlinear filtering [14], [23], [26]. We represent solutions of these SPDEs as a conditional expectation over characteristics (they are solutions of SDEs). We first approximate the characteristics in the weak sense and then apply the Monte Carlo technique to simulate the conditional expectation over the approximate characteristics. This way allows us to again exploit ideas from the well-developed theory of weak-sense numerical integration of SDEs (see, e.g. [18]). Our approach to numerical solution of linear SPDEs differs from the one often exploited in nonlinear filtering (see, e.g. [13] and the references therein). The approach used in [13] works in the reverse order to ours: first the expectations are approximated via a Monte Carlo procedure and then paths of the signal are discretized. In our approach the use of weak schemes for the characteristics arises naturally, while in the other approach one usually exploits mean-square approximations of the signal [13]. Let us mention that in [13] a more general nonlinear filtering problem is studied than (1.1)–(1.3); there, in particular, the case of correlated noise in the observation and signal is treated. The correlated case is considered within our approach in [19] and [20]. We note that in the uncorrelated case, (1.1)–(1.3), a higher order of convergence of Euler-type methods is proved in this paper (see also [25]) compared to that which can be proved for the correlated case [13], [19], [20].

We emphasize that the independent trajectories obtained by the considered methods are approximate characteristics of the backward filtering equations and they serve only for Monte Carlo simulation of the conditional expectations. In contrast, the particle methods (see [4], [5], [9], [10], [13], and the references therein) produce empirical measures approximating solutions to the Zakai or Kushner–Stratonovich equation which can then be used to evaluate the conditional expectations.

All the considered methods have such advantages as a capability in principle to solve the problems of large dimensionality, reliable error control, and recurrency. Their efficiency is achieved due to the numerical procedures which use effective schemes of numerical integration of SDEs and variance reduction techniques. Some other numerical approaches to nonlinear filtering are available in [2], [4], [5], [9], [10], [12], [14], [15], and [22] (see also the references therein).

Section 2 contains some preliminary material. In Section 3 we consider numerical methods for the backward pathwise filtering equation, while in Section 4 we deal with approximation of solutions to the backward SPDE of nonlinear filtering. Section 5 is devoted to variance reduction techniques. The results obtained are supported by numerical experiments which are presented in Section 6.

2. Preliminaries

In this section we recall some known facts from the theory of nonlinear filtering in the form suitable for our purposes.

2.1. Kallianpur–Striebel formula

We define the stochastic process $\eta(t)$ as

$$\eta^{-1}(t) := \exp\left\{-\int_0^t \beta^\top(X(s))\,\mathrm{d}v(s) - \frac{1}{2}\int_0^t |\beta(X(s))|^2\,\mathrm{d}s\right\}$$
$$= \exp\left\{-\int_0^t \beta^\top(X(s))\,\mathrm{d}y(s) + \frac{1}{2}\int_0^t |\beta(X(s))|^2\,\mathrm{d}s\right\}. \tag{2.1}$$

According to our assumptions, we have

$$\mathrm{E}[\eta^{-1}(t)] = 1, \qquad 0 \le t \le T.$$

Using $\eta^{-1}(T)$, we introduce the new probability measure $\tilde{\mathrm{P}}$ on $(\Omega, \mathcal{F})$,

$$\tilde{\mathrm{P}}(\Gamma) = \int_\Gamma \eta^{-1}(T)\,\mathrm{dP}(\omega).$$

The measures P and $\tilde{\mathrm{P}}$ are mutually absolutely continuous and the Radon–Nikodym derivative $\mathrm{dP}/\mathrm{d}\tilde{\mathrm{P}}$ is equal to $\eta(T)$:

$$\frac{\mathrm{dP}}{\mathrm{d}\tilde{\mathrm{P}}}(X(\cdot), y(\cdot)) = \eta(T) = \exp\left\{\int_0^T \beta^\top(X(s))\,\mathrm{d}y(s) - \frac{1}{2}\int_0^T |\beta(X(s))|^2\,\mathrm{d}s\right\}.$$

We can show, by the Girsanov theorem, that there exists a standard Wiener process $(w(t), \tilde{v}(t))$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \tilde{\mathrm{P}})$ such that its part $w(t)$ coincides with that in (1.1) and that the process $(X(s), y(s))$ satisfies the following system of Itô equations:

$$\mathrm{d}X = \alpha(X)\,\mathrm{d}s + \sigma(X)\,\mathrm{d}w(s), \qquad X(0) = x, \tag{2.2}$$
$$\mathrm{d}y = \mathrm{d}\tilde{v}(s), \qquad y(0) = 0. \tag{2.3}$$

Thus, the processes $X(s)$ and $y(s)$ are independent on $(\Omega, \mathcal{F}, \mathcal{F}_s, \tilde{\mathrm{P}})$, and $y(s)$ is a Wiener process.

The following formula holds for a function $F(x, y)$:

$$\begin{aligned}\mathrm{E}[F(X(t), y(t))] &= \tilde{\mathrm{E}}[F(X(t), y(t))\eta(T)] \\ &= \tilde{\mathrm{E}}[\tilde{\mathrm{E}}[F(X(t), y(t))\eta(T) \mid \mathcal{F}_t]] \\ &= \tilde{\mathrm{E}}[F(X(t), y(t))\,\tilde{\mathrm{E}}[\eta(T) \mid \mathcal{F}_t]] \\ &= \tilde{\mathrm{E}}[F(X(t), y(t))\eta(t)],\end{aligned} \tag{2.4}$$

where $\tilde{\mathrm{E}}$ is the expectation with respect to $\tilde{\mathrm{P}}$.

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Formula (2.4) can be rewritten as

$$\mathrm{E}[F(X(t),\, y(t))]|_{(1.1)-(1.2)} = \mathrm{E}\left[F(X(t),\, y(t)) \exp\left\{\int_0^t \beta^\top(X(s))\, \mathrm{d}y(s) - \frac{1}{2}\int_0^t |\beta(X(s))|^2\, \mathrm{d}s\right\}\right]\Big|_{(2.2)-(2.3)} \tag{2.5}$$

or

$$\mathrm{E}[F(X(t),\, y(t))]|_{(1.1)-(1.2)} = \mathrm{E}[F(X(t),\, y(t))\eta(t)]|_{(2.7)}, \tag{2.6}$$

where the system (2.7) has the form

$$\begin{aligned} &\mathrm{d}X = \alpha(X)\, \mathrm{d}s + \sigma(X)\, \mathrm{d}w(s), \qquad X(0) = x, \\ &\mathrm{d}y = \mathrm{d}\tilde{v}(s), \qquad y(0) = 0, \\ &\mathrm{d}\eta = \beta^\top(X)\eta\, \mathrm{d}\tilde{v}(s), \qquad \eta(0) = 1. \end{aligned} \tag{2.7}$$

Here $\mathrm{E}[\cdot]|_{(1.1)-(1.2)}$, $\mathrm{E}[\cdot]|_{(2.2)-(2.3)}$, and $\mathrm{E}[\cdot]|_{(2.7)}$ denote the expectation with respect to the corresponding system of SDEs. Representations (2.5) and (2.6) are constructive: unlike (2.4), they admit direct application of the Monte Carlo technique using methods of weak approximation of SDEs (see Section 4).

Let us recall a general version of Bayes' formula (see, e.g. [16, Chapter 7, Section 9] and [26, Chapter 6, Section 1.1]).

Lemma 2.1. *Suppose that two mutually absolutely continuous measures Q and $\tilde{Q}$ are given on $(\Omega, \mathcal{F})$. If ξ is a Q-integrable random variable and $\mathcal{G}$ is some σ-subalgebra of $\mathcal{F}$, then, almost surely,*

$$\mathrm{E}[\xi \mid \mathcal{G}] = \frac{\tilde{\mathrm{E}}[\xi\, \mathrm{d}Q/\mathrm{d}\tilde{Q} \mid \mathcal{G}]}{\tilde{\mathrm{E}}[\mathrm{d}Q/\mathrm{d}\tilde{Q} \mid \mathcal{G}]}. \tag{2.8}$$

Clearly, (2.4) is a particular case of (2.8). Another particular case is the well-known Kallianpur–Striebel formula for the mean (1.3):

$$\mathrm{E}[f(X(t)) \mid y(s),\ 0 \le s \le t] = \frac{\tilde{\mathrm{E}}[f(X(t))\eta(t) \mid y(s),\ 0 \le s \le t]}{\tilde{\mathrm{E}}[\eta(t) \mid y(s),\ 0 \le s \le t]}, \tag{2.9}$$

where

$$\eta(t) = \exp\left\{\int_0^t \beta^\top(X_{0,x}(s))\, \mathrm{d}y(s) - \frac{1}{2}\int_0^t |\beta(X_{0,x}(s))|^2\, \mathrm{d}s\right\}$$

and $X(s) = X_{0,x}(s)$ is the solution of (2.2).

The numerator and denominator in the Kallianpur–Striebel formula, (2.9), can be written in the form

$$\tilde{\mathrm{E}}[f(X(t))\eta(t) \mid y(s),\ 0 \le s \le t] = \mathrm{E}\left[f(X_{0,x}(t)) \exp\left\{\int_0^t \beta^\top(X_{0,x}(s))\, \mathrm{d}y(s) - \frac{1}{2}\int_0^t |\beta(X_{0,x}(s))|^2\, \mathrm{d}s\right\}\right]\Big|_{(2.2)}, \tag{2.10}$$

$$\tilde{\mathrm{E}}[\eta(t) \mid y(s),\ 0 \le s \le t] = \mathrm{E}\left[\exp\left\{\int_0^t \beta^\top(X_{0,x}(s))\, \mathrm{d}y(s) - \frac{1}{2}\int_0^t |\beta(X_{0,x}(s))|^2\, \mathrm{d}s\right\}\right]\Big|_{(2.2)}. \tag{2.11}$$

The sign '$|_{(2.2)}$' means that $X_{0,x}(s)$ is the solution of (2.2) under a fixed trajectory y, i.e. the averaging here is carried out with respect to X only. These formulae are true due to the

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independence of $X(\cdot)$ and $y(\cdot)$ on $(\Omega, \mathcal{F}, \tilde{\mathrm{P}})$. In what follows we will also use the following notation for the conditional expectation:

$$\tilde{\mathrm{E}}^{y}[F(X(\cdot), y(\cdot))] := \tilde{\mathrm{E}}[F(X(\cdot), y(\cdot)) \mid y(s'),\ 0 \le s' \le s],$$

where $F(X(\cdot), y(\cdot))$ is a functional of $X(s')$ and $y(s')$, $0 \le s' \le s$.

Remark 2.1. Owing to the Itô formula, we have

$$\begin{aligned}\int_0^t \beta^{\top}(X_{0,x}(s))\,\mathrm{d}y(s) = \beta^{\top}(X_{0,x}(t))y(t) - \int_0^t \sum_{i=1}^{d} \frac{\partial \beta^{\top}}{\partial x^i}\alpha_i y(s)\,\mathrm{d}s \\ - \frac{1}{2}\int_0^t \sum_{i,l=1}^{d}\sum_{k=1}^{d_1} \sigma_{ik}\sigma_{lk}\frac{\partial^2 \beta^{\top}}{\partial x^i \partial x^l} y(s)\,\mathrm{d}s \\ - \int_0^t \sum_{i=1}^{d} \frac{\partial \beta^{\top}}{\partial x^i} y(s) \sum_{j=1}^{d_1}\sigma_{ij}\,\mathrm{d}w_j(s),\end{aligned}$$

which allows us to consider $y(\cdot)$ in (2.10) and (2.11) to be fixed.

2.2. Backward pathwise filtering equations and probabilistic representation of their solutions

Define the $d \times d$ dimensional matrix $a = \{a_{ij}\}$ by

$$a(x) = \sigma(x)\sigma^{\top}(x), \qquad a_{ij}(x) = \sum_{k=1}^{d_1}\sigma_{ik}(x)\sigma_{jk}(x).$$

Owing to the Itô formula, the process $\eta(t)$ can be transformed into

$$\begin{aligned}\eta(t) &= \exp\{\beta^{\top}(X_{0,x}(t))y(t)\} \\ &\quad \times \exp\left\{-\int_0^t \sum_{i=1}^{d}\frac{\partial \beta^{\top}}{\partial x^i}\alpha_i y(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t \sum_{i,l=1}^{d} a_{il}\frac{\partial^2 \beta^{\top}}{\partial x^i \partial x^l} y(s)\,\mathrm{d}s \right. \\ &\qquad \left. - \frac{1}{2}\int_0^t |\beta|^2\,\mathrm{d}s\right\} \exp\left\{-\int_0^t \sum_{i=1}^{d}\frac{\partial \beta^{\top}}{\partial x^i} y(s)\sum_{j=1}^{d_1}\sigma_{ij}\,\mathrm{d}w_j(s))\right\} \\ &:= \exp\{\beta^{\top}(X_{0,x}(t))y(t)\}\zeta(t), \qquad (2.12)\end{aligned}$$

where all the functions α, β, $\partial\beta/\partial x^i$, $\partial^2\beta/\partial x^i \partial x^l$, and σ in the integrands have $X_{0,x}(s)$ as their arguments.

Introduce the system of SDEs for X and the scalar ζ:

$$\mathrm{d}X = \alpha(X)\,\mathrm{d}s' + \sigma(X)\,\mathrm{d}w(s'), \qquad X(s) = x,\ s' \ge s, \qquad (2.13)$$

$$\begin{aligned}\mathrm{d}\zeta &= -\zeta\left(\sum_{i=1}^{d}\frac{\partial \beta^{\top}}{\partial x^i}\alpha_i y(s') + \frac{1}{2}\sum_{i,l=1}^{d} a_{il}\frac{\partial^2 \beta^{\top}}{\partial x^i \partial x^l} y(s')\right)\mathrm{d}s' \\ &\quad + \tfrac{1}{2}\zeta(|\sigma^{\top}\mathrm{grad}(\beta^{\top}y(s'))|^2 - |\beta|^2)\,\mathrm{d}s' \\ &\quad - \zeta(\sigma^{\top}\mathrm{grad}(\beta^{\top}y(s')))^{\top}\,\mathrm{d}w(s'), \qquad \zeta(s) = z. \qquad (2.14)\end{aligned}$$

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In (2.14) we use the short notation

$$\frac{\partial^2 \beta^\top}{\partial x^i \partial x^l} y(s) = \sum_{k=1}^{r} \frac{\partial^2 \beta_k}{\partial x^i \partial x^l} y^k(s), \qquad \sigma^\top \mathrm{grad}(\beta^\top y) = \sum_{i=1}^{d} \sigma_{ij} \sum_{k=1}^{r} \frac{\partial \beta_k}{\partial x^i} y^k(s).$$

Let us fix the obtained observation $y(s),\ 0 \le s \le t$. Consider the function

$$u_g(s,x) = \mathrm{E}[g(X_{s,x}(t)) \exp\{\beta^\top(X_{s,x}(t))y(t)\}\zeta_{s,x,1}(t)]|_{(2.13)-(2.14)}, \tag{2.15}$$

where g is a scalar function on $\mathbb{R}^d$, $x \in \mathbb{R}^d$ is deterministic now, and $X_{s,x}(s')$, $\zeta_{s,x,1}(s')$ is the solution of system (2.13)–(2.14) starting from $(x, 1)$ at the instant s. Clearly, for a deterministic $X(0) = x$, the values $u_f(0,x)$ and $u_{\mathbf{1}}(0,x)$ (here the function $\mathbf{1} = \mathbf{1}(x)$ is identically equal to 1) coincide with the numerator and denominator of (2.9), respectively. Hence,

$$\begin{aligned}\hat{f}(t, y(\cdot)) &= \mathrm{E}[f(X(t)) \mid y(s),\ 0 \le s \le t] \\ &= \frac{\mathrm{E}[f(X_{0,x}(t)) \exp\{\beta^\top(X_{0,x}(t))y(t)\}\zeta_{0,x,1}(t)]|_{(2.13)-(2.14)}}{\mathrm{E}[\exp\{\beta^\top(X_{0,x}(t))y(t)\}\zeta_{0,x,1}(t)]|_{(2.13)-(2.14)}} \\ &= \frac{u_f(0,x)}{u_{\mathbf{1}}(0,x)},\end{aligned} \tag{2.16}$$

where x is deterministic.

