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A NEW REFINEMENT OF YOUNG'S INEQUALITY

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Abstract A classical theorem due to Young states that the cosine polynomial

$$C_n(x) = 1 + \sum_{k=1}^{n} \frac{\cos(kx)}{k}$$

is positive for all $n \geqslant 1$ and $x \in (0, \pi)$. We prove the following refinement. For all $n \geqslant 2$ and $x \in [0, \pi]$ we have

$$\tfrac{1}{6} + c(\pi - x)^2 \leqslant C_n(x),$$

with the best possible constant factor

$$c = \min_{0 \leqslant t < \pi} \frac{5 + 6\cos(t) + 3\cos(2t)}{6(\pi - t)^2} = 0.069\ldots.$$

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1. Introduction

The function

$$C_n(x) = 1 + \sum_{k=1}^{n} \frac{\cos(kx)}{k} \quad (n \in \mathbb{N};\ x \in \mathbb{R})$$

is known as Young's cosine polynomial. In 1913, Young [**15**] proved that

$$0 < C_n(x) \quad (n \in \mathbb{N};\ 0 < x < \pi). \tag{1.1}$$

Since $C_1(\pi) = 0$, we conclude that the lower bound in (1.1) is sharp. However, if we assume that $n > 1$, then inequality (1.1) can be improved. In 1997, Brown and Koumandos [**8**] showed that

$$\tfrac{1}{6} \leqslant C_n(x) \quad (2 \leqslant n \in \mathbb{N};\ 0 < x < \pi), \tag{1.2}$$

where the constant $\frac{1}{6}$ is the best possible. An application of the Rogosinski–Szegö inequality,

$$0 < \frac{1}{2} + \sum_{k=1}^{n} \frac{\cos(kx)}{k+1} \quad (n \in \mathbb{N};\ 0 < x < \pi) \tag{1.3}$$

(see [**7**,**14**]), easily yields a positive lower bound for $C_n(x)$, which depends on x. We have

$$\tfrac{1}{2}(\cos(x/2))^2 < C_n(x) \quad (n \in \mathbb{N};\ 0 < x < \pi). \tag{1.4}$$

Indeed, if we denote the expression on the right-hand side of (1.3) by $R_n(x)$, then summation by parts and (1.3) lead to

$$C_n(x) - \tfrac{1}{2}(\cos(x/2))^2 = \tfrac{3}{4}(1+\cos(x)) + \sum_{k=2}^{n} \frac{\cos(kx)}{k} = \sum_{k=2}^{n} \frac{R_k(x)}{k(k+1)} + \frac{n+2}{n+1} R_n(x) > 0.$$

Young's inequality and inequalities for other trigonometric polynomials have received enormous attention: numerous papers providing new proofs, various extensions, improvements, and counterparts, as well as interesting applications of (1.1) and related results, have been published. We refer to [**1**–**11**] and [**12**, Chapter 4], where historical remarks and many references on this subject can be found.

It is our aim to present a refinement of (1.1), (1.2) and (1.4). More precisely, we provide the largest (positive) real number c such that

$$\tfrac{1}{6} + c(\pi - x)^2 \leqslant C_n(x) \quad (2 \leqslant n \in \mathbb{N};\ 0 \leqslant x \leqslant \pi).$$

We have used the computer program Maple V (Releases 5.1, 6.01 and 9) to find the numerical values given in this paper.

2. Lemmas

In order to prove our main result we need several technical lemmas, which we collect in this section. In what follows, let $\delta = \delta_n = 2\pi/(2n+1)$ $(n \in \mathbb{N})$.