If $X(0) = \xi$ is random with density $\varphi(\cdot)$ then the numerator is equal to the integral $\int u_f(0,x)\varphi(x)\,\mathrm{d}x$ and the denominator is equal to the integral $\int u_{\mathbf{1}}(0,x)\varphi(x)\,\mathrm{d}x$, i.e.

$$\begin{aligned}\hat{f}(t, y(\cdot)) &= \mathrm{E}[f(X(t)) \mid y(s),\ 0 \le s \le t] \\ &= \frac{\mathrm{E}[f(X_{0,\xi}(t)) \exp\{\beta^\top(X_{0,\xi}(t))y(t)\}\zeta_{0,\xi,1}(t)]|_{(2.13)-(2.14)}}{\mathrm{E}[\exp\{\beta^\top(X_{0,\xi}(t))y(t)\}\zeta_{0,\xi,1}(t)]|_{(2.13)-(2.14)}} \\ &= \frac{\int u_f(0,x)\varphi(x)\,\mathrm{d}x}{\int u_{\mathbf{1}}(0,x)\varphi(x)\,\mathrm{d}x}.\end{aligned}$$

It is well known that the function $u_g(s,x)$ is the solution to the Cauchy problem for the linear parabolic equation

$$\frac{\partial u}{\partial s} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b_i(s,x)\frac{\partial u}{\partial x^i} + c(s,x)u = 0, \qquad s < t, \tag{2.17}$$

with the following condition on the right-hand side of the time interval $[0, t]$:

$$u(t,x) = g(x)\exp\{\beta^\top(x)y(t)\} := G(t,x). \tag{2.18}$$

In (2.17) we have

$$b(s,x) = \{b_i(s,x)\} = \alpha(x) - a(x)\mathrm{grad}(\beta^\top(x)y(s)), \tag{2.19}$$

$$\begin{aligned}c(s,x) = {} & -\alpha^\top(x)\mathrm{grad}(\beta^\top(x)y(s)) - \frac{1}{2}\sum_{i,l=1}^{d} a_{il}(x)\frac{\partial^2 \beta^\top(x)}{\partial x^i \partial x^l} y(s) \\ & - \tfrac{1}{2}|\beta(x)|^2 + \tfrac{1}{2}|\sigma^\top(x)\mathrm{grad}(\beta^\top(x)y(s))|^2.\end{aligned} \tag{2.20}$$

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Formula (2.17) is known as the backward pathwise filtering equation (see, e.g. [24]). The probabilistic representation of its solution satisfying condition (2.18) is given by (2.13)–(2.15). Formula (2.15) is known as the Feynman–Kac formula. Thus, finding the estimate $\hat{f}(t)$ amounts to evaluating means of the form

$$u_{g,\varphi} := \int u_g(0,x)\varphi(x)\,\mathrm{d}x = \tilde{\mathrm{E}}^y[G(t, X_{0,\xi}(t))\zeta_{0,\xi,1}(t)]. \tag{2.21}$$

2.3. Backward SPDE for nonlinear filtering

We fix a time moment t and introduce the function

$$v_g(s,x) = \tilde{\mathrm{E}}^y[g(X_{s,x}(t))\eta_{s,x,1}(t)], \tag{2.22}$$

where g is a scalar function on $\mathbb{R}^d$, $x \in \mathbb{R}^d$ is deterministic, and $X_{s,x}(s')$, $\eta_{s,x,1}(s')$ is the solution of the system

$$\begin{aligned} \mathrm{d}X &= \alpha(X)\,\mathrm{d}s' + \sigma(X)\,\mathrm{d}w(s'), \qquad X(s) = x, \\ \mathrm{d}\eta &= \beta^\top(X)\eta\,\mathrm{d}y(s'), \qquad \eta(s) = 1. \end{aligned} \tag{2.23}$$

Here $w(s')$ and $y(s')$ are independent standard Wiener processes on $(\Omega, \mathcal{F}, \mathcal{F}_s, \tilde{\mathrm{P}})$.

The function $v_g(s,x)$ is the solution of the Cauchy problem for the backward linear SPDE (see, e.g. [14], [23], and [26, Chapter 6, Section 3])

$$-\mathrm{d}v = \left[\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 v}{\partial x^i \partial x^j} + \sum_{i=1}^{d}\alpha_i(x)\frac{\partial v}{\partial x^i}\right]\mathrm{d}s + \beta^\top(x)v *\,\mathrm{d}y, \qquad s < t, \tag{2.24}$$

$$v(t,x) = g(x). \tag{2.25}$$

The notation '$*\,\mathrm{d}y$' means backward Itô integral. We recall [26, pp. 36–37] that to define this integral we introduce the 'backward' Wiener process $\tilde{y}(s) := y(t) - y(t-s)$, $s \le t$, and a decreasing family of σ-subalgebras $\mathcal{F}_t^s$, $0 \le s \le t$, induced by the increments $y(t) - y(s')$, $s' \ge s$. A σ-algebra induced by $\tilde{y}(s')$, $s' \le s$, coincides with $\mathcal{F}_t^{t-s}$. Then the backward Itô integral is defined as the Itô integral with respect to $\tilde{y}(s)$:

$$\int_s^{s'} \psi(s'') *\,\mathrm{d}y(s'') := \int_{t-s'}^{t-s} \psi(t-s'')\,\mathrm{d}\tilde{y}(s''), \qquad s \le s' \le t,$$

where $\psi(t-s)$, $s \le t$, is an $\mathcal{F}_t^{t-s}$-adapted square-integrable function. The process $v(s,x)$ is $\mathcal{F}_t^s$-adapted.

The solution $v(s,x)$ to the SPDE (2.24)–(2.25) relates to the solution $u(s,x)$ of the backward pathwise PDE (2.17)–(2.18) [3], [7], [16], [23], [26],

$$v(s,x) = \exp\{-\beta^\top(x)y(s)\}u(s,x), \tag{2.26}$$

and, in particular, $v(0,x) = u(0,x)$. Indeed, by the same arguments as those used in (2.12) we obtain

$$\eta_{s,x,1}(t) = \exp\{\beta^\top(X_{s,x}(t))y(t) - \beta^\top(x)y(s)\}\zeta_{s,x,1}(t).$$

We note that in (2.12), $\eta(t) = \eta_{0,x,1}(t)$ and $\zeta = \zeta_{0,x,1}(t)$. Comparing (2.15) and (2.22), we arrive at (2.26). We note that (2.26) can also be verified by direct substitution of $v(s,x)$ from (2.26) into (2.24)–(2.25).

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As pointed out before, (2.17) has the form of a usual (not stochastic) PDE whose coefficients depend on $y(s)$. Clearly, (2.17)–(2.18) is well defined for any continuous function $y(s)$. Then the solution of (2.17)–(2.18) can be interpreted in the pathwise sense, i.e. we can consider the PDE problem (2.17)–(2.18) with the fixed observation sample path $y(s)$, $0 \le s \le t$. In its turn, the pathwise interpretation of the PDE (2.17)–(2.18), together with (2.26), allows us to interpret the solution to the SPDE (2.24)–(2.25) in the pathwise sense as well [3], [7], [8]. We also note that (2.26) implies some properties of $v(s, x)$. Under the above assumptions, the function $v(s, x)$ is smooth in x and the function and its derivatives with respect to x are continuous in s.

If $X(0) = \xi$ is a random variable with density $\varphi(\cdot)$ then we can write (cf. (2.9)–(2.11) and (2.22))

$$\hat{f}(t, y(\cdot)) = \mathrm{E}[f(X(t)) \mid y(s),\ 0 \le s \le t] = \frac{v_{f,\varphi}}{v_{\mathbf{1},\varphi}}, \tag{2.27}$$

where

$$v_{g,\varphi} := \int v_g(0, x)\varphi(x)\,\mathrm{d}x = \tilde{\mathrm{E}}^y[g(X_{0,\xi}(t))\eta_{0,\xi,1}(t)].$$

3. Integrators for SDEs with coefficients nonsmooth in the time variable

In Subsection 2.2, computing the estimate $\hat{f}(t)$ is reduced to finding $u_{g,\varphi}$ from (2.21), which can be evaluated using the weak-sense numerical integration of system (2.13)–(2.14) together with the Monte Carlo technique. More specifically, we have

$$u_{g,\varphi} \simeq \tilde{\mathrm{E}}^y[G(t, \bar{X}_{0,\xi}(t))\bar{\zeta}_{0,\xi,1}(t)] \simeq \frac{1}{M}\sum_{m=1}^{M} G\big(t, \bar{X}^{(m)}_{0,x^{(m)}}(t)\big)\bar{\zeta}^{(m)}_{0,x^{(m)},1}(t), \tag{3.1}$$

where the first approximate equality involves an error due to replacing X and ζ by $\bar{X}$ and $\bar{\zeta}$ (the error is related to the approximate integration of system (2.13)–(2.14)), while the error in the second approximate equality comes from the Monte Carlo technique; $x^{(m)}$, $m = 1, \dots, M$, are independent realizations of the random variable ξ distributed according to the density $\varphi(\cdot)$, and $\bar{X}^{(m)}_{0,x^{(m)}}(t)$ and $\bar{\zeta}^{(m)}_{0,x^{(m)},1}(t)$ are independent realizations of $\bar{X}_{0,\xi}(t)$ and $\bar{\zeta}_{0,\xi,1}(t)$.

Since the coefficients of the backward pathwise filtering equation (2.17)–(2.18) depend on the observation sample path $y(s)$, $0 \le s \le t$, SDEs arising in its probabilistic representations (e.g. (2.13)–(2.14)) have continuous but not differentiable coefficients with respect to time. Then their numerical approximation requires special attention and in this section we consider such approximations.

3.1. Integral methods for SDEs with coefficients nonsmooth in the time variable

Consider the following system of Itô SDEs:

$$\mathrm{d}X = A(s, X)\,\mathrm{d}s + B(s, X)\,\mathrm{d}w(s), \qquad X(s_0) = x,\ s_0 \le s \le T, \tag{3.2}$$

where $A(s, x)$ is an n-dimensional vector, $B(s, x)$ is an $n \times n_1$-matrix, and $w(s)$ is an n_1-dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_s, \mathrm{P})$. We assume that the components of $A(s, x)$ and $B(s, x)$ are continuous in s and sufficiently smooth in $x \in \mathbb{R}^n$, and that the spatial derivatives of the components of A and B up to a sufficiently high order are uniformly bounded on $[s_0, T] \times \mathbb{R}^n$. System (2.13)–(2.14) can be written in the form (3.2). We note that in this section we use the notation $X(s) = X_{s_0,x}(s)$ for a different process and $(\Omega, \mathcal{F}, \mathcal{F}_s, \mathrm{P})$ for a different space than in the previous sections, but this should not

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lead to any confusion. Weak methods for systems of the form (3.2) with nondifferentiable in s coefficients were proposed in [27] and [28]. Here we revise them and develop them further. In particular, our proofs of convergence of the integral methods differ from those in [27] and [28]. To the best of the authors' knowledge, the analysis of the Euler method given in Subsection 3.2 is done for the first time.

We start with a technical lemma.

Lemma 3.1. *Let a function $f(s,x)$, $s_0 \le s \le T$, $x \in \mathbb{R}^n$, be twice continuously differentiable in x and have growth in x at ∞ not faster than polynomial. Then the following formula holds:*

$$f(s, X_{s_0,x}(s)) = f(s,x) + \int_{s_0}^{s} \mathcal{L}^{s'} f(s, X(s'))\,\mathrm{d}s' + \sum_{j=1}^{n_1} \int_{s_0}^{s} \Lambda_j^{s'} f(s, X(s'))\,\mathrm{d}w_j(s'), \quad (3.3)$$

where $X(s) = X_{s_0,x}(s)$ is the solution of (3.2) and

$$\mathcal{L}^{s'} f(s,x) := \sum_{i=1}^{n} A^i(s',x) \frac{\partial}{\partial x^i} f(s,x) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{l=1}^{n_1} B^{il}(s',x) B^{jl}(s',x) \frac{\partial^2}{\partial x^i \partial x^j} f(s,x),$$

$$\Lambda_j^{s'} f(s,x) := \sum_{i=1}^{n} B^{ij}(s',x) \frac{\partial}{\partial x^i} f(s,x), \qquad j = 1, \dots, n_1.$$

Proof. Fix s and apply the Itô formula to $f(s, X_{s_0,x}(s'))$, $s_0 \le s' \le s$:

$$\mathrm{d} f(s, X_{s_0,x}(s')) = \mathcal{L}^{s'} f(s, X(s'))\,\mathrm{d}s' + \sum_{j=1}^{n_1} \Lambda_j^{s'} f(s, X(s'))\,\mathrm{d}w_j(s'). \quad (3.4)$$

Integrating (3.4) from s_0 to s, we obtain (3.3).

Now we proceed to constructing numerical methods for (3.2) in the usual way [18, Chapter 1]. Using Lemma 3.1, we expand the solution of (3.2):

$$X_{s_0,x}(s_0 + h) = x + \int_{s_0}^{s_0+h} A(s',x)\,\mathrm{d}s' + \int_{s_0}^{s_0+h} B(s',x)\,\mathrm{d}w(s') + \rho, \quad (3.5)$$

where

$$\begin{aligned}\rho = {} & \int_{s_0}^{s_0+h} \int_{s_0}^{s'} \mathcal{L}^{s''} A(s', X(s''))\,\mathrm{d}s''\,\mathrm{d}s' + \sum_{j=1}^{n_1} \int_{s_0}^{s_0+h} \int_{s_0}^{s'} \Lambda_j^{s''} A(s', X(s''))\,\mathrm{d}w_j(s'')\,\mathrm{d}s' \\ & + \int_{s_0}^{s_0+h} \int_{s_0}^{s'} \mathcal{L}^{s''} B(s', X(s''))\,\mathrm{d}s''\,\mathrm{d}w(s') \\ & + \sum_{j=1}^{n_1} \int_{s_0}^{s_0+h} \int_{s_0}^{s'} \Lambda_j^{s''} B(s', X(s''))\,\mathrm{d}w_j(s'')\,\mathrm{d}w(s'). \end{aligned} \quad (3.6)$$

Here application of the operators to a vector or matrix is understood as their application to each element of the corresponding vector or matrix.

Truncating expansion (3.5) we obtain the one-step approximation

$$\bar{X}_{s_0,x}(s_0 + h) = x + \int_{s_0}^{s_0+h} A(s',x)\,\mathrm{d}s' + \int_{s_0}^{s_0+h} B(s',x)\,\mathrm{d}w(s'). \quad (3.7)$$

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We note that the last term in (3.7) is the vector whose components are Gaussian random variables with zero mean and covariance matrix

$$R(s_0, x, h) = \int_{s_0}^{s_0+h} B(s', x)B^\top(s', x)\,\mathrm{d}s'.$$

We partition the time interval $[s_0, T]$ into N equal parts with step $h = (T - s_0)/N$ and $s_k = s_0 + kh$, $k = 0, \dots, N$. Based on the one-step approximation, (3.7), we obtain the Euler-type method

$$\begin{aligned} X_0 &= x, \\ X_{k+1} &= X_k + \int_{s_k}^{s_{k+1}} A(s', X_k)\,\mathrm{d}s' + \sigma(s_k, X_k, h)\xi_k, \qquad k = 0, \dots, N, \end{aligned} \tag{3.8}$$

where $\sigma(s_k, X_k, h)$ is an $n \times n$-matrix such that $\sigma(s, x, h)\sigma^\top(s, x, h) = R(s, x, h)$, $\xi_k = (\xi_k^1, \dots, \xi_k^n)^\top$, ξ_k^i are independent and identically distributed (i.i.d.) Gaussian random variables with zero mean and unit variance, and ξ_k are independent of X_k. We call methods of this type (i.e. which contain integrals of the coefficients with respect to time) *integral methods*.

Theorem 3.1. *Method (3.8) for system (3.2) has the mean-square order $\frac{1}{2}$, i.e.*

$$\mathrm{E}[(X_{s_0,x}(s_k) - X_k)^2]^{1/2} \le Kh^{1/2}, \qquad k = 0, \dots, N,$$

where the constant K does not depend on the discretization step h.

Let $F(x)$ have four continuous derivatives and let it, together with its derivatives, have growth at ∞ not faster than polynomial. Method (3.8) for system (3.2) has the weak order 1, i.e.

$$|\,\mathrm{E}[F(X_{s_0,x}(s_k))] - \mathrm{E}[F(X_k)]| \le Kh, \qquad k = 0, \dots, N,$$

where the constant K does not depend on the discretization step h.

Proof. The mean-square order $\frac{1}{2}$ of method (3.8) follows from the estimates of the remainder in (3.5),

$$\mathrm{E}[\rho] = O(h^2) \quad \text{and} \quad \mathrm{E}[\rho^2] = O(h^2),$$

and the fundamental theorem of mean-square convergence [18, p. 4].

To prove the order of weak convergence, introduce the vectors

$$\Delta = X_{s_0,x}(s_0 + h) - x \quad \text{and} \quad \bar{\Delta} = \bar{X}_{s_0,x}(s_0 + h) - x.$$

It is not difficult to prove that

$$\mathrm{E}\left[\prod_{j=1}^{q} \Delta^{i_j} - \prod_{j=1}^{q} \bar{\Delta}^{i_j}\right] = O(h^2), \qquad q = 1, 2, 3,$$

$$\mathrm{E}\left[\prod_{j=1}^{4} |\bar{\Delta}^{i_j}|\right] = O(h^2),$$

where Δ^{i_j} and $\bar{\Delta}^{i_j}$, $i_j = 1, \dots, n$, are i_j-components of the vectors Δ and $\bar{\Delta}$, respectively. Then, due to the general weak convergence theorem [18, p. 100], we find that method (3.8) is of weak order 1. This completes the proof.

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As it is common in the general theory of weak numerical integration [18, Chapter 2], we can use discrete random variables instead of the Gaussian random variables in (3.8) and obtain the first order of weak convergence as well. Consider the weak scheme

$$\begin{gathered} X_0 = x, \\ X_{k+1} = X_k + \int_{s_k}^{s_{k+1}} A(s', X_k)\,\mathrm{d}s' + \sigma(s_k, X_k, h)\eta_k, \qquad k = 0, \dots, N, \end{gathered} \tag{3.9}$$

where the random vectors $\eta_k = (\eta_k^1, \dots, \eta_k^n)^\top$ are such that

$$\begin{gathered} \mathrm{E}[\eta_k] = 0, \qquad \mathrm{E}[\eta_k \eta_k^T] = I, \\ \mathrm{E}[\eta_k^i \eta_k^j \eta_k^l] = 0, \quad i, j, l = 1, \dots, n, \qquad \mathrm{E}\Bigl[\prod_{j=1}^{4} \eta_k^{i_j}\Bigr] = O(h^2), \end{gathered} \tag{3.10}$$

the η_k are independent of X_k, and I is the unit matrix here. For instance, we can take η, whose components are i.i.d. random variables with the law

$$\mathrm{P}(\eta^i = \pm 1) = \tfrac{1}{2}. \tag{3.11}$$

It is not difficult to prove the following theorem.

Theorem 3.2. *Method (3.9)–(3.10) for system (3.2) is of weak order* 1.

We note that the coefficients of SDEs from Subsection 2.2 can be written as a sum of products of functions sufficiently smooth in both time s and spatial variables x and nonsmooth functions which depend of s only (see, e.g. (2.13)–(2.14)). In this case, methods (3.8) and (3.9)–(3.10) take a simpler form.

Consider the system

$$\mathrm{d}X = \tilde{A}(s, X)A(s)\,\mathrm{d}s + \tilde{B}(s, X)B(s)\,\mathrm{d}w(s), \qquad X(s_0) = x, \tag{3.12}$$

where $\tilde{A}(s, x)$ and $\tilde{B}(s, x)$ are $n \times n$-matrices whose components are sufficiently smooth in s and satisfy the previously stated assumptions with respect to x, while the n-dimensional vector $A(s)$ and the $n \times n_1$-matrix $B(s)$ consist of functions continuous in s. As an example, we give the first-order weak method for (3.12):

$$\begin{gathered} X_0 = x, \\ X_{k+1} = X_k + \tilde{A}(s_k, X_k) \int_{s_k}^{s_{k+1}} A(s')\,\mathrm{d}s' + \tilde{B}(s_k, X_k)\sigma(s_k, h)\eta_k, \qquad k = 0, \dots, N, \end{gathered} \tag{3.13}$$

where $\sigma(s_k, h)$ is an $n \times n$-dimensional matrix such that

$$\sigma(s_k, h)\sigma^\top(s_k, h) = \int_{s_k}^{s_{k+1}} B(s')B^\top(s')\,\mathrm{d}s'; \tag{3.14}$$

the random vectors η_k are defined as those in method (3.9)–(3.10).

Since we assume that the functions $A(s)$ and $B(s)$ are given (which, in particular, corresponds to fixing the sample path $y(s)$ in the filtering problem), the integrals in (3.13)–(3.14) can be computed once for all Monte Carlo runs.

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Remark 3.1. By continuing the expansion (3.5)–(3.6) we can construct a more accurate one-step approximation and, as a result, obtain a weak scheme of order 2 for systems (3.2) and (3.12) (cf. [27] and [28]).

Remark 3.2. Suppose that the error $R = \mathrm{E}[F(X_{s_0,x}(T))] - \mathrm{E}[F(X_N)]$ of, for example, method (3.9)–(3.10) can be expanded in powers of the time step h, i.e.

$$R = C_0 h + O(h^2), \tag{3.15}$$

then it becomes possible to use the *Talay–Tubaro extrapolation method* (see it in the case of smooth coefficients in, e.g. [18, Chapter 2]) and to obtain a method of order 2 by applying method (3.9)–(3.10) twice with different time steps. Namely, let $\bar{u}^{h_1}$ and $\bar{u}^{h_2}$ be approximations of $u(s_0, x) = \mathrm{E}[F(X_{s_0,x}(T))]$ calculated according to method (3.9)–(3.10) with the time steps h_1 and h_2, respectively. Then

$$\bar{u}_{\mathrm{imp}} = \bar{u}^{h_1} \frac{h_2}{h_2 - h_1} - \bar{u}^{h_2} \frac{h_1}{h_2 - h_1}, \qquad u(s_0, x) = \bar{u}_{\mathrm{imp}} + O(h^2). \tag{3.16}$$

We leave the proof of (3.15) for further study; however, our experiments in Subsection 6.1 confirm the rule expressed by (3.16).

In this paper the nonsmoothness in s of the coefficients of SDEs is due to their dependence on the observation process $y(s)$ (see, e.g. (2.13)–(2.14)), and $A(s)$ and $B(s)$ in (3.12) can, in fact, be written as some functions of $y(s)$: $A(s) = \breve{A}(y(s))$ and $B(s) = \breve{B}(y(s))$. Output of practical devices can include integrals of the observation process $y(s)$ and, thus, they can provide integral methods like (3.13) with the needed values of the integrals ($\int_{s_k}^{s_{k+1}} A(s')\,\mathrm{d}s'$ and $\int_{s_k}^{s_{k+1}} B(s')B^{\top}(s')\,\mathrm{d}s'$ in (3.13)); otherwise, these integrals have to be computed numerically. It turns out that if we approximate these integrals rather roughly (simply by the left-rectangle rule with the time step h), the convergence order of the resulting method (which is the Euler method) is arbitrarily close to 1 (see Theorem 3.4, below).

3.2. The Euler method

To clarify the matter, we consider the following particular case of system (3.2):

$$\mathrm{d}X = A(y(s), X)\,\mathrm{d}s + B(y(s), X)\,\mathrm{d}w(s), \qquad X(0) = x, \tag{3.17}$$

where $w(s)$ and $y(s)$ are independent standard Wiener processes on $(\Omega, \mathcal{F}, \mathcal{F}_s, \mathrm{P})$. Note that system (2.13)–(2.14) is of the form (3.17). We recall that in this section we use the notation $X(s) = X_{0,x}(s)$ for a different process and $(\Omega, \mathcal{F}, \mathcal{F}_s, \mathrm{P})$ for a different space than in Sections 1 and 2.

We assume that the components of $A(y, x)$ and $B(y, x)$ are sufficiently smooth in $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^r$, that the derivatives of the components of A and B up to a sufficiently high order are bounded in $x \in \mathbb{R}^n$, and that the coefficients and their derivatives have growth in y at ∞ not faster than polynomial. We note that the assumptions made imply, in particular, that moments of $A^i(y(s), X(s))$ and $B^{ij}(y(s), X(s))$ and of their derivatives in x and y are bounded.

Let us apply the standard Euler method to (3.17):

$$X_{k+1} = X_k + A(y(s_k), X_k)h + B(y(s_k), X_k)h^{1/2}\zeta_k, \qquad k = 0, \dots, N-1, \tag{3.18}$$

where $\zeta_k = (\zeta_k^1, \dots, \zeta_k^{n_1})^{\top}$ is a vector whose components are i.i.d. random variables with the law $\zeta^i \sim \mathcal{N}(0, 1)$ or $\mathrm{P}(\zeta^i = \pm 1) = \frac{1}{2}$.

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Since $y(s)$ is a Wiener process, we have, for any $0 < \varepsilon < \frac{1}{2}$,

$$|y(s) - y(s')| \le C|s - s'|^{1/2-\varepsilon} \quad \text{for any } s, s' \in [0, T] \text{ almost surely (a.s.),}$$

where C is a.s. a finite random variable.

When the coefficients of (3.2) are Hölder continuous in s with exponent $\frac{1}{2} - \varepsilon$ (and satisfy the other conditions imposed on the coefficients in this section), a straightforward modification of the proof of Theorem 3.1 shows that the standard Euler method applied to (3.2) has the mean-square order and weak order equal to $\frac{1}{2} - \varepsilon$. The latter implies that, for almost every trajectory $y(\cdot)$, scheme (3.18) converges with weak order $\frac{1}{2} - \varepsilon$, at least. But a more accurate analysis leads to a more precise weak order of convergence of (3.18) (see Theorem 3.4, below).

Theorem 3.3. *Let $F(y, x)$ have four continuous derivatives and let it, together with its derivatives, satisfy the growth condition of the form*

$$|F(y, x)| \le C\, \mathrm{e}^{\gamma|y|}(1 + |x|^{\varkappa}) \tag{3.19}$$

for some positive constants C, γ, and $\varkappa$. Then method (3.18) for system (3.17) satisfies the inequality

$$(\mathrm{E}[|\,\mathrm{E}^y[F(y(s_k), X_{0,x}(s_k))] - \mathrm{E}^y[F(y(s_k), X_k)]|^{2p}])^{1/2p} \le Kh, \qquad k = 0, \dots, N, \tag{3.20}$$

for $p \ge 1$, where the constant K does not depend on the discretization step h.

We note that in the case of (2.18), the function $F(y, x)$ is equal to $g(x)\exp\{\beta^\top(x)y\}$ with bounded and smooth g and β, and this function satisfies the conditions of Theorem 3.3, in particular, (3.19) holds for it. Furthermore, since coefficients of (2.13)–(2.14) satisfy the assumptions stated after (3.17), Theorem 3.3 is applicable in the case of the backward filtering equations considered in Subsection 2.2.

Proof of Theorem 3.3. Introduce the family of σ-subalgebras $\mathcal{F}_s^y$, $0 \le s \le T$, induced by the Wiener process $y(s')$, the operator

$$L^y f(x) := \sum_{i=1}^{n} A^i(y, x)\frac{\partial}{\partial x^i} f(x) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{l=1}^{n_1} B^{il}(y, x)B^{jl}(y, x)\frac{\partial^2}{\partial x^i \partial x^j} f(x),$$

and the function

$$u(s, x) = \mathrm{E}[F(y(T), X_{s,x}(T)) \mid y(s') - y(s),\ s \le s' \le T] =: \mathrm{E}^y[F(y(T), X_{s,x}(T))].$$

We note that $u(s, x)$ is $\mathcal{F}_T^y$-measurable, it depends on $y(s')$, $s \le s' \le T$; in fact, the more precise notation for this conditional expectation is $u(s, x; y(\cdot))$, but we use the shorter $u(s, x)$ notation. We also note that the more precise shorter notation for $\mathrm{E}[\cdot \mid y(s') - y(s),\ s \le s' \le T]$ would be, for example, $\mathrm{E}_s^y[\cdot]$, but everywhere in the proof the precise dependence on $y(\cdot)$ can be easily restored from the context.

Owing to the assumptions made, the function $u(s, x)$ is sufficiently smooth in x and it, together with its derivatives, has moments of a sufficiently high order which grow at ∞ in x not faster than polynomial. Furthermore, under the assumptions, moments of a sufficiently high order of both $X(s)$ and its approximation X_k exist. The moments of X_k are bounded due to the assumption that the coefficients of (3.18) are globally Lipschitz and due to Lemma 2.2.2 of [18, p. 102].

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We note that below we use the same letter K without any index for various constants which do not depend on h.

Using the standard technique (see [18, p. 100]), we can write the error from (3.20) for $k = N$ in the form

$$R_N := \mathrm{E}^y[F(y(T), X_{0,x}(T))] - \mathrm{E}^y[F(y(T), X_N)] = \mathrm{E}^y\Bigg[\sum_{i=0}^{N-1} \rho(s_i, X_i)\Bigg], \tag{3.21}$$

where

$$\rho(s_i, X_i) = \mathrm{E}^y[u(s_{i+1}, X_{s_i,X_i}(s_{i+1})) - u(s_{i+1}, \bar{X}_{s_i,X_i}(s_{i+1})) \mid X_i],$$

where $\bar{X}_{s_i,X_i}(s_{i+1}) = X_{i+1}$.

The rest of the proof of the theorem is divided into five steps.

Step 1. Let us analyze the one-step error

$$\rho(s, x) := \mathrm{E}^y[u(s + h, X_{s,x}(s + h))] - \mathrm{E}^y[u(s + h, \bar{X}_{s,x}(s + h))].$$

Applying Lemma 3.1, using the properties of Itô integrals and the independence of $w(s')$ and $y(s')$, we can obtain

$$\mathrm{E}^y[u(s + h, X_{s,x}(s + h))] = u(s + h, x) + \int_s^{s+h} L^{y(s')}u(s + h, x)\,\mathrm{d}s' + O(h^2).$$

Here and in what follows, the random variable $O(h^2)$ is such that $\mathrm{E}[|O(h^2)|^{2p}] \le K(x)h^{4p}$ for any $p \ge 1$ with $K(x)$ growing at ∞ not faster than polynomial.

Expanding $u(s + h, \bar{X}_{s,x}(s + h))$ around x up to the fourth derivatives and using the independence of $w(s')$ and $y(s')$, we obtain

$$\mathrm{E}^y[u(s + h, \bar{X}_{s,x}(s + h))] = u(s + h, x) + L^{y(s)}u(s + h, x)h + O(h^2).$$

Therefore,

$$\begin{aligned}
\rho(s, x) &= \mathrm{E}^y[u(s + h, X_{s,x}(s + h))] - \mathrm{E}^y[u(s + h, \bar{X}_{s,x}(s + h))] \\
&= \int_s^{s+h} (L^{y(s')}u(s + h, x) - L^{y(s)}u(s + h, x))\,\mathrm{d}s' + O(h^2) \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x^i}u(s + h, x) \int_s^{s+h} (A^i(y(s'), x) - A^i(y(s), x))\,\mathrm{d}s' \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{l=1}^{n_1} \frac{\partial^2}{\partial x^i \partial x^j}u(s + h, x) \\
&\qquad \times \int_s^{s+h} [B^{il}(y(s'), x)B^{jl}(y(s'), x) - B^{il}(y(s), x)B^{jl}(y(s), x)]\,\mathrm{d}s' + O(h^2).
\end{aligned} \tag{3.22}$$

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By the Itô formula we obtain

$$A^i(y(s'),x) - A^i(y(s),x) = \int_s^{s'} \sum_{j=1}^r \frac{\partial}{\partial y^j} A^i(y(s''),x)\,\mathrm{d}y^j(s'') + \frac{1}{2}\int_s^{s'} \sum_{j=1}^r \frac{\partial^2}{(\partial y^j)^2} A^i(y(s''),x)\,\mathrm{d}s'' = \sum_{j=1}^r \frac{\partial}{\partial y^j} A^i(y(s),x)(y^j(s') - y^j(s)) + O(h),$$

where $\mathrm{E}[|O(h)|^{2p}] \le K(x)h^{2p}$. The analogous relations can be obtained for the differences $B^{il}(y(s'),x)B^{jl}(y(s'),x) - B^{il}(y(s),x)B^{jl}(y(s),x)$. Hence, we can write the one-step error in the form

$$\rho(s,x) = \sum_{j=1}^r U_j(s,x) \int_s^{s+h} (y^j(s') - y^j(s))\,\mathrm{d}s' + O(h^2), \tag{3.23}$$

where the $U_j(s,x)$ are combinations of $B^{il}(y(s),x)$, their derivatives, and derivatives of $u(s+h,x)$ and $A^i(y(s),x)$. We note that the $U_j(s,x)$ are random, they depend on $y(s')$, $s \le s' \le T$, and have moments of a sufficiently high order.