Lemma 2.1. *Let*

$$\phi(t) = \frac{5 + 6\cos(t) + 3\cos(2t)}{6(\pi - t)^2}.$$

Then we have

$$\min_{0 \leqslant t < \pi} \phi(t) = 0.069\ldots.$$

Proof. Let $c_0 = 0.069$ and

$$f(t) = \tfrac{5}{6} + \cos(t) + \tfrac{1}{2}\cos(2t) - c_0(\pi - t)^2.$$

Differentiation gives

$$f'(t) = -\sin(t) - \sin(2t) + 2c_0(\pi - t), \qquad f''(t) = -\cos(t) - 2\cos(2t) - 2c_0$$

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and

$$f'''(t) = \sin(t)[1 + 8\cos(t)].$$

We set $t_0 = \arccos(-\frac{1}{8}) = 1.69\dots$. Then

$$f'''(t) \begin{cases} > 0 & \text{for } 0 < t < t_0, \\ < 0 & \text{for } t_0 < t < \pi. \end{cases}$$

This implies that f'' is strictly increasing on $[0, t_0]$ and strictly decreasing on $[t_0, \pi]$. Since $f''(0) = -3.13\dots$, $f''(t_0) = 1.92\dots$ and $f''(\pi) = -1.13\dots$, we conclude that there are real numbers $r, s \in (0, \pi)$ such that f'' is negative on $[0, r) \cup (s, \pi]$ and positive on (r, s). We have

$$f''(0.9659) < 0 < f''(0.9660) \quad \text{and} \quad f''(2.5298) > 0 > f''(2.5299).$$

Thus,

$$r = 0.9659\dots \quad \text{and} \quad s = 2.5298\dots.$$

Let $t \in [r, s]$ and let $\tilde{t} = 1.997$. Since f is strictly convex on $[r, s]$ and $f'(\tilde{t}) > 0$, we get

$$f(t) \geqslant f(\tilde{t}) + (t - \tilde{t})f'(\tilde{t}) \geqslant f(\tilde{t}) + (0.9659 - \tilde{t})f'(\tilde{t}) = 0.000\,14\dots.$$

We have $f(0) = 1.65\dots$ and $f(\pi) = \frac{1}{3}$. Since f is strictly concave on $[0, r]$ and on $[s, \pi]$, we conclude from $f(r) > 0$ and $f(s) > 0$ that $f(t) > 0$ for $t \in [0, r) \cup (s, \pi]$, too.

Thus, we obtain

$$0.069 < \min_{0 \leqslant t < \pi} \phi(t).$$

And, since $\phi(1.967) = 0.0698\dots$, we get

$$\min_{0 \leqslant t < \pi} \phi(t) \leqslant 0.0698\dots.$$

Hence, $\min_{0 \leqslant t < \pi} \phi(t) = 0.069\dots$. □

Lemma 2.2. *For $n = 3, 4$ and $x \in (0, \pi)$ we have*

$$\tfrac{1}{6} + \tfrac{7}{100}(\pi - x)^2 < C_n(x). \tag{2.1}$$

Proof. First, we assume that $n = 3$. Let

$$u(x) = C_3(x) - 1 \quad \text{and} \quad v(x) = \tfrac{5}{6} - \tfrac{7}{100}(\pi - x)^2.$$

If $0 < x \leqslant \pi/3$, then $u(x) + v(x) \geqslant u(\pi/3) + v(0) = 0.059\dots$. If $\pi/3 \leqslant x \leqslant \pi/2$, then $u(x) + v(x) \geqslant u(\pi/2) + v(\pi/3) = 0.026\dots$. Next, let $w(x) = u(x) + v(x)$ and $\pi/2 \leqslant x < \pi$. Differentiation gives

$$w'''(x) = 4\sin(x)(9(\cos(x))^2 + 2\cos(x) - 2).$$

This implies that w'' is strictly decreasing on $[\pi/2, \tilde{x}]$ and strictly increasing on $[\tilde{x}, \pi]$, where $\tilde{x} = \arccos(-(1 + \sqrt{19})/9) = 2.20\dots$. Since $w''(\pi/2) > 0$, $w''(\tilde{x}) < 0$ and

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$w''(\pi) > 0$, we conclude that there exist numbers $\tilde{x}_1$ and $\tilde{x}_2$ such that w' is strictly increasing on $[\pi/2, \tilde{x}_1]$ and on $[\tilde{x}_2, \pi]$, and w' is strictly decreasing on $[\tilde{x}_1, \tilde{x}_2]$. We have $w'(\pi/2) > 0$ and $w'(\pi) = 0$. This implies that there is a number $\tilde{x}_3 \in (\tilde{x}_1, \tilde{x}_2)$ such that w' is positive on $[\pi/2, \tilde{x}_3)$ and negative on $(\tilde{x}_3, \pi)$. Since $w(\pi/2) > 0$ and $w(\pi) = 0$, we obtain that w is positive on $[\pi/2, \pi)$.