Step 2. Substituting (3.23) into (3.21) and using the independence of $w(s')$ and $y(s')$, we obtain

$$R_N = \sum_{i=0}^{N-1} \sum_{j=1}^r \mathrm{E}^y[U_j(s_i,X_i)] \int_{s_i}^{s_{i+1}} (y^j(s') - y^j(s_i))\,\mathrm{d}s' + O(h) = \sum_{i=0}^{N-1} \sum_{j=1}^r \psi_{i,j}\, I_{i,j} + O(h), \tag{3.24}$$

where the

$$\psi_{i,j} = \mathrm{E}^y[U_j(s_i,X_i)]$$

are $\mathcal{F}_T^y$-measurable random variables which have moments of a sufficiently high order bounded by a constant independent of h,

$$I_{i,j} := \int_{s_i}^{s_{i+1}} (y^j(s') - y^j(s_i))\,\mathrm{d}s',$$

and $O(h)$ is such that

$$\mathrm{E}[|O(h)|^{2p}] \le K(x)h^{2p}.$$

Introduce the martingales

$$\varphi_{i,j}(s) = \mathrm{E}[\psi_{i,j} \mid \mathcal{F}_s^y], \qquad 0 \le s \le T,\ i = 0,\dots,N-1.$$

Owing to the martingale representation theorem, they can be written in the form

$$\varphi_{i,j}(s) = \mathrm{E}[\psi_{i,j}] + \int_0^s Z_{i,j}^\top(s')\,\mathrm{d}y(s') \quad \text{a.s.},$$

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where the $Z_{i,j}(s)$ are square-integrable $\mathcal{F}_s^y$-adapted processes. Furthermore,

$$\psi_{i,j} = \mathrm{E}[\psi_{i,j} \mid \mathcal{F}_T^y] = \varphi_{i,j}(T) = \varphi_{i,j}(s_i) + \int_{s_i}^{s_{i+1}} Z_{i,j}^\top(s')\,\mathrm{d}y(s') + \int_{s_{i+1}}^{T} Z_{i,j}^\top(s')\,\mathrm{d}y(s').$$

Then we have (see (3.24))

$$\begin{aligned}\Delta &:= \sum_{i=0}^{N-1}\sum_{j=1}^{r}\psi_{i,j}\,I_{i,j}\\ &= \sum_{j=1}^{r}\Biggl(\sum_{i=0}^{N-1}\varphi_{i,j}(s_i)I_{i,j} + \sum_{i=0}^{N-1} I_{i,j}\int_{s_i}^{s_{i+1}} Z_{i,j}^\top(s')\,\mathrm{d}y(s')\\ &\qquad + \sum_{i=0}^{N-1} I_{i,j}\int_{s_{i+1}}^{T} Z_{i,j}^\top(s')\,\mathrm{d}y(s')\Biggr). \qquad (3.25)\end{aligned}$$

Introduce the notation

$$S_{1,j} = 0, \qquad S_{k,j} = \sum_{i=0}^{k-1}\varphi_{i,j}(s_i)I_{i,j},$$

$$P_{1,j} = 0, \qquad P_{k,j} = \sum_{i=0}^{k-1}\int_{s_i}^{s_{i+1}} Z_{i,j}^\top(s')\,\mathrm{d}y(s')I_{i,j},$$

for $k = 2,\dots,N$ and $j = 1,\dots,r$. The terms in the third sum in (3.25),

$$\sum_{i=0}^{N-1}\int_{s_{i+1}}^{T} Z_{i,j}^\top(s')\,\mathrm{d}y(s')I_{i,j} := Q_{N,j},$$

can be rearranged in the following way:

$$\begin{aligned}Q_{N,j} &= \sum_{i=0}^{N-1}\int_{s_{i+1}}^{T} Z_{i,j}^\top(s')\,\mathrm{d}y(s')I_{i,j}\\ &= \sum_{i=0}^{N-1}\sum_{l=i+1}^{N-1}\int_{s_l}^{s_{l+1}} Z_{i,j}^\top(s')\,\mathrm{d}y(s')I_{i,j}\\ &= \sum_{l=1}^{N-1}\sum_{i=0}^{l-1}\int_{s_l}^{s_{l+1}} Z_{i,j}^\top(s')\,\mathrm{d}y(s')I_{i,j}\\ &= \sum_{l=1}^{N-1}\sum_{i=0}^{l-1} J_{i,j,l}I_{i,j},\end{aligned}$$

where

$$J_{i,j,l} = \int_{s_l}^{s_{l+1}} Z_{i,j}^\top(s')\,\mathrm{d}y(s').$$

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We introduce the notation

$$Q_{1,j} = 0, \qquad Q_{k,j} = \sum_{l=1}^{k-1} \sum_{i=0}^{l-1} J_{i,j,l} I_{i,j}, \qquad k = 2, \dots, N,\ j = 1, \dots, r.$$

Now we can rewrite (3.25) as

$$\Delta = \sum_{j=1}^{r} (S_{N,j} + P_{N,j} + Q_{N,j}).$$

Hence, for any $p > 0$,

$$\mathrm{E}[|\Delta|^{2p}] \le K \sum_{j=1}^{r} (\mathrm{E}[|S_{N,j}|^{2p}] + \mathrm{E}[|P_{N,j}|^{2p}] + \mathrm{E}[|Q_{N,j}|^{2p}]). \tag{3.26}$$

Step 3. Let p be integer. We have

$$\begin{aligned} \mathrm{E}[|S_{k+1,j}|^{2p}] &= \mathrm{E}[(S_{k,j} + \varphi_{k,j}(s_k) I_{k,j})^{2p}] \\ &= \mathrm{E}[(S_{k,j}^2 + 2 S_{k,j} \varphi_{k,j}(s_k) I_{k,j} + [\varphi_{k,j}(s_k) I_{k,j}]^2)^p] \\ &\le \mathrm{E}[|S_{k,j}|^{2p}] + 2p\, \mathrm{E}[S_{k,j}^{2p-1} \varphi_{k,j}(s_k) I_{k,j}] \\ &\quad + K \sum_{l=2}^{2p} \mathrm{E}[|S_{k,j}|^{2p-l} |\varphi_{k,j}(s_k) I_{k,j}|^l]. \end{aligned} \tag{3.27}$$

The second term in (3.27) is equal to 0. Indeed

$$\mathrm{E}[S_{k,j}^{2p-1} \varphi_{k,j}(s_k) I_{k,j}] = \mathrm{E}[S_{k,j}^{2p-1} \varphi_{k,j}(s_k)\, \mathrm{E}[I_{k,j} \mid \mathcal{F}_{s_k}^y]] = 0. \tag{3.28}$$

Using the elementary inequality

$$ab = h\left(\frac{ab}{h}\right) \le h\left(\frac{a^p}{p} + \frac{b^q h^{-q}}{q}\right), \qquad a, b > 0,\ p, q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \tag{3.29}$$

we obtain

$$\begin{aligned} &\mathrm{E}[|S_{k,j}|^{2p-l} |\varphi_{k,j}(s_k) I_{k,j}|^l] \\ &\quad \le h\, \mathrm{E}\left[\frac{2p-l}{2p} |S_{k,j}|^{2p} + \frac{l}{2p} |\varphi_{k,j}(s_k) I_{k,j}|^{2p} h^{-2p/l}\right], \qquad l = 2, \dots, 2p-1. \end{aligned} \tag{3.30}$$

Owing to the properties of the Itô integral, $I_{k,j}$, and the boundedness of moments of $\varphi_{k,j}$, we have

$$\mathrm{E}[|\varphi_{k,j}(s_k) I_{k,j}|^{2p}] \le K h^{3p}. \tag{3.31}$$

Relations (3.27)–(3.31) imply that

$$\mathrm{E}[|S_{k+1,j}|^{2p}] \le \mathrm{E}[|S_{k,j}|^{2p}] + Kh\, \mathrm{E}[|S_{k,j}|^{2p}] + Kh^{2p+1}, \qquad \mathrm{E}[|S_{1,j}|^{2p}] = 0,$$

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whence (see, e.g. [18, Lemma 1.1.6, p. 7])

$$\mathrm{E}[|S_{N,j}|^{2p}] \le Kh^{2p}. \tag{3.32}$$

Note that then, by Jensen's inequality, this inequality holds for noninteger p as well.

Step 4. Since

$$\mathrm{E}\left[\left|\int_{s_i}^{s_{i+1}} Z_{i,j}^{\top}(s')\,\mathrm{d}y(s')I_{i,j}\right|^{2p}\right] \le Kh^{4p},$$

we obtain, for the second term in (3.26),

$$\mathrm{E}[|P_{N,j}|^{2p}] \le Kh^{2p}. \tag{3.33}$$

Finally, we obtain an accurate estimate for the third term in (3.26). This is the most difficult step (step 5, below) of the proof. We need two preliminary lemmas before we proceed with its estimation.

Lemma 3.2. *Let ς_{i+1} be an $\mathcal{F}_{s_{i+1}}$-measurable random variable with moments of a sufficiently high order independent of h. Then, for any integer $m \ge 1$,*

$$\mathrm{E}[I_{i,j}^{2m}\varsigma_{i+1} \mid \mathcal{F}_{s_i}] = h^{3m}\tilde{\varsigma}_i, \qquad \mathrm{E}[I_{i,j}^{2m-1}\varsigma_{i+1} \mid \mathcal{F}_{s_i}] = h^{3m-1}\check{\varsigma}_i,$$

where $\tilde{\varsigma}_i$ and $\check{\varsigma}_i$ are some $\mathcal{F}_{s_i}$-measurable random variables with moments of a sufficiently high order which are bounded by a constant independent of h.

Proof. The first equality can be obtained using the properties of Itô integrals and the conditional expectation. To prove the second inequality, introduce the martingale

$$\gamma(s) = \mathrm{E}[\varsigma_{i+1} \mid \mathcal{F}_s], \qquad s \le s_{i+1}.$$

Then, by the martingale representation theorem we obtain

$$\varsigma_{i+1} = \gamma(s_i) + \int_{s_i}^{s_{i+1}} \tilde{Z}^{\top}(s')\,\mathrm{d}y(s'),$$

where $\tilde{Z}(s')$ is a square-integrable $\mathcal{F}_{s'}^{y}$-adapted process.

Owing to the properties of Itô integrals, we have

$$\mathrm{E}[I_{i,j}^{2m-1}\gamma(s_i) \mid \mathcal{F}_{s_i}] = \gamma(s_i)\,\mathrm{E}[I_{i,j}^{2m-1} \mid \mathcal{F}_{s_i}] = 0.$$

Furthermore, we can obtain

$$\mathrm{E}\left[I_{i,j}^{2m-1}\int_{s_i}^{s_{i+1}} \tilde{Z}_{i,j}^{\top}(s')\,\mathrm{d}y(s') \;\middle|\; \mathcal{F}_{s_i}\right] = h^{3m-1}\check{\varsigma}_i.$$

Lemma 3.2 is proved.

Lemma 3.3. *We have, for $p \ge 1$,*

$$\mathrm{E}\left[\left|\sum_{i=0}^{k-1} I_{i,j}J_{i,j,k}\right|^{2p}\right] \le Kh^{3p},$$

where K is independent of h.

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Proof. Let p be integer. We have

$$\begin{aligned} A &:= \mathrm{E}\Bigg[\sum_{i=0}^{k-1} I_{i,j}\, J_{i,j,k}\Bigg]^{2p} \\ &= \sum_{i_1=0}^{k-1} \cdots \sum_{i_{2p}=0}^{k-1} \mathrm{E}\Bigg[\prod_{l=1}^{2p} I_{i_l,j} \prod_{l=1}^{2p} J_{i_l,j,k}\Bigg] \\ &= \sum_{i_1=0}^{k-1} \cdots \sum_{i_{2p}=0}^{k-1} \mathrm{E}\Bigg[\prod_{l=1}^{2p} I_{i_l,j}\, \mathrm{E}\Bigg[\prod_{l=1}^{2p} J_{i_l,j,k} \;\Bigg|\; \mathcal{F}_{\max_{1\le l\le 2p} s_{i_l+1}}\Bigg]\Bigg]. \end{aligned}$$

Using the properties of Itô integrals and the conditional expectation, we obtain

$$\mathrm{E}\Bigg[\prod_{l=1}^{2p} J_{i_l,j,k} \;\Bigg|\; \mathcal{F}_{\max_{0\le l<k} s_{i_l+1}}\Bigg] = h^p\, \varsigma_{j,\{i_1,\dots,i_{2p}\}}\Big(\max_{1\le l\le 2p} s_{i_l+1}\Big),$$

where $\varsigma_{j,\{i_1,\dots,i_{2p}\}}(\max_{1\le l\le 2p} s_{i_l+1})$ is an $\mathcal{F}_{\max_{1\le l\le 2p} s_{i_l+1}}$-measurable random variable with finite moments independent of h. Hence,

$$A = h^p \sum_{i_1=0}^{k-1} \cdots \sum_{i_{2p}=0}^{k-1} \mathrm{E}\Bigg[\prod_{l=1}^{2p} I_{i_l,j}\, \varsigma_{j,\{i_1,\dots,i_{2p}\}}\Big(\max_{l} s_{i_l+1}\Big)\Bigg].$$

Let q be the number of distinguished indices i_l in $\{i_1,\dots,i_{2p}\}$. For example, if all the i_l in $\{i_1,\dots,i_{2p}\}$ are different then $q = 2p$ and if all the i_l in $\{i_1,\dots,i_{2p}\}$ are the same then $q = 1$, etc. Introduce

$$B_{\{i_1,\dots,i_{2p}\}} := \mathrm{E}\Bigg[\prod_{l=1}^{2p} I_{i_l,j}\, \varsigma_{j,\{i_1,\dots,i_{2p}\}}\Big(\max_{l} s_{i_l+1}\Big)\Bigg].$$

For each q, the number of the corresponding $B_{\{i_1,\dots,i_{2p}\}}$ in A is bounded by $Kk^q \le KN^q$ (recall that $h = T/N$), where K depends on p and is independent of N.

Lemma 3.2 implies the following rule: if $I_{i_k,j}$ appears in $\prod_{l=1}^{2p} I_{i_l,j}$ with even power $2m$, its contribution (i.e. the contribution of $I_{i_k,j}^{2m}$) to the expectation $B_{\{i_1,\dots,i_{2p}\}}$ is of order h^{3m}; if $I_{i_k,j}$ appears in $\prod_{l=1}^{2p} I_{i_l,j}$ with odd power $2m-1$, its contribution (i.e. the contribution of $I_{i_k,j}^{2m-1}$) to the expectation is of order h^{3m-1}. In particular, $I_{i_k,j}^2$ contributes h^3, while $I_{i_k,j}$ with a unique i_k contributes h^2. Then we can deduce the following estimates:

$$\begin{aligned} B_{\{i_1,\dots,i_{2p}\}} &\le Kh^{3p} \quad \text{for } 1 \le q \le p, \\ B_{\{i_1,\dots,i_{2p}\}} &\le Kh^{2p+q} \quad \text{for } p < q \le 2p. \end{aligned}$$

We note that, for some particular combinations $\{i_1,\dots,i_{2p}\}$, the terms $B_{\{i_1,\dots,i_{2p}\}}$ can be of smaller order than h^{3p} and h^{2p+q}, respectively. To clarify how Lemma 3.2 is used here, let us

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give a particular example. Let $p = 2$ and $i_1 < i_2$. By Lemma 3.2 we obtain

$$\begin{aligned}
\mathrm{E}[I_{i_1,j} I_{i_2,j}^3 \varsigma_{j,\{i_1,i_2\}}(s_{i_2+1})] &= \mathrm{E}[I_{i_1,j}\, \mathrm{E}[\mathrm{E}[I_{i_2,j}^3 \varsigma_{j,\{i_1,i_2\}}(s_{i_2+1}) \mid \mathcal{F}_{s_{i_2}}] \mid \mathcal{F}_{s_{i_1+1}}]] \\
&= h^5\, \mathrm{E}[I_{i_1,j}\, \mathrm{E}[\breve{\varsigma}_{j,\{i_1,i_2\}}(s_{i_2}) \mid \mathcal{F}_{s_{i_1+1}}]] \\
&= h^5\, \mathrm{E}[I_{i_1,j} \breve{\varsigma}_{j,\{i_1,i_2\}}(s_{i_1+1})] \\
&= h^5\, \mathrm{E}[\mathrm{E}[I_{i_1,j} \breve{\varsigma}_{j,\{i_1,i_2\}}(s_{i_1+1}) \mid \mathcal{F}_{s_{i_1}}]] \\
&= h^7\, \mathrm{E}[\breve{\varsigma}_{i_1}].
\end{aligned}$$

Summarizing,

$$A \le h^p K \sum_{q=1}^{p} h^{3p} N^q + h^p K \sum_{q=p+1}^{2p} h^{2p+q} N^q \le K h^{3p}.$$

Lemma 3.3 is proved.

Step 5. (The final step in the proof of Theorem 3.3.) Now we are in position to obtain an estimate for the third term in (3.26). Let p be integer. We have

$$\begin{aligned}
\mathrm{E}[|Q_{k+1,j}|^{2p}] &= \mathrm{E}\left[\left(Q_{k,j} + \sum_{i=0}^{k-1} J_{i,j,k} I_{i,j}\right)^{2p}\right] \\
&= \mathrm{E}\left[\left(Q_{k,j}^2 + 2Q_{k,j} \sum_{i=0}^{k-1} J_{i,j,k}\, I_{i,j} + \left(\sum_{i=0}^{k-1} J_{i,j,k}\, I_{i,j}\right)^2\right)^p\right] \\
&\le \mathrm{E}[|Q_{k,j}|^{2p}] + 2p\, \mathrm{E}\left[Q_{k,j}^{2p-1} \sum_{i=0}^{k-1} J_{i,j,k}\, I_{i,j}\right] \\
&\quad + K \sum_{m=2}^{2p} \mathrm{E}\left[|Q_{k,j}|^{2p-m} \left|\sum_{i=0}^{k-1} J_{i,j,k} I_{i,j}\right|^m\right]. \qquad (3.34)
\end{aligned}$$

The second term in (3.34) is equal to 0 because

$$\mathrm{E}\left[Q_{k,j}^{2p-1} \sum_{i=0}^{k-1} J_{i,j,k}\, I_{i,j}\right] = \mathrm{E}\left[Q_{k,j}^{2p-1} \sum_{i=0}^{k-1} I_{i,j}\, \mathrm{E}[J_{i,j,k} \mid \mathcal{F}_{s_k}^y]\right] = 0. \qquad (3.35)$$

Using inequality (3.29), we obtain

$$\begin{aligned}
&\mathrm{E}\left[|Q_{k,j}|^{2p-m} \left|\sum_{i=0}^{k-1} J_{i,j,k}\, I_{i,j}\right|^m\right] \\
&\quad \le h\, \mathrm{E}\left[\frac{2p-m}{2p} |Q_{k,j}|^{2p} + \frac{m}{2p} \left|\sum_{i=0}^{k-1} J_{i,j,k}\, I_{i,j}\right|^{2p} h^{-2p/m}\right], \qquad m = 2, \dots, 2p-1. \qquad (3.36)
\end{aligned}$$

Owing to Lemma 3.3, we have

$$\left|\sum_{i=0}^{k-1} J_{i,j,k}\, I_{i,j}\right|^{2p} \le K h^{3p}. \qquad (3.37)$$

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Relations (3.34)–(3.37) imply that

$$\mathrm{E}[|Q_{k+1,j}|^{2p}] \le \mathrm{E}[|Q_{k,j}|^{2p}] + Kh\,\mathrm{E}[|Q_{k,j}|^{2p}] + Kh^{2p+1}, \qquad \mathrm{E}[|Q_{1,j}|^{2p}] = 0,$$

whence

$$\mathrm{E}[|Q_{N,j}|^{2p}] \le Kh^{2p}. \tag{3.38}$$

Thus, it follows from (3.32), (3.33), and (3.38) that $\mathrm{E}[\Delta^{2p}] \le Kh^{2p}$ (see (3.26)), and, therefore, $\mathrm{E}[R_N^{2p}] \le Kh^{2p}$ (see (3.24)). Theorem 3.3 is proved.

Using Theorem 3.3, we prove the following almost-sure convergence result for the Euler method (3.18).

Theorem 3.4. *For almost every trajectory $y(\cdot)$ and any $\varepsilon > 0$, method (3.18) converges with weak order $1-\varepsilon$, i.e.*

$$|\,\mathrm{E}^y[F(y(T), X_{0,x}(T))] - \mathrm{E}^y[F(y(T), X_N)]| \le Ch^{1-\varepsilon} \quad a.s., \tag{3.39}$$

where C is a.s. a finite random variable independent of h.

Proof. Define

$$R := |\,\mathrm{E}^y[F(y(T), X_{0,x}(T))] - \mathrm{E}^y[F(y(T), X_N)]|.$$

Theorem 3.3 and Markov's inequality imply that

$$\mathrm{P}(R > h^\gamma) \le \frac{\mathrm{E}[R^{2p}]}{h^{2p\gamma}} \le Kh^{2p(1-\gamma)}.$$

Then, for any $\gamma = 1-\varepsilon$, there is a sufficiently large $p \ge 1$ such that (recall that $h = T/N$)

$$\sum_{N=1}^{\infty} \mathrm{P}\left(R > \frac{T^\gamma}{N^\gamma}\right) \le KT^{2p(1-\gamma)} \sum_{N=1}^{\infty} \frac{1}{N^{2p(1-\gamma)}} < \infty.$$

Hence, owing to the Borel–Cantelli lemma, the random variable $\varkappa := \sup_{h>0} h^{-\gamma}|R|$ is a.s. finite, which implies (3.39).

Remark 3.3. We can use other Euler-type methods to simulate (3.17) and prove their convergence analogously to Theorem 3.4. For instance, the error estimate (3.39) holds for the scheme

$$X_{k+1} = X_k + A(y(s_{k+1}), X_k)h + B(y(s_{k+1}), X_k)h^{1/2}\zeta_k, \qquad k = 0, \dots, N-1. \tag{3.40}$$

Remark 3.4. We observe in numerical experiments that methods (3.13) and (3.18) directly applied to system (2.13)–(2.14) can demonstrate numerically unstable behavior in some situations. We can see that system (2.13)–(2.14) has the peculiar structure. The second component of its solution can be written as the exponential function

$$\zeta_{s,x,1}(s') = \exp\{Z_{s,x,0}(s')\}, \tag{3.41}$$

where $Z_{s,x,z}(s')$ satisfies the equation

$$\begin{aligned} \mathrm{d}Z = &-\left(\sum_{i=1}^{d} \frac{\partial \beta^\top}{\partial x^i}\alpha_i y(s') + \frac{1}{2}\sum_{i,l=1}^{d} a_{il}\frac{\partial^2 \beta^\top}{\partial x^i \partial x^l} y(s') + \frac{1}{2}|\beta|^2\right)\mathrm{d}s' \\ &- (\sigma^\top \mathrm{grad}(\beta^\top y(s')))^\top \mathrm{d}w(s'), \qquad Z(s) = z. \end{aligned} \tag{3.42}$$

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Obviously, $\zeta_{s,x,1}(s')$ is positive. But methods (3.13) and (3.18) applied to (2.13)–(2.14), in general, do not preserve the positivity, which is apparently the reason for their occasional failure. We are avoiding this problem when we apply methods (3.13) and (3.18) to system (2.13), (3.42) and compute (cf. (2.15))

$$u_g(s,x) = \tilde{\mathrm{E}}^y[g(X_{s,x}(t))\exp\{\beta^\top(X_{s,x}(t))y(t) + Z_{s,x,0}(t)\}]. \tag{3.43}$$

In our experiments, this way of computing $u_g(s,x)$ gives quite accurate results, even in the cases when the direct application of methods (3.13) and (3.18) to (2.13)–(2.14) fails.

We should note that, strictly speaking, we cannot directly use Theorems 3.1–3.4 for functions $F(y,x,z) := g(x)\exp\{\beta^\top(x)y + z\}$ from (3.43) because it has exponential growth in z. The growth condition on F in Theorems 3.1–3.4 were imposed in order to ensure existence of corresponding moments required in the proofs of these theorems. At the same time, for Theorems 3.1–3.4 to be applicable, existence of such moments is sufficient. Since the conditions imposed on the coefficients of (2.13)–(2.15) in the introduction (in particular, $g(x)$, $\beta(x)$, and their derivatives are assumed to be bounded) ensure existence of all the moments of $F(y(T), X(T), Z(T))$ and its derivatives, the results of Theorems 3.1–3.4 are applicable in the case (3.42)–(3.43).

3.3. Recurrency of algorithms

We emphasize that the considered methods ensure a recurrent solution of the nonlinear filtering problem. In this connection we simulate the sample at time s_k:

$$X_k^{(m)} = \bar{X}^{(m)}_{0,x^{(m)}}(s_k), \qquad Z_k^{(m)} = \bar{Z}^{(m)}_{0,x^{(m)},0}(s_k), \qquad m = 1,\dots,M.$$

Then we can estimate the desired quantity $\hat{f}(s_k)$:

$$\hat{f}(s_k) \simeq \bar{f}(s_k) = \frac{(1/M)\sum_{m=1}^M f(X_k^{(m)})\exp\{\beta^\top(X_k^{(m)})y(s_k) + Z_k^{(m)}\}}{(1/M)\sum_{m=1}^M \exp\{\beta^\top(X_k^{(m)})y(s_k) + Z_k^{(m)}\}},$$

and continue the procedure to obtain the sample

$$X_{k+1}^{(m)} = \bar{X}^{(m)}_{s_k,X_k^{(m)}}(s_{k+1}), \qquad Z_{k+1}^{(m)} = \bar{Z}^{(m)}_{s_k,X_k^{(m)},Z_k^{(m)}}(s_{k+1}),$$

at the next time moment, which can be used for finding $\hat{f}(s_{k+1})$, and so on. This note is valid for Section 4 as well.

4. Numerical methods for the backward SPDE

This section is devoted to the numerical evaluation of the function $v_g(s,x)$ from (2.22)–(2.23). As discussed in Subsection 2.3, this function is the solution of the corresponding backward linear SPDE (2.24)–(2.25) and the numerator and denominator in the Kallianpur–Striebel formula, (2.9), have the form $\int v_g(0,x)\varphi(x)\,\mathrm{d}x$ (see (2.27)). To cover the case of other probabilistic representations for the SPDEs solution used for variance reduction in Section 5, we deal here with a slightly more general system than (2.23). We consider

$$v_g(s,x) = \tilde{\mathrm{E}}^y[g(X_{s,x}(t))\eta_{s,x,1}(t)],$$

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where $X_{s,x}(s')$, $\eta_{s,x,1}(s')$ is the solution of the system

$$\begin{aligned} \mathrm{d}X &= [\alpha(X) - \sigma(X)\lambda(s', X)]\,\mathrm{d}s' + \sigma(X)\,\mathrm{d}w(s'), \qquad X(s) = x, \\ \mathrm{d}\eta &= \lambda^\top(s', X)\eta\,\mathrm{d}w(s') + \beta^\top(X)\eta\,\mathrm{d}y(s'), \qquad \eta(s) = 1. \end{aligned} \tag{4.1}$$

Here $\lambda(t, x)$ is a d_1-dimensional vector whose components are assumed to be sufficiently smooth and to satisfy the boundedness conditions as before.

Taking into account Remark 3.4, we also introduce the scalar $Z_{s,x,z}(s')$, $s' \geq s$, satisfying the equation

$$\mathrm{d}Z = -\tfrac{1}{2}(|\lambda|^2 + |\beta|^2)\,\mathrm{d}s' + \lambda^\top(s', X)\,\mathrm{d}w(s') + \beta^\top(X)\,\mathrm{d}y(s'), \qquad Z(s) = z, \tag{4.2}$$

and we then write

$$v_g(s, x) = \tilde{\mathrm{E}}^y[g(X_{s,x}(t))\exp\{Z_{s,x,0}(t)\}]. \tag{4.3}$$

Recall that the expectation in (4.3) is with respect to $w(\cdot)$ only.

Now we apply the Euler method to (4.1) and (4.2):

$$\begin{aligned} X_{k+1} &= X_k + [\alpha(X_k) - \sigma(X_k)\lambda(s_k, X_k)]h + \sigma(X_k)h^{1/2}\zeta_k, \qquad X_0 = x, \\ Z_{k+1} &= Z_k - \tfrac{1}{2}(|\lambda(s_k, X_k)|^2 + |\beta(X_k)|^2)h + \lambda^\top(s_k, X_k)h^{1/2}\zeta_k \\ &\quad + \beta^\top(X_k)\Delta_k y, \qquad Z_0 = z,\ k = 0, \ldots, N-1,\ h = t/N, \end{aligned} \tag{4.4}$$

where $\zeta_k = (\zeta_k^1, \ldots, \zeta_k^{d_1})^\top$ is a vector whose components are i.i.d. random variables with the law $\zeta^i \sim \mathcal{N}(0, 1)$ or $\mathrm{P}(\zeta^i = \pm 1) = \frac{1}{2}$ and $\Delta_k y := y(s_{k+1}) - y(s_k)$. Then

$$\begin{aligned} v_g(s, x) &= \tilde{\mathrm{E}}^y[g(X_{s,x}(t))\exp\{Z_{s,x,0}(t)\}] \\ &\simeq \bar{v}_g(s, x) \\ &= \tilde{\mathrm{E}}^y[g(X_N)\exp\{Z_N\}] \\ &\simeq \frac{1}{M}\sum_{m=1}^{M} g(X_N^{(m)})\exp\{Z_N^{(m)}\}, \end{aligned} \tag{4.5}$$

where $X_N^{(m)}$ and $Z_N^{(m)}$ are independent realizations of X_N and Z_N.

Let us establish correspondence between method (4.4) for solving the backward SPDE and the methods applied to (2.13) and (3.42) for solving the backward pathwise filtering equation (see Subsection 2.3 for the relation between the backward SPDE (2.24)–(2.25) and the backward pathwise PDE (2.17)–(2.18)). For simplicity, we set $\lambda = 0$. We have

$$\begin{aligned} v_g(0, x) &\simeq \bar{v}_g(0, x) \\ &= \tilde{\mathrm{E}}^y[g(X_N)\exp\{Z_N\}] \\ &= \tilde{\mathrm{E}}^y\left[g(X_N)\exp\left\{-\frac{h}{2}\sum_{k=0}^{N-1}|\beta(X_k)|^2 + \sum_{k=0}^{N-1}\beta^\top(X_k)\Delta_k y\right\}\right]. \end{aligned} \tag{4.6}$$

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We expand $\beta^\top(X_{k+1})$ at X_k and obtain

$$\beta^\top(X_k)y(s_{k+1}) = \beta^\top(X_{k+1})y(s_{k+1}) - h\sum_{i=1}^{d}\frac{\partial\beta^\top}{\partial x^i}(X_k)\,y(s_{k+1})\alpha_i(X_k) - \frac{h}{2}\sum_{i,l=1}^{d}a_{il}(X_k)\frac{\partial^2\beta^\top}{\partial x^i\partial x^l}(X_k)\,y(s_{k+1}) - h^{1/2}\sum_{i=1}^{d}\frac{\partial\beta^\top}{\partial x^i}(X_k)\,y(s_{k+1})\sum_{j=1}^{d_1}\sigma_{ij}(X_k)\zeta_k^j + r_k, \tag{4.7}$$

where the reminder r_k is a combination of the coefficients α_i, σ_{ij}, β_i, the derivatives of β_i, and $y(s_{k+1})$, and it is such that, for a fixed $y(\cdot)$, we have

$$\tilde{\mathrm{E}}^y[r_k] = O(h^2),\quad \tilde{\mathrm{E}}^y[r_k^2] = O(h^3),\quad \tilde{\mathrm{E}}^y[r_jr_k] = O(h^4)\quad \text{for } j\neq k,\quad \tilde{\mathrm{E}}^y[r_k^4] = O(h^6),$$
$$\tilde{\mathrm{E}}^y[r_i^2r_jr_k] = O(h^7)\quad \text{for nonequal } i,j,k,\qquad \tilde{\mathrm{E}}^y[r_j^3r_k] = O(h^8)\quad \text{for } j\neq k,$$
$$\text{and}\quad \tilde{\mathrm{E}}^y[r_ir_jr_kr_l] = O(h^8)\quad \text{for nonequal } i,j,k,l. \tag{4.8}$$

Making use of (4.7), we rearrange (4.6) and obtain

$$\bar v_g(0,x) = \tilde{\mathrm{E}}^y\Bigg[g(X_N)\exp\Bigg\{\beta^\top(X_N)\,y(s_N) - \frac{h}{2}\sum_{k=0}^{N-1}|\beta(X_k)|^2 - \sum_{k=0}^{N-1}\Bigg[h\sum_{i=1}^{d}\frac{\partial\beta^\top}{\partial x^i}(X_k)\alpha_i(X_k)\,y(s_{k+1}) + \frac{h}{2}\sum_{i,l=1}^{d}a_{il}(X_k)\frac{\partial^2\beta^\top}{\partial x^i\partial x^l}(X_k)\,y(s_{k+1}) + h^{1/2}\sum_{i=1}^{d}\frac{\partial\beta^\top}{\partial x^i}(X_k)\,y(s_{k+1})\sum_{j=1}^{d_1}\sigma_{ij}(X_k)\zeta_k^j\Bigg] + \sum_{k=0}^{N-1}r_k\Bigg\}\Bigg]. \tag{4.9}$$

When we apply method (3.40) to system (2.13), (3.42), we obtain (see also (3.43))

$$u_g(0,x) \simeq \bar u_g(0,x) = \tilde{\mathrm{E}}^y\Bigg[g(X_N)\exp\Bigg\{\beta^\top(X_N)\,y(s_N) - \frac{h}{2}\sum_{k=0}^{N-1}|\beta(X_k)|^2 - \sum_{k=0}^{N-1}\Bigg[h\sum_{i=1}^{d}\frac{\partial\beta^\top}{\partial x^i}(X_k)\,\alpha_i(X_k)\,y(s_{k+1}) + \frac{h}{2}\sum_{i,l=1}^{d}a_{il}(X_k)\frac{\partial^2\beta^\top}{\partial x^i\partial x^l}(X_k)\,y(s_{k+1}) + h^{1/2}\sum_{i=1}^{d}\frac{\partial\beta^\top}{\partial x^i}(X_k)\,y(s_{k+1})\sum_{j=1}^{d_1}\sigma_{ij}(X_k)\zeta_k^j\Bigg]\Bigg\}\Bigg]. \tag{4.10}$$

Recall that $u_g(0,x) = v_g(0,x)$. Comparing (4.9) and (4.10), we see that method (4.4) coincides with (3.40) applied to system (2.13), (3.42) up to terms of higher order.

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Lemma 4.1. *For almost every trajectory $y(\cdot)$,*

$$|\bar{u}_g(0,x) - \bar{v}_g(0,x)| \le Kh, \tag{4.11}$$

where the constant K does not depend on the discretization step h.

Proof. Denote by $\bar{\psi}$ the expression under the expectation in (4.10). We have

$$R := |\bar{u}_g(0,x) - \bar{v}_g(0,x)| = \left|\tilde{\mathrm{E}}^y\left[\bar{\psi}\left(1 - \exp\left\{\sum_{k=0}^{N-1} r_k\right\}\right)\right]\right|.$$

We recall that the coefficients of the SDE systems are assumed to be bounded together with their derivatives and that the function g has growth not faster than polynomial at ∞. Under these assumptions, moments of $\exp\{\sum_{k=0}^{N-1} r_k\}$ and moments of $\bar{\psi}$ up to a sufficiently high order are finite.