Next, let $n = 4$. If $x \in [0, \pi/8] \cup [3\pi/8, 5\pi/8] \cup [7\pi/8, \pi)$, then we have $C_4(x) \geqslant C_3(x)$. Let

$$z(x) = C_4(x) - 1.$$

If $\pi/8 \leqslant x \leqslant 3\pi/8$, then

$$z(x) + v(x) \geqslant z(3\pi/8) + v(\pi/8) = 0.025\ldots.$$

If $5\pi/8 \leqslant x \leqslant 3\pi/4$, then

$$z(x) + v(x) \geqslant z(3\pi/4) + v(5\pi/8) = 0.014\ldots.$$

And if $3\pi/4 \leqslant x \leqslant 7\pi/8$, then

$$z(x) + v(x) \geqslant z(\arccos(-(1+\sqrt{5})/4)) + v(3\pi/4) = 0.036\ldots.$$

The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let*

$$a_k(x) = 2\int_0^{1/2} \cos((k-t)x) \int_0^t \int_0^t \frac{1}{(k-t+s+v)^3}\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t \tag{2.2}$$

and

$$b_k(x) = 2\int_0^{1/2} \sin(kx)\sin(tx) \int_0^t \frac{1}{(k+s)^2}\,\mathrm{d}s\,\mathrm{d}t. \tag{2.3}$$

Then we have, for all integers $n \geqslant 1$ and real numbers $x \in (0, \pi)$,

$$\left|\sum_{k=n+1}^{\infty} a_k(x)\right| < \frac{1}{8n^2} \quad \text{and} \quad \left|\sum_{k=n+1}^{\infty} b_k(x)\right| < \frac{\pi}{12n^2}. \tag{2.4}$$

Proof. Since $t \mapsto \log((2t+1)/(2t-1)) - 1/t$ is positive and strictly decreasing on $(\frac{1}{2}, \infty)$, we obtain

$$\begin{aligned}
\left|\sum_{k=n+1}^{\infty} a_k(x)\right| &\leqslant \sum_{k=n+1}^{\infty} \int_0^{1/2} \int_0^t \int_0^t \frac{2}{(k-t+s+v)^3}\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t \\
&= \sum_{k=n+1}^{\infty} \left(\log\frac{2k+1}{2k-1} - \frac{1}{k}\right) \\
&\leqslant \int_n^{\infty} \left(\log\frac{2t+1}{2t-1} - \frac{1}{t}\right)\mathrm{d}t \\
&= 1 + \log(2n) - (n+\tfrac{1}{2})\log(2n+1) + (n-\tfrac{1}{2})\log(2n-1).
\end{aligned}$$

Let $t \geqslant 1$ and let

$$j(t) = \frac{1}{8t^2} - 1 - \log(2t) + (t + \tfrac{1}{2})\log(2t+1) - (t - \tfrac{1}{2})\log(2t-1).$$

Then we get

$$j'(t) = \log\frac{2t+1}{2t-1} - \frac{1}{t} - \frac{1}{4t^3} \quad \text{and} \quad j''(t) = \frac{8t^2-3}{4t^4(4t^2-1)}.$$

We have $j''(t) > 0$ and $\lim_{t\to\infty} j(t) = \lim_{t\to\infty} j'(t) = 0$. Hence, j is positive on $[1, \infty)$. This proves the first inequality of (2.4).

Summation by parts and the inequality

$$\biggl|\sum_{k=n+1}^{n+m} \sin(kx)\biggr| \leqslant \frac{1}{\sin(x/2)}$$

(see [**13**, p. 250]) yield

$$\biggl|\sum_{k=n+1}^{\infty} \sin(kx)\biggl(\frac{1}{k} - \frac{1}{k+t}\biggr)\biggr| \leqslant \frac{1}{\sin(x/2)}\biggl(\frac{1}{n+1} - \frac{1}{n+1+t}\biggr) \quad (t > 0).$$

Thus, we get

$$\begin{aligned}
\biggl|\sum_{k=n+1}^{\infty} b_k(x)\biggr| &= \biggl|2\int_0^{1/2} \sin(tx) \sum_{k=n+1}^{\infty} \sin(kx)\biggl(\frac{1}{k} - \frac{1}{k+t}\biggr)\,\mathrm{d}t\biggr| \\
&\leqslant \frac{2}{\sin(x/2)} \int_0^{1/2} \sin(tx)\biggl(\frac{1}{n+1} - \frac{1}{n+1+t}\biggr)\,\mathrm{d}t \\
&\leqslant \frac{2x}{\sin(x/2)} \int_0^{1/2} t\biggl(\frac{1}{n+1} - \frac{1}{n+1+t}\biggr)\,\mathrm{d}t \\
&< \frac{x}{12\sin(x/2)}\frac{1}{n^2} \\
&\leqslant \frac{\pi}{12n^2}.
\end{aligned}$$

□

Lemma 2.4. *Let*

$$\mathrm{Ci}(x) = -\int_x^{\infty} \frac{\cos(t)}{t}\,\mathrm{d}t \quad \text{and} \quad W(x) = \mathrm{Ci}((n+\tfrac{1}{2})x) \quad (n \in \mathbb{N};\ x > 0). \tag{2.5}$$

The function W is strictly increasing on $(0, \delta/2]$ and $[(4k-1)\delta/2, (4k+1)\delta/2]$, and W is strictly decreasing on $[(4k-3)\delta/2, (4k-1)\delta/2]$, where $k \geqslant 1$ is an integer. Moreover, the sequence

$$k \mapsto W((4k-1)\delta/2) \quad (k = 1, 2, \dots)$$

is strictly increasing.

Proof. Let $k \geqslant 1$ be an integer and let $x > 0$. Differentiation gives

$$W'(x) = \frac{\cos(\pi x/\delta)}{x}.$$

Hence, if $x \in (0, \delta/2) \cup ((4k-1)\delta/2, (4k+1)\delta/2)$, then $W'(x) > 0$. And, if $x \in ((4k-3)\delta/2, (4k-1)\delta/2)$, then $W'(x) < 0$. Further, we have

$$W((4k+3)\delta/2) - W((4k-1)\delta/2) = \pi \int_{(2k-1/2)\pi}^{(2k+1/2)\pi} \frac{\cos(t)}{t(t+\pi)}\,\mathrm{d}t > 0.$$

□

Lemma 2.5. *For all integers $n \geqslant 1$ and $x \in (0, \pi)$ we have*

$$-\frac{2}{x}\sin(x/2)\sum_{k=n+1}^{\infty}\frac{\cos(kx)}{k} = \mathrm{Ci}((n+\tfrac{1}{2})x) + \sum_{k=n+1}^{\infty} a_k(x) + \sum_{k=n+1}^{\infty} b_k(x), \tag{2.6}$$

where $a_k(x)$, $b_k(x)$ and $\mathrm{Ci}(x)$ are defined in (2.2), (2.3) and (2.5), respectively.

Proof. The idea comes from [**16**, Chapter V, p. 192]. Let $k \geqslant 1$ be an integer and let $x \in (0, \pi)$. Then we have

$$\frac{2}{x}\sin(x/2)\frac{\cos(kx)}{k} = \int_{k-1/2}^{k+1/2}\frac{\cos(xt)}{t}\,\mathrm{d}t - \int_{k-1/2}^{k+1/2}\left(\frac{1}{t}-\frac{1}{k}\right)\cos(xt)\,\mathrm{d}t.$$

This leads to

$$\frac{2}{x}\sin(x/2)\sum_{k=n+1}^{\infty}\frac{\cos(kx)}{k} = \int_{n+1/2}^{\infty}\frac{\cos(xt)}{t}\,\mathrm{d}t - \sum_{k=n+1}^{\infty}\int_{k-1/2}^{k+1/2}\left(\frac{1}{t}-\frac{1}{k}\right)\cos(xt)\,\mathrm{d}t.$$