By the Cauchy–Bunyakovskii inequality we obtain

$$R \le (\tilde{\mathrm{E}}^y[(\bar{\psi})^2])^{1/2}\left(\tilde{\mathrm{E}}^y\left[\left(1 - \exp\left\{\sum_{k=0}^{N-1} r_k\right\}\right)^2\right]\right)^{1/2}.$$

Expanding the exponent and using the Cauchy–Bunyakovskii inequality again, we obtain

$$\tilde{\mathrm{E}}^y\left[\left(1 - \exp\left\{\sum_{k=0}^{N-1} r_k\right\}\right)^2\right] = \tilde{\mathrm{E}}^y\left[\left(\sum_{k=0}^{N-1} r_k\, \mathrm{e}^{\theta}\right)^2\right] \le \left(\tilde{\mathrm{E}}^y\left[\left(\sum_{k=0}^{N-1} r_k\right)^4\right]\tilde{\mathrm{E}}^y[\mathrm{e}^{2\theta}]\right)^{1/2},$$

where θ is an intermediate (random) point between 0 and $\sum_{k=0}^{N-1} r_k$. Using relations (4), we obtain $\tilde{\mathrm{E}}^y[(\sum_{k=0}^{N-1} r_k)^4] = O(h^4)$. Hence, (4.11) holds.

Lemma 4.1 and Theorems 3.3 and 3.4 together with Remark 3.4 imply convergence of method (4.4), which is stated in the following theorem.

Theorem 4.1. *Method (4.4) satisfies the following inequality for $p \ge 1$:*

$$(\tilde{\mathrm{E}}[|\tilde{\mathrm{E}}^y[g(X_{s,x}(t))\exp\{Z_{s,x,0}(t)\}] - \tilde{\mathrm{E}}^y[g(X_N)\exp\{Z_N\}]|^{2p}])^{1/2p} \le Kh, \tag{4.12}$$

where the constant K does not depend on the discretization step h.

For almost every trajectory $y(\cdot)$ and any $\varepsilon > 0$, method (4.4) converges with weak order $1-\varepsilon$, i.e.

$$|\tilde{\mathrm{E}}^y[g(X_{s,x}(t))\exp\{Z_{s,x,0}(t)\}] - \tilde{\mathrm{E}}^y[g(X_N)\exp\{Z_N\}]| \le Ch^{1-\varepsilon} \quad a.s.,$$

where C is a.s. a finite random variable independent of h.

Remark 4.1. In [25], inequality (4.12) is proved for $p = 1$ and $\lambda = 0$.

Remark 4.2. The methods from this section are applicable to more general nonlinear filtering problems than (1.1)–(1.2), whereas the pathwise approach places some limitations on the classes of nonlinear filtering problems which can be treated by it.

5. Variance reduction

Variance reduction is of crucial importance for effectiveness of any Monte Carlo procedure. In this section we exploit various probabilistic representations of solutions to the considered problems to reduce the Monte Carlo error.

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5.1. Variance reduction in the pathwise approach: deterministic initial data

The error of the Monte Carlo method, (3.1), is evaluated by

$$\bar{\rho} = c\,\frac{[\mathrm{var}^y(G(t,\bar{X}_{0,\xi}(t))\bar{\zeta}_{0,\xi,1}(t))]^{1/2}}{M^{1/2}},$$

where, for example, the values $c = 1, 2, 3$ correspond to the fiducial probabilities 0.68, 0.95, and 0.997, respectively. We recall (see (2.18)) that

$$G(t,x) = g(t,x)\exp\{\beta^\top(x)y(t)\}.$$

Since $\mathrm{var}^y(G(t,\bar{X}_{0,\xi}(t))\bar{\zeta}_{0,\xi,1}(t))$ is close to $\mathrm{var}^y(G(t,X_{0,\xi}(t))\zeta_{0,\xi,1}(t))$, we can assume that the error of the Monte Carlo method is estimated by

$$\rho = c\frac{[\mathrm{var}^y(G(t,X_{0,\xi}(t))\zeta_{0,\xi,1}(t))]^{1/2}}{M^{1/2}}.$$

If $\mathrm{var}^y(G(t,X_{0,\xi}(t))\zeta_{0,\xi,1}(t))$ is large then to achieve a satisfactory accuracy we have to simulate a very large number of trajectories. Fortunately, there exist other probabilistic representations for $u_{g,\varphi}$ (see (2.21)) which allow us to reach smaller variances.

We begin with the case when ξ is deterministic, i.e. $\xi = x$. To obtain various probabilistic representations of the solution to problem (2.17)–(2.18), we introduce the system (see [17], [18, p. 126], and [21])

$$\mathrm{d}X = b(s',X)\,\mathrm{d}s' - \sigma(X)\mu(s',X)\,\mathrm{d}s' + \sigma(X)\,\mathrm{d}w(s'), \qquad X(s) = x, \tag{5.1}$$

$$\mathrm{d}\mathcal{X} = c(s',X)\mathcal{X}\,\mathrm{d}s' + \mu^\top(s',X)\mathcal{X}\,\mathrm{d}w(s'), \qquad \mathcal{X}(s) = \chi, \tag{5.2}$$

$$\mathrm{d}\mathbb{X} = \mathbb{F}^\top(s',X)\mathcal{X}\,\mathrm{d}w(s'), \qquad \mathbb{X}(s) = \tilde{x}, \tag{5.3}$$

where $b(s,x)$ is given in (2.19), $c(s,x)$ is given in (2.20), $\mathcal{X}$ and $\mathbb{X}$ are scalars, and μ and $\mathbb{F}$ are column-vector functions of dimension d_1 satisfying some regularity conditions (e.g. they have bounded derivatives with respect to x^i up to some order). We should note that X and w in (5.1) differ from X and w in (1.1); however, this does not lead to any ambiguity.

In particular, if in (5.1)–(5.3) we take

$$\mu(s',x) = -\sigma^\top(x)\mathrm{grad}(\beta^\top(x)y(s')), \tag{5.4}$$

we obtain the system

$$\mathrm{d}X = \alpha(X)\,\mathrm{d}s' + \sigma(X)\,\mathrm{d}w(s'), \qquad X(s) = x,$$

$$\mathrm{d}\mathcal{X} = c(s',X)\mathcal{X}\,\mathrm{d}s' - (\sigma^\top(X)\mathrm{grad}(\beta^\top(X)Y(s')))^\top\mathcal{X}\,\mathrm{d}w(s'), \qquad \mathcal{X}(s) = \chi,$$

$$\mathrm{d}\mathbb{X} = \mathbb{F}^\top(s',X)\mathcal{X}\,\mathrm{d}w(s'), \qquad \mathbb{X}(s) = \tilde{x}.$$

We see that the first two equations of this system coincide with system (2.13)–(2.14).

The solution to problem (2.17)–(2.18) has the following probabilistic representation:

$$u(s,x) = \tilde{\mathrm{E}}^y[G(t,X_{s,x}(t))\mathcal{X}_{s,x,1}(t) + \mathbb{X}_{s,x,1,0}(t)], \tag{5.5}$$

where $X_{s,x}(s')$, $\mathcal{X}_{s,x,1}(s')$, $\mathbb{X}_{s,x,1,0}(s')$, $s' \geq s$, is the solution of the Cauchy problem, (5.1)–(5.3). The usual probabilistic representation (when $\mu = 0$ and $\mathbb{F} = 0$) is a particular

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case of (5.5) (see, e.g. [11]). The representation for $\mu \neq 0$ and $\mathbb{F} = 0$ follows from Girsanov's theorem, and (5.5) follows from the equality

$$\tilde{\mathrm{E}}^y\left[\int_s^t \mathbb{F}^\top(s', X(s'))\mathcal{X}(s')\,\mathrm{d}w(s')\right] = 0 \quad \text{for any } \mathbb{F}.$$

Consider the random variable

$$\begin{aligned}\Gamma &= \Gamma_{s,x}\\ &= \Gamma_{s,x}(t)\\ &:= G(t, X_{s,x}(t))\mathcal{X}_{s,x,1}(t) + \mathbb{X}_{s,x,1,0}(t)\\ &= g(X_{s,x}(t))\exp\{\beta^\top(X_{s,x}(t))y(t)\}\mathcal{X}_{s,x,1}(t) + \mathbb{X}_{s,x,1,0}(t).\end{aligned} \tag{5.6}$$

It is clear that

$$u(s,x) = \tilde{\mathrm{E}}^y[\Gamma_{s,x}], \qquad u_g(0,x) = u(0,x) = \tilde{\mathrm{E}}^y[\Gamma_{0,x}]. \tag{5.7}$$

The corresponding Monte Carlo estimate of $u(s,x)$ has the form

$$u(s,x) \simeq \bar{u}(s,x) = \frac{1}{M}\sum_{m=1}^{M} \bar{\Gamma}_{0,x}^{(m)}, \tag{5.8}$$

where the $\bar{\Gamma}_{0,x}^{(m)}$, $m = 1, \ldots, M$, are independent realizations of $\bar{\Gamma}_{0,x}$:

$$\bar{\Gamma}_{0,x} = G(t, \bar{X}_{0,x}(t))\bar{\mathcal{X}}_{0,x,1}(t) + \bar{\mathbb{X}}_{0,x,1,0}(t).$$

The Monte Carlo error of estimate (5.8) is of order $(M^{-1}\operatorname{var}^y\Gamma)^{1/2}$, where M is the number of simulations of the random variable Γ. While the mean $\tilde{\mathrm{E}}^y[\Gamma]$ does not depend on the choice of μ and $\mathbb{F}$, the variance $\operatorname{var}^y\Gamma = \tilde{\mathrm{E}}^y[\Gamma^2] - (\tilde{\mathrm{E}}^y[\Gamma])^2$ does. Thus, by decreasing the variance $\operatorname{var}^y\Gamma$, the Monte Carlo error can be reduced. The following theorem is proved in [21] (see also [17] and [18, p. 129]).

Theorem 5.1. *Let μ and $\mathbb{F}$ be such that, for any $x \in \mathbb{R}^d$, there exists a solution to system (5.1)–(5.3) on the interval $[s,t]$. Then the variance* $\operatorname{var}^y\Gamma$ *is equal to*

$$\operatorname{var}^y\Gamma = \tilde{\mathrm{E}}^y\left[\int_s^t \mathcal{X}_{s,x,1}^2(s')\sum_{j=1}^{d_1}\left(\sum_{i=1}^{d}\sigma_{ij}\frac{\partial u}{\partial x^i} + u\mu_j + \mathbb{F}_j\right)^2 \mathrm{d}s'\right], \tag{5.9}$$

provided that the expectation in (5.9) exists. In (5.9), all the functions σ_{ij}, μ_j, $\mathbb{F}_j$, u, and $\partial u/\partial x^i$ have $(s', X_{s,x}(s'))$ as their argument.

In particular, if μ and $\mathbb{F}$ are such that

$$\sum_{i=1}^{d}\sigma_{ij}\frac{\partial u}{\partial x^i} + u\mu_j + \mathbb{F}_j = 0, \qquad j = 1, \ldots, d_1, \tag{5.10}$$

then $\operatorname{var}^y\Gamma = 0$*, i.e. Γ is deterministic.*

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We recall that if we set $\mathbb{F} = 0$ here then we obtain the method of importance sampling, and if we set $\mu = 0$ then we obtain the method of control variates. Theorem 5.1 establishes the combining method of variance reduction [18, Section 2.4.2].

Obviously, μ and $\mathbb{F}$ satisfying (5.10) cannot be constructed without knowing $u(s', x)$, $s \le s' \le t$, $x \in \mathbb{R}^d$. Nevertheless, the theorem claims a general possibility of variance reduction by a proper choice of the functions μ_j and $\mathbb{F}_j$, $j = 1, \dots, d_1$. Theorem 5.1 can be used, for example, if we know a function $u^0(s', x)$ connected with an approximating problem and being close to $u(s', x)$. In this case we take any $\mu_j^0, \mathbb{F}_j^0$, $j = 1, \dots, d_1$, satisfying

$$\sum_{i=1}^{d} \sigma_{ij} \frac{\partial u^0}{\partial x^i} + u^0 \mu_j^0 + \mathbb{F}_j^0 = 0,$$

and then the variance $\operatorname{var}^y \Gamma$ is, although not 0, small.

5.2. Variance reduction in the pathwise approach: random initial data

Now let us return to variance reduction for $u_{g,\varphi}$. Clearly, along with (2.21), $u_{g,\varphi}$ has the following probabilistic representations:

$$u_{g,\varphi} = \tilde{\mathrm{E}}^y[\Gamma_{0,\xi}],$$

where ξ is a random variable with density φ and $\Gamma_{s,x}$ is defined in (5.6). This representation gives the following estimate:

$$u_{g,\varphi} \simeq \frac{1}{M} \sum_{m=1}^{M} \bar{\Gamma}^{(m)}_{0,x^{(m)}},$$

where $x^{(m)}$ is a sample distributed according to $\varphi(\cdot)$ and

$$\bar{\Gamma}^{(m)}_{0,x^{(m)}} = G\big(t, \bar{X}^{(m)}_{0,x^{(m)}}(t)\big)\bar{\mathcal{X}}^{(m)}_{0,x^{(m)},1}(t) + \bar{\mathbb{X}}^{(m)}_{0,x^{(m)},1,0}(t). \tag{5.11}$$

We have (recall that $u_g(0, x) = u(0, x) = \tilde{\mathrm{E}}^y[\Gamma_{0,x}]$ (see (5.7)))

$$\begin{aligned}
\operatorname{var}^y(\Gamma_{0,\xi}) &= \tilde{\mathrm{E}}^y[\Gamma_{0,\xi}^2] - (\tilde{\mathrm{E}}^y[\Gamma_{0,\xi}])^2 \\
&= \tilde{\mathrm{E}}^y[\tilde{\mathrm{E}}^y[\Gamma_{0,\xi}^2 \mid \xi]] - (\tilde{\mathrm{E}}^y[\tilde{\mathrm{E}}^y[\Gamma_{0,\xi} \mid \xi]])^2 \\
&= \int_{\mathbb{R}^d} \tilde{\mathrm{E}}^y[\Gamma_{0,x}^2]\varphi(x)\,\mathrm{d}x - \left[\int_{\mathbb{R}^d} \tilde{\mathrm{E}}^y[\Gamma_{0,x}]\varphi(x)\,\mathrm{d}x\right]^2 \\
&= \int_{\mathbb{R}^d} \tilde{\mathrm{E}}^y[\Gamma_{0,x}^2]\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} (\tilde{\mathrm{E}}^y[\Gamma_{0,x}])^2\varphi(x)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} (\tilde{\mathrm{E}}^y[\Gamma_{0,x}])^2\varphi(x)\,\mathrm{d}x - \left(\int_{\mathbb{R}^d} \tilde{\mathrm{E}}^y[\Gamma_{0,x}]\varphi(x)\,\mathrm{d}x\right)^2 \\
&= \int_{\mathbb{R}^d} \operatorname{var}^y(\Gamma_{0,x})\varphi(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} u_g^2(0, x)\varphi(x)\,\mathrm{d}x - \left(\int_{\mathbb{R}^d} u_g(0, x)\varphi(x)\,\mathrm{d}x\right)^2 \\
&= \int_{\mathbb{R}^d} \operatorname{var}^y(\Gamma_{0,x})\varphi(x)\,\mathrm{d}x + \operatorname{var}(u(0, \xi)).
\end{aligned} \tag{5.12}$$

The variance $\operatorname{var}^y(\Gamma_{0,x})$ in (5.12) can be reduced due to Theorem 5.1 if a suitable function $u^0(s', x)$, $0 \le s' \le t$, $x \in \mathbb{R}^d$, is known. The second term, $\operatorname{var}(u(0, \xi))$, is connected with

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the Monte Carlo error in evaluation of the deterministic integral $\int u(0,x)\varphi(x)\,\mathrm{d}x$. To reduce $\mathrm{var}(u(0,\xi))$, we can use the same function at $s'=0$, the function $u^0(x) := u^0(0,x)$. Let us assume (without any essential loss of generality) that the function $u^0(x)$ is positive and, in addition, that the value of the integral

$$I^0 = \int_{\mathbb{R}^d} u^0(x)\varphi(x)\,\mathrm{d}x$$

is known. The direct Monte Carlo evaluation is given by

$$\int_{\mathbb{R}^d} u(0,x)\varphi(x)\,\mathrm{d}x \simeq \frac{1}{M}\sum_{m=1}^{M} u(0,x^{(m)}), \tag{5.13}$$

where the $x^{(m)}$, $m=1,\dots,M$, are independent realizations of the random variable distributed according to the density $\varphi(x)$, while the Monte Carlo evaluation with variance reduction is given by

$$\int_{\mathbb{R}^d} u(0,x)\varphi(x)\,\mathrm{d}x = I^0\int_{\mathbb{R}^d}\frac{u(0,x)}{u^0(x)}\frac{u^0(x)\varphi(x)}{I^0}\,\mathrm{d}x \simeq \frac{1}{M}\sum_{m=1}^{M}\frac{I^0 u(0,x^{(m)})}{u^0(x^{(m)})}, \tag{5.14}$$

where the $x^{(m)}$, $m=1,\dots,M$, are independent realizations of the random variable distributed according to the density

$$\psi(x) = \frac{u^0(x)\varphi(x)}{I^0}. \tag{5.15}$$

We note that both M and $x^{(m)}$ are, in general, different in (5.13) and (5.14). We also note that there is no bias in both (5.13) and (5.14).