Since

$$\int_{n+1/2}^{\infty}\frac{\cos(xt)}{t}\,\mathrm{d}t = -\,\mathrm{Ci}((n+\tfrac{1}{2})x)$$

and

$$\int_{k-1/2}^{k+1/2}\left(\frac{1}{t}-\frac{1}{k}\right)\cos(xt)\,\mathrm{d}t = a_k(x) + b_k(x)$$

(see also [**10**]), we conclude that (2.6) is valid. □

Lemma 2.6. *The function*

$$\Delta(x) = \frac{2}{x}\sin(x/2)(\tfrac{5}{6} - \tfrac{7}{100}(\pi - x)^2 - \log(2\sin(x/2))) \tag{2.7}$$

is strictly decreasing on $(0, \pi]$.

Proof. Let

$$\Delta_1(x) = \frac{2}{x}\sin(x/2) \quad \text{and} \quad \Delta_2(x) = \tfrac{5}{6} - \tfrac{7}{100}(\pi - x)^2 - \log(2\sin(x/2)).$$

A short calculation reveals that Δ_1 and Δ_2 are positive and strictly decreasing on $(0, \pi]$. This implies that $\Delta = \Delta_1\Delta_2$ is also strictly decreasing on $(0, \pi]$. □

3. Main result

We are now in a position to present a new lower bound for $C_n(x)$. In particular, we sharpen the inequalities (1.2) and (1.4), and we obtain an improvement of Young's inequality (1.1).

Theorem. *For all integers $n \geqslant 2$ and real numbers $x \in [0,\pi]$ we have*

$$\tfrac{1}{6} + c(\pi - x)^2 \leqslant C_n(x), \tag{3.1}$$

with the best possible constant factor

$$c = \min_{0 \leqslant t < \pi} \frac{5 + 6\cos(t) + 3\cos(2t)}{6(\pi - t)^2} = 0.069\ldots. \tag{3.2}$$

Proof. Let c be the real number given in (3.2). From Lemmas 2.1 and 2.2 we conclude that (3.1) holds for $n = 2, 3, 4$. Moreover, the case $n = 2$ yields that in (3.1) the constant factor c cannot be replaced by a larger number. We suppose that $n \geqslant 5$. First, let $0 \leqslant x \leqslant \delta/2$. Since $\cos(kx) > 0$ for $k = 1, \ldots, n$ and $\frac{5}{6} - \frac{7}{100}(\pi - x)^2 > 0$, it follows that (3.1) is valid. Next, let $\delta/2 \leqslant x \leqslant \pi$. Further, let $a_k(x)$, $b_k(x)$ and $\Delta(x)$ be defined in (2.2), (2.3) and (2.7), respectively. The identities

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k} = -\log(2\sin(x/2)) \quad (0 < x < \pi)$$

and (2.6) imply that it suffices to prove that

$$0 \leqslant \mathrm{Ci}((n + \tfrac{1}{2})x) + \sum_{k=n+1}^{\infty} a_k(x) + \sum_{k=n+1}^{\infty} b_k(x) + \Delta(x) = \Theta_n(x), \quad \text{say.}$$

Using Lemmas 2.3, 2.4 and 2.6 we obtain the following:

(i) if $\delta/2 \leqslant x \leqslant 5\delta/2$, then

$$\Theta_n(x) \geqslant \mathrm{Ci}(3\pi/2) - (\tfrac{1}{8} + \pi/12)\tfrac{1}{25} + \Delta(5\pi/11) = 0.114\ldots;$$

(ii) if $5\delta/2 \leqslant x \leqslant 9\delta/2$, then

$$\Theta_n(x) \geqslant \mathrm{Ci}(7\pi/2) - (\tfrac{1}{8} + \pi/12)\tfrac{1}{25} + \Delta(9\pi/11) = 0.013\ldots;$$

and

(iii) if $9\delta/2 \leqslant x \leqslant \pi$, then

$$\Theta_n(x) \geqslant \mathrm{Ci}(11\pi/2) - (\tfrac{1}{8} + \pi/12)\tfrac{1}{25} + \Delta(\pi) = 0.016\ldots.$$

This completes the proof of the theorem. □

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