The above arguments lead to the following estimate:

$$u_{g,\varphi} \simeq \frac{I^0}{M}\sum_{m=1}^{M}\frac{\bar\Gamma^{(m)}_{0,x^{(m)}}}{u^0(x^{(m)})}, \tag{5.16}$$

where $x^{(m)}$ is distributed according to $\psi(x)$ in (5.15) and $\bar\Gamma^{(m)}_{0,x^{(m)}}$ is defined in (5.11). The bias of this estimate is equal to the error of approximate integration (see Section 3), and the Monte Carlo error is evaluated by

$$\rho = c\frac{[\mathrm{var}^y(I^0\Gamma_{0,\xi}/u^0(\xi))]^{1/2}}{M^{1/2}},$$

where ξ is distributed according to ψ in (5.15). It is not difficult to obtain

$$\mathrm{var}^y\left(\frac{I^0\Gamma_{0,\xi}}{u^0(\xi)}\right) = \int_{\mathbb{R}^d}\left(\frac{I^0}{u^0(x)}\right)^2\mathrm{var}^y(\Gamma_{0,x})\psi(x)\,\mathrm{d}x + \mathrm{var}\left(\frac{I^0 u(0,\xi)}{u^0(\xi)}\right). \tag{5.17}$$

Thus, if $u^0(x) = u(0,x)$, the variance $\mathrm{var}(I^0u(0,\xi)/u^0(\xi))$ in (5.17) vanishes. Combining this conclusion with Theorem 5.1, we arrive at the following theorem.

Theorem 5.2. *If μ and $\mathbb{F}$ satisfy (5.10), if $u(0,x)$ is positive, and if $\psi(x)$ is chosen as*

$$\psi(x) = \frac{u(0,x)\varphi(x)}{I}, \qquad I = \int_{\mathbb{R}^d} u(0,x)\varphi(x)\,\mathrm{d}x,$$

then the Monte Carlo error of (5.16) vanishes, i.e. $\mathrm{var}^y(I\Gamma_{0,\xi}/u(0,\xi)) = 0$.

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We conclude that, by a proper choice of μ and $\mathbb{F}$, the Monte Carlo error can be reduced. At the same time, the question of how to find them in concrete filtering problems constructively requires further investigation.

5.3. Variance reduction in the SPDE approach

As discussed in Subsection 2.3, we can realize the Kallianpur–Striebel formula by computing the SPDEs solution $v_{g,\varphi}$ (see (2.22), (2.27), and (4.5)):

$$\begin{aligned} v_{g,\varphi} &= \tilde{\mathrm{E}}^y[g(X_{0,\xi}(t))\eta_{0,\xi,1}(t)] \\ &= \tilde{\mathrm{E}}^y[g(X_{0,\xi}(t))\exp\{Z_{0,\xi,0}(t)\}] \\ &\simeq \bar{v}_{g,\varphi} \\ &= \tilde{\mathrm{E}}^y[g(\bar{X}_{0,\xi}(t))\exp\{\bar{Z}_{0,\xi,0}(t)\}] \\ &\simeq \frac{1}{M}\sum_{m=1}^{M} g(\bar{X}^{(m)}_{0,x^{(m)}}(t))\exp\{\bar{Z}^{(m)}_{0,x^{(m)},0}(t)\}, \end{aligned} \tag{5.18}$$

where ξ is a random variable with density $\varphi(\cdot)$. The Monte Carlo error of (5.18) is evaluated by

$$\rho = c\frac{[\mathrm{var}^y(g(X_{0,\xi}(t))\exp\{Z_{0,\xi,0}(t)\})]^{1/2}}{M^{1/2}}. \tag{5.19}$$

To reduce it, we can use the same ideas as in Subsections 5.1 and 5.2.

Let $\xi = x$ be deterministic. Introduce the system

$$\begin{aligned} &\mathrm{d}X = [\alpha(X) - \sigma(X)\lambda(s', X)]\,\mathrm{d}s' + \sigma(X)\,\mathrm{d}w(s'), \qquad X(s) = x, \\ &\mathrm{d}\eta = \lambda^\top(s', X)\eta\,\mathrm{d}w(s') + \beta^\top(X)\eta\,\mathrm{d}y(s'), \qquad \eta(s) = 1, \\ &\mathrm{d}\tilde{\mathbb{X}} = \tilde{\mathbb{F}}^\top(s', X)\eta\,\mathrm{d}w(s'), \qquad \tilde{\mathbb{X}}(s) = 0, \end{aligned} \tag{5.20}$$

where λ and $\mathbb{F}$ are column-vector functions of dimension d_1 satisfying some regularity conditions. By similar arguments as in Subsection 5.1 we obtain (see also (4.3))

$$\begin{aligned} v_g(s,x) &= \mathrm{E}^y[g(X_{s,x}(t))\eta_{s,x,1}(t)]|_{(2.23)} \\ &= \mathrm{E}^y[g(X_{s,x}(t))\eta_{s,x,1}(t) + \tilde{\mathbb{X}}_{s,x,1,0}(t)]|_{(5.20)} \\ &= \mathrm{E}^y[g(X_{s,x}(t))\exp\{Z_{s,x,0}(t)\} + \tilde{\mathbb{X}}_{s,x,1,0}(t)]|_{(5.20),(5.22)}, \end{aligned} \tag{5.21}$$

where

$$\mathrm{d}Z = -\tfrac{1}{2}(|\lambda|^2 + |\beta|^2)\,\mathrm{d}s' + \lambda^\top(s', X)\,\mathrm{d}w(s') + \beta^\top(X)\,\mathrm{d}y(s'), \qquad Z(s) = 0. \tag{5.22}$$

For $\lambda = 0$ and $\tilde{\mathbb{F}} = 0$, representation (5.18)–(5.20) coincides with (2.22)–(2.23). The variety of probabilistic representations (5.18)–(5.20) for the backward SPDE (2.24)–(2.25) can be exploited for variance reduction.

Let us take λ and $\tilde{\mathbb{F}}$ in the form

$$\lambda(s', x) = \mu(s', x) + \sigma^\top(x)\mathrm{grad}(\beta^\top(x)y(s')), \qquad \tilde{\mathbb{F}}(s', x) = \exp\{-\beta^\top(x)y(s')\}\mathbb{F}(s', x). \tag{5.23}$$

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Then it is not difficult to see that the first equation in (5.20) coincides with (5.1). Furthermore, solving the linear equations for η in (5.20) and for $\mathcal{X}$ in (5.2), and then using calculations analogous to those in (2.12), we obtain

$$\eta_{s,x,1}(t) = \exp\{\beta^\top(X_{s,x}(t))y(t) - \beta^\top(x)y(s)\}\mathcal{X}_{s,x,1}(t) \tag{5.24}$$

(we note that $\eta(t)$ in (2.12) corresponds to $\eta_{0,x,1}(t)$ from (5.20) with $\lambda = 0$). Equalities (5.23) and (5.24) imply that

$$\tilde{\mathbb{X}}_{s,x,1,0}(t) = \exp\{-\beta^\top(x)y(s)\}\mathbb{X}_{s,x,1,0}(t).$$

Consequently,

$$\tilde{\Gamma}_{s,x}(t) = \exp\{-\beta^\top(x)y(s)\}\Gamma_{s,x}(t),$$

where $\Gamma_{s,x}(t)$ is defined in (5.6) and (see (5.21))

$$\tilde{\Gamma}_{s,x}(t) := g(X_{s,x}(t))\eta_{s,x,1}(t) + \tilde{\mathbb{X}}_{s,x,1,0}(t).$$

Therefore, Theorem 5.1 and the relation (see (2.26))

$$v(s,x) = \exp\{-\beta^\top(x)y(s)\}u(s,x)$$

imply the following theorem.

Theorem 5.3. *Let λ and $\mathbb{F}$ be such that, for any $x \in \mathbb{R}^d$, there exists a solution to system (5.20) on the interval $[0,t]$. Then the variance $\operatorname{var}^y \tilde{\Gamma}_{s,x}(t)$ is equal to*

$$\operatorname{var}^y(\tilde{\Gamma}_{s,x}(t)) = \tilde{\mathrm{E}}^y\left[\int_s^t \eta_{s,x,1}^2 \sum_{j=1}^{d_1}\Big(\sum_{i=1}^{d}\sigma_{ij}\frac{\partial v}{\partial x^i} + v\lambda_j + \tilde{\mathbb{F}}_j\Big)^2 \mathrm{d}s'\right], \tag{5.25}$$

provided that the expectation in (5.25) exists. In (5.25), all the functions σ_{ij}, λ_j, $\tilde{\mathbb{F}}_j$, β_i, y_i, v, and $\partial v/\partial x^i$ have $(s', X_{s,x}(s'))$ as their argument.

We see that if $\lambda(s,x)$ and $\tilde{\mathbb{F}}(s,x)$ are such that

$$\sum_{i=1}^{d}\sigma_{ij}\frac{\partial v}{\partial x^i} + v\lambda_j + \tilde{\mathbb{F}}_j = 0, \qquad j = 1,\dots,d_1, \tag{5.26}$$

then the right-hand side of (5.25) is 0 and, consequently, the variance is 0. We recall that $v(s',x)$ depends on $y(s'')$, $s' \le s'' \le t$. However, we need to require that $\lambda(s',x)$ does not depend on $y(s'')$, $s' \le s'' \le t$, otherwise $X(s')$ in (5.20) depends on $y(s'')$, $s' \le s'' \le t$, and we are then faced with the problem of interpreting the term $\beta^\top(X(s'))\,\mathrm{d}y(s')$ in (5.22). At the same time, dependence of the function $\tilde{\mathbb{F}}(s',x)$ on $y(s'')$ does not cause any trouble in interpreting the third equation in (5.20), and the identity (5.26) can, in principle, be reached.

Now consider the Monte Carlo error (5.19) in computing $v_{g,\varphi}$, i.e. when the initial data are random. Analogously to (5.16), we can propose the following estimate instead of (5.18):

$$\begin{aligned} v_{g,\varphi} &\simeq \bar{v}_{g,\varphi} \\ &= \tilde{\mathrm{E}}^y[g(\bar{X}_{0,\xi}(t))\exp\{\bar{Z}_{0,\xi,0}(t)\} + \bar{\mathbb{X}}_{0,\xi,1,0}(t)] \\ &\simeq \frac{I^0}{M}\sum_{m=1}^{M}\frac{1}{v^0(x^{(m)})}\big(g(\bar{X}^{(m)}_{0,x^{(m)}}(t))\exp\big\{\bar{Z}^{(m)}_{0,x^{(m)},0}(t)\big\} + \bar{\mathbb{X}}^{(m)}_{0,x^{(m)},1,0}(t)\big), \end{aligned} \tag{5.27}$$

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where $x^{(m)}$ is distributed according to

$$\psi(x) = \frac{v^0(x)\varphi(x)}{I_0}, \qquad v^0(x) > 0,\ x \in \mathbb{R}^d,\ I_0 = \int_{\mathbb{R}^d} v^0(x)\varphi(x)\,\mathrm{d}x.$$

The variance of (5.27) can be estimated by

$$\mathrm{var}^y\left(\frac{I^0\tilde{\Gamma}_{0,\xi}(t)}{v^0(\xi)}\right),$$

where ξ is distributed according to ψ. Analogously to (5.17), we obtain

$$\mathrm{var}^y\left(\frac{I^0\tilde{\Gamma}_{0,\xi}(t)}{v^0(\xi)}\right) = \int_{\mathbb{R}^d}\left(\frac{I^0}{v^0(x)}\right)^2 \mathrm{var}^y(\tilde{\Gamma}_{0,x}(t))\psi(x)\,\mathrm{d}x + \mathrm{var}\left(\frac{I^0 v(0,\xi)}{v^0(\xi)}\right).$$

The variance of the estimate in (5.27) can be made smaller than the variance of the estimate in (5.18) by a proper choice of $\lambda(s,x)$, $\tilde{\mathbb{F}}(s,x)$, and $v^0(x)$.

6. Numerical experiments

In this section we test the proposed algorithms on two models for which the solution to the filtering problem is known exactly. First we consider the linear filtering problem for which the exact solution is given by the Kalman–Bucy filter. The second model is nonlinear, its exact solution is obtained by the Beneš filter.

6.1. Linear filtering

Consider the linear stochastic system

$$\mathrm{d}X = \alpha X\,\mathrm{d}s' + \sigma\,\mathrm{d}w(s'), \qquad X(0) = \xi, \tag{6.1}$$

$$\mathrm{d}y = \beta X\,\mathrm{d}s' + \mathrm{d}v(s'), \qquad y(0) = 0, \tag{6.2}$$

where X and y are scalars, α, σ, and β are parameters, and the random variable ξ is normally distributed, i.e. $\mathcal{N}(m_0, P_0)$.

We are interested in computing the estimate

$$\hat{X}(t) := \mathrm{E}[X(t) \mid y(s'),\ 0 \le s' \le t]. \tag{6.3}$$

According to the Kalman–Bucy filter, the estimate $\hat{X}(s)$ satisfies the system

$$\begin{aligned} \mathrm{d}\hat{X} &= \alpha\hat{X}\,\mathrm{d}s' + \beta P(s')(\mathrm{d}y - \beta\hat{X}\,\mathrm{d}s'), \qquad \hat{X}(0) = m_0,\\ \dot{P} &= 2\alpha P - \beta^2 P^2 + \sigma^2, \qquad P(0) = P_0. \end{aligned} \tag{6.4}$$

In our tests we fix the observation $y(t)$, which is obtained as a result of simulation of (6.1)–(6.2), by the mean-square Euler method with small time step ($h = 0.0001$) for a particular, fixed realization of the Wiener process $(w(s'), v(s'))$. We compare the algorithms proposed in the previous sections with the reference value of $\hat{X}$ obtained by an accurate simulation of the Kalman–Bucy filter, (6.4), which is done by the Euler method with $h = 0.0001$.

In Tables 1 and 2 we present results obtained by the three numerical algorithms: the 'pathwise, integral method', the 'pathwise, Euler method', and the 'SPDE method', which are described below. The first two algorithms solve the filtering problem via the backward

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TABLE 1: *Linear filtering.* Simulation of $\hat{X}(t)$ from (6.3) by various algorithms (see their description in the text) and with various time steps h. The reference value of $\hat{X}(t)$ is found due to the Kalman–Bucy filter, (6.4). Here $\alpha = -1$, $\sigma = 1$, $\beta = -1$, $t = 1$, $m_0 = 0$, and $P_0 = 1$. The expectations are computed by the Monte Carlo technique simulating $M = 400\,000$ independent realizations. The '$\pm$' reflects the Monte Carlo error only, it does not reflect the error of the method. All simulations are done along the same observation path $y(t)$. The corresponding reference value is $-0.377\,42$.

h	Pathwise, integral method	Pathwise, Euler method	SPDE method
0.20	-0.4032 ± 0.0024	-0.4132 ± 0.0023	-0.3311 ± 0.0023
0.10	-0.3860 ± 0.0023	-0.4101 ± 0.0023	-0.3688 ± 0.0023
0.05	-0.3807 ± 0.0023	-0.3933 ± 0.0023	-0.3732 ± 0.0023
0.02	-0.3794 ± 0.0023	-0.3854 ± 0.0023	-0.3778 ± 0.0023
0.01	-0.3774 ± 0.0023	-0.3790 ± 0.0023	-0.3752 ± 0.0023

TABLE 2: *Linear filtering.* Simulation of $\hat{X}(t)$ from (6.3). Here $\alpha = 1$, $\sigma = 1$, $\beta = 1$, $t = 1$, $m_0 = 0$, and $P_0 = 1$. The expectations are computed by the Monte Carlo technique simulating $M = 400\,000$ independent realizations. The corresponding reference value is 2.8655.

h	Pathwise, integral method	Pathwise, Euler method	SPDE method
0.20	3.303 ± 0.039	3.570 ± 0.041	2.698 ± 0.027
0.10	3.083 ± 0.034	3.300 ± 0.038	2.850 ± 0.029
0.05	2.968 ± 0.031	3.114 ± 0.034	2.890 ± 0.030
0.02	2.903 ± 0.030	2.957 ± 0.031	2.869 ± 0.030
0.01	2.888 ± 0.030	2.905 ± 0.030	2.861 ± 0.029

pathwise filtering equation (see Subsection 2.2). To this end, we introduce the following system (cf. (5.1)–(5.3)):

$$\mathrm{d}X = b(s', X)\,\mathrm{d}s' - \sigma\mu(s', X)\,\mathrm{d}s' + \sigma\,\mathrm{d}w(s'), \qquad X(s) = x, \tag{6.5}$$

$$\mathrm{d}\mathcal{X} = c(s', X)\mathcal{X}\,\mathrm{d}s' + \mu(s', X)\mathcal{X}\,\mathrm{d}w(s'), \qquad \mathcal{X}(s) = \chi, \tag{6.6}$$

$$\mathrm{d}\mathbb{X} = \mathbb{F}(s', X)\mathcal{X}\,\mathrm{d}w(s'), \qquad \mathbb{X}(s) = \tilde{x}, \tag{6.7}$$

where (cf. (2.19)–(2.20))

$$b(s', x) = \alpha x - \sigma^2\beta y(s'), \qquad c(s', x) = -\alpha x\beta y(s') - \frac{\beta^2 x^2}{2} + \frac{\sigma^2}{2}\beta^2 y^2(s'). \tag{6.8}$$

We also introduce $Z_{s,x,0}(s')$ such that (see (3.41)–(3.42) and Remark 3.4)

$$\mathcal{X}_{s,x,1}(s') = \exp\{Z_{s,x,0}(s')\}.$$

Then $Z_{s,x,z}(s')$ satisfies the equation

$$\mathrm{d}Z = \bigl(c(s', X) - \tfrac{1}{2}\mu^2(s', X)\bigr)\,\mathrm{d}s' + \mu(s', X)\,\mathrm{d}w(s'), \qquad Z(s) = z. \tag{6.9}$$

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The estimate $\hat{X}(t)$ in (6.3) is found as (see (2.16), (2.21), and (3.43))

$$\hat{X}(t) = \frac{\tilde{\mathrm{E}}^y[X_{0,\xi}(t)\exp\{\beta X_{0,\xi}(t)y(t)\}\mathcal{X}_{s,\xi,1}(s') + \mathbb{X}_{0,\xi,0,0}(t)]}{\tilde{\mathrm{E}}^y[\exp\{\beta X_{0,\xi}(t)y(t)\}\mathcal{X}_{s,\xi,1}(s') + \mathbb{X}_{0,\xi,0,0}(t)]}$$
$$= \frac{\tilde{\mathrm{E}}^y[X_{0,\xi}(t)\exp\{\beta X_{0,\xi}(t)y(t) + Z_{0,\xi,0}(t)\} + \mathbb{X}_{0,\xi,0,0}(t)]}{\tilde{\mathrm{E}}^y[\exp\{\beta X_{0,\xi}(t)y(t) + Z_{0,\xi,0}(t)\} + \mathbb{X}_{0,\xi,0,0}(t)]}. \tag{6.10}$$

The results in Tables 1 and 2 are obtained for

$$\mu(s', x) = -\sigma\beta y(s') \quad \text{and} \quad \mathbb{F} = 0. \tag{6.11}$$

For this μ, (6.5) for $X(s')$ takes the natural form (6.1) (cf. (5.4)).

The first algorithm, the 'pathwise, integral method', gives the results obtained by the integral method (3.13) applied to (6.5), (6.9), and (6.11), while the second algorithm, the 'pathwise, Euler method', presents the results of the Euler method (3.18) for (6.5), (6.9), and (6.11). In the realizations of both (3.13) and (3.18) we use the discrete random variables (see (3.11)).

The third algorithm, the 'SPDE method', corresponds to simulation by the Euler method (4.4) with $\lambda = 0$. Here this method takes the form

$$X_{k+1} = X_k + \alpha X_k h + \sigma h^{1/2}\zeta_k, \qquad X_0 = \xi,$$
$$Z_{k+1} = Z_k - \tfrac{1}{2}\beta^2 {X_k}^2 h + \beta^\top(X_k)\Delta_k y, \qquad Z_0 = 0,$$

where $\Delta_k y := y(s_{k+1}) - y(s_k)$ and ζ_k are i.i.d. random variables with the law $\mathrm{P}(\zeta^i = \pm 1) = \frac{1}{2}$.

The data in Table 1 correspond to the case of a stable solution of (6.1)–(6.2), while the data in Table 2 correspond to an unstable solution. The results are in good agreement with the theoretical results: all the algorithms demonstrate first order of convergence. It is quite natural that the pathwise integral method gives better results than the pathwise Euler method. We also note that the 'SPDE method' produces the most accurate results.

Let us demonstrate the use of the Talay–Tubaro extrapolation method (see Remark 3.2). Using the data from Table 1 for the pathwise integral method, we obtain

$$\bar{u}_{\mathrm{imp}} = 2\bar{u}^{0.05} - \bar{u}^{0.1} = -0.3754,$$

and comparing this with the exact solution, we see that it is more accurate than $\bar{u}^{0.05}$ and $\bar{u}^{0.1}$. Expansion (3.15) can also be exploited for the error control. For instance, in this example

$$C_0 \approx -\frac{\bar{u}^{0.1} - \bar{u}^{0.05}}{0.1 - 0.05} = -\frac{-0.3860 + 0.3807}{0.1 - 0.05} \approx 0.1.$$

Note that $u_{\mathrm{exact}} - \bar{u}^{0.1} \approx 0.01$.

In the stable case (as in Table 1) the integral method (3.13) applied to (6.5), (6.6), and (6.11) produces results similar to those of the integral method (3.13) applied to (6.5), (6.9), and (6.11). But, in the unstable case (as in Table 2), the integral method (3.13) applied to (6.5), (6.6), and (6.11) demonstrates numerical instability (see also the discussion before Remark 3.4). For instance, for the same parameters as in Table 2, it gives 13 ± 32 for $h = 0.1$, 14 ± 54 for $h = 0.05$, 2.260 ± 0.023 for $h = 0.02$, and 2.827 ± 0.029 for $h = 0.001$. For larger time t, the numerical instability becomes worse. Then, in general, it is preferable to use representation (6.5), (6.9), (6.11) rather than (6.5), (6.6), (6.11).

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The aim of the next experiment is to illustrate the variance reduction techniques from Section 5. Let $u_g(s', x)$ be the solution of the Cauchy problem, (2.17)–(2.18), with $d = 1$ and $a_{11}(x) = \sigma^2$, and $b(s', x)$ and $c(s', x)$ given in (6.8). We can verify that, for $g(x) = x$,

$$u_g(s', x) = (v_0(s')x + v_1(s')) \exp\{\tfrac{1}{2} Q(s')x^2 + q(s')x\},$$

where $v_0(s')$, $v_1(s')$, $Q(s')$, and $q(s')$, $0 \le s' \le t$, satisfy

$$\dot{Q} = \beta^2 - 2\alpha Q - \sigma^2 Q^2, \qquad Q(t) = 0, \tag{6.12}$$

$$\dot{q} = y(s')\beta(\alpha + \sigma^2 Q) - q(\sigma^2 Q + \alpha), \qquad q(t) = \beta y(t), \tag{6.13}$$

$$\dot{v}_0 = -v_0\big(\tfrac{3}{2}\sigma^2 Q + \tfrac{1}{2}\sigma^2 q^2 + \alpha - \sigma^2 \beta y(s')q + \tfrac{1}{2}\sigma^2\beta^2 y^2(s')\big), \qquad v_0(t) = 1,$$

$$\dot{v}_1 = -\tfrac{1}{2}\sigma^2 v_1(Q + q^2 - 2\beta y(s')q + \beta^2 y^2(s')) - \sigma^2 v_0(q - \beta y(s')), \qquad v_1(t) = 0.$$

We can also find that, for $g(x) = 1$,

$$u_{\mathbf{1}}(s', x) = v_2(s') \exp\{\tfrac{1}{2} Q(s')x^2 + q(s')x\},$$

where $Q(s')$, $q(s')$, and $v_2(s')$, $0 \le s' \le t$, satisfy (6.12)–(6.13) and

$$\dot{v}_2 = -\tfrac{1}{2}\sigma^2 v_2(Q + q^2 - 2\beta y(s')q + \beta^2 y^2(s')), \qquad v_2(t) = 1.$$

Now we choose

$$\mu(s', x) = 0 \tag{6.14}$$

and

$$F_g(s', x) = -\sigma \frac{\partial u_g}{\partial x}(s', x), \qquad F_{\mathbf{1}}(s', x) = -\sigma \frac{\partial u_{\mathbf{1}}}{\partial x}(s', x). \tag{6.15}$$

We simulate the numerator of (6.10) by substituting F_g from (6.15) into (6.7) and the denominator of (6.10) by substituting $F_{\mathbf{1}}$ from (6.15) into (6.7).

We see that to produce results of the same quality we need $M = 4 \cdot 10^5$ independent trajectories without variance reduction and $M = 4 \cdot 10^4$ independent realizations in the variance reduction case (compare Tables 1 and 3). We note that the variance reduction technique used does not give zero variance since we do not exploit the possibility of reducing the variance associated with the distribution of the initial data (see Subsection 5.2). When we set $P_0 = 0$

TABLE 3: *Linear filtering—an illustration of variance reduction techniques.* Simulation of $\hat{X}(t)$ from (6.3) by the integral method (3.13) applied to (6.5), (6.9), and (6.14)–(6.15). The expectations are computed by the Monte Carlo technique simulating $M = 40\,000$ independent realizations. The other parameters are the same as those in Table 1. The corresponding reference value is $-0.377\,42$.

h	Pathwise, integral method
0.20	-0.3864 ± 0.0023
0.10	-0.3802 ± 0.0022
0.05	-0.3782 ± 0.0021
0.02	-0.3773 ± 0.0022
0.01	-0.3770 ± 0.0022

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(i.e. the initial data are deterministic), this variance reduction technique gives zero variance (see Subsection 5.1). Then, taking into account the error of numerical integration, the Monte Carlo error in simulations using this variance reduction technique becomes proportional to $\sqrt{h}/\sqrt{M}$. Indeed, for $P_0 = 0$, $M = 4000$, and the other parameters as those in Table 1, we have -0.323 ± 0.015 for $h = 0.1$, -0.3797 ± 0.003 for $h = 0.01$, and -0.3856 ± 0.0001 for $h = 0.001$ (the reference solution here is $-0.386\,12$). These experiments demonstrate a principal possibility of variance reduction for the considered algorithms. A further study is required to make the discussed variance reduction techniques more practical.

6.2. Beneš' filter

Consider the nonlinear system

$$\mathrm{d}X = \tanh(X)\,\mathrm{d}s' + \mathrm{d}w(s'), \qquad X(0) = 0, \tag{6.16}$$

$$\mathrm{d}y = X\,\mathrm{d}s' + \mathrm{d}v(s'), \qquad y(0) = 0, \tag{6.17}$$

where X and y are scalars.

We consider the estimate

$$\hat{X}(t) := \mathrm{E}[X(t) \mid y(s'),\ 0 \le s' \le t]. \tag{6.18}$$

The exact solution of this filtering problem is given by (see [1])

$$\hat{X}(t) = m(t) + P(t)\tanh(m(t)), \tag{6.19}$$

where $m(s')$ and $P(s')$, $0 \le s' \le t$, satisfy

$$\mathrm{d}m = P\,\mathrm{d}y - mP\,\mathrm{d}s', \qquad m(0) = 0,$$

$$P(s') = \tanh(s').$$

We obtain the observation $y(t)$ by simulating (6.16)–(6.17) using the mean-square Euler method with small time step ($h = 0.0001$) for a particular, fixed realization of the Wiener process $(w(s'), v(s'))$. Here we test the Euler method (4.4) with $\lambda = 0$ (the 'SPDE method') by comparing its results with the reference value of $\hat{X}$ obtained by an accurate simulation of the Beneš filter, (6.19), which is done by the Euler method with $h = 0.0001$. The results are presented in Table 4. We can see that the method produces quite accurate results even for large time steps.

TABLE 4: *Beneš' filter.* Simulation of $\hat{X}(t)$ from (6.18) at $t = 2$ by the Euler method (4.4) with $\lambda = 0$. The expectations are computed by the Monte Carlo technique simulating $M = 400\,000$ independent realizations. All simulations are done along the same observation path $y(t)$. The corresponding reference value is $-1.660\,44$.

h	SPDE method
0.20	-1.7893 ± 0.0024
0.10	-1.7273 ± 0.0022
0.05	-1.6744 ± 0.0022
0.02	-1.6700 ± 0.0021
0.01	-1.6618 ± 0.0021

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References

[1] Beneš, V. E. (1981). Exact finite-dimensional filters for certain diffusions with nonlinear drift. *Stochastics* **5,** 65–92.

[2] Bensoussan, A., Glowinski, R. and Răşcanu, A. (1990). Approximation of the Zakai equation by the splitting up method. *SIAM J. Control Optimization* **28,** 1420–1431.

[3] Clark, J. M. C. (1978). The design of robust approximations to the stochastic differential equations of nonlinear filtering. In *Communication Systems and Random Process Theory* (Proc. 2nd NATO Adv. Study Inst., Darlington, 1977) ed. J. K. Skwirzynski, Sijthoff and Noordhoff, pp. 721–734.

[4] Crisan, D. (2003). Exact rates of convergence for a branching particle approximation to the solution of the Zakai equation. *Ann. Prob.* **31,** 693–718.

[5] Crisan, D. and Lyons, T. (1999). A particle approximation of the solution of the Kushner–Stratonovich equation. *Prob. Theory Relat. Fields* **115,** 549–578.

[6] Davis, M. H. A. (1981). New approach for filtering for nonlinear systems. *Institution IEE Proc. D* **128,** 166–172.

[7] Davis, M. H. A. (1981). Pathwise nonlinear filtering. In *Stochastic Systems: The Mathematics of Filtering and Identification and Applications* (Proc. NATO Adv. Study Inst., Les Arcs, 1980), eds M. Hazewinkel and J. C. Willems, Reidel, Dordrecht, pp. 505–528.

[8] Davis, M. H. A. (1982). A pathwise solution of the equations of nonlinear filtering. *Theory Prob. Appl.* **27,** 160–167.

[9] Del Moral, P. (1996). Nonlinear filtering: interacting particle solution. *Markov Process. Relat. Fields* **2,** 555–579.

[10] Del Moral, P. and Miclo, L. (2000). Branching and interacting particle systems approximations of Feynman–Kac formulae with applications to non-linear filtering. In *Séminaire de Probabilités XXXIV* (Lecture Notes in Math. **1729**), Springer, Berlin, pp. 1–145.

[11] Dynkin, E. B. (1965). *Markov Processes*. Springer, New York.

[12] Itô, K. and Rozovskii, B. (2000). Approximation of the Kushner equation for nonlinear filtering. *SIAM J. Control Optimization* **38,** 893–915.

[13] Kurtz, T. G. and Xiong, J. (2001). Numerical solutions for a class of SPDEs with application to filtering. In *Stochastics in Finite and Infinite Dimensions*, Birkhäuser, Boston, pp. 233–258.

[14] Kushner, H. J. (1977). *Probability Methods for Approximations in Stochastic Control and for Elliptic Equations*. Academic Press, New York.

[15] Le Gland, F. (1992). Splitting-up approximation for SPDEs and SDEs with application to nonlinear filtering. In *Stochastic Partial Differential Equations and Their Applications* (Lecture Notes Control and Inform. Sci. **176**), Springer, Berlin, pp. 177–187.

[16] Liptser, R. S. and Shiryayev, A. N. (1977). *Statistics of Random Processes*. I. Springer, New York.

[17] Milstein, G. N. and Schoenmakers, J. G. M. (2002). Numerical construction of a hedging strategy against the multi-asset European claim. *Stoch. Stoch. Reports* **73,** 125–157.

[18] Milstein, G. N. and Tretyakov, M. V. (2004). *Stochastic Numerics for Mathematical Physics*. Springer, Berlin.

[19] Milstein, G. N. and Tretyakov, M. V. (2009). Averaging over characteristics with innovation approach in nonlinear filtering. In *Handbook on Nonlinear Filtering*, eds D. Crisan and B. Rozovskii, Oxford University Press (to appear).

[20] Milstein, G. N. and Tretyakov, M. V. (2009). Solving parabolic stochastic partial differential equations via averaging over characteristics. To appear in *Math. Comput.*

[21] Milstein, G. N., Schoenmakers, J. G. M. and Spokoiny, V. (2004). Transition density estimation for stochastic differential equations via forward-reverse representations. *Bernoulli* **10,** 281–312.

[22] Newton, N. J. (2000). Observation sampling and quantisation for continuous-time estimators. *Stoch. Process. Appl.* **87,** 311–337.

[23] Pardoux, E. (1981). Nonlinear filtering, prediction and smoothing. In *Stochastic Systems: The Mathematics of Filtering and Identification and Applications* (NATO Adv. Study Inst., Les Arcs, 1980), eds M. Hazewinkel and J. C. Willems, Reidel, Dordrecht, pp. 529–557.

[24] Pardoux, E. (1982). Équations du filtrage non linéaire, de la prédiction et du lissage. *Stochastics* **6,** 193–231.
[25] Picard, J. (1984). Approximation of nonlinear filtering problems and order of convergence. In *Filtering and Control of Random Processes* (Lecture Notes Control Inform. Sci. **61**), Springer, Berlin, pp. 219–236.
[26] Rozovskii, B. L. (1990). *Stochastic Evolution Systems*. Kluwer, Dordrecht.
[27] Talay, D. (1984). Efficient numerical schemes for the approximation of expectations of functionals of the solution of a SDE and applications. In *Filtering and Control of Random Processes* (Lecture Notes Control Inform. Sci. **61**), Springer, Berlin, pp. 294–313.
[28] Talay, D. (1986). Discrétisation d'une équation différentielle stochastique et calcul approché d'espérances de fonctionnelles de la solution. *RAIRO Modél. Math. Anal. Numér.* **20,** 141–179.