

ON THE DERIVATIVES OF ZETA-FUNCTIONS OF CERTAIN CUSP FORMS, II

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Abstract. The discrete universality of the derivative and logarithmic derivative of zeta-functions of normalized eigenforms is obtained. This is used to estimate the number of zeros of the derivatives in the critical strip. For the proof the method of functional limit theorems in the sense of weak convergence of probability measures is applied.

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1. Introduction. Let $F(z)$ be a normalized eigenform of weight κ for all Hecke operators. Then $F(z)$ has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}$$

with multiplicative coefficients $c(m)$. Denote by $s = \sigma + it$ a complex variable, and attach to the form $F(z)$ the zeta-function

$$\varphi(s; F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

In view of multiplicativity of the coefficients $c(m)$, the function $\varphi(s; F)$ has the Euler product expansion over primes

$$\varphi(s; F) = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\alpha(p) + \beta(p) = c(p)$. By Deligne's estimates [2] both the Dirichlet series and product converge absolutely in the half-plane $\sigma > \frac{\kappa+1}{2}$ and define there a holomorphic function with no zeros. Moreover, the function $\varphi(s; F)$ is analytically continuable to an entire function and satisfies the functional equation of the Riemann type

$$(2\pi)^{-s} \Gamma(s) \varphi(s; F) = (-1)^{\kappa/2} (2\pi)^{s-\kappa} \Gamma(\kappa-s) \varphi(\kappa-s; F).$$

S. M. Voronin [14] was the first to prove (continuous) universality for the Riemann zeta-function, and discrete universality was invented by A. Reich [13] for Dedekind

zeta-functions. In [5] we began to study value-distribution of the derivative as well as of the logarithmic derivative of the function $\varphi(s; F)$. We proved the universality in the Voronin sense [14] for the functions $\varphi'(s; F)$ and $\frac{\varphi'}{\varphi}(s; F)$, and obtained that $\varphi'(s; F)$ has infinitely many zeros in the critical strip $\{s \in \mathbb{C} : \kappa/2 < \sigma < (\kappa + 1)/2\}$. To state more precisely the mentioned results we need some notation. Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(\dots) = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \dots\},$$

where in place of dots a condition satisfied by τ is to be written. Then in [5] the following statements have been obtained. Let $D = \{s \in \mathbb{C} : \kappa/2 < \sigma < (\kappa + 1)/2\}$, and let $\psi(s; F)$ denote $\varphi'(s; F)$ or $\frac{\varphi'}{\varphi}(s; F)$.

THEOREM A. *Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous on K function which is analytic in the interior of K . Then, for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \nu_T \left(\sup_{s \in K} |\psi(s + i\tau; F) - f(s)| < \varepsilon \right) > 0.$$

THEOREM B. *For every $\sigma_1, \sigma_2, \kappa/2 < \sigma_1 < \sigma_2 < (\kappa + 1)/2$, there exists a constant $c = c(\sigma_1, \sigma_2) > 0$ such that, for sufficiently large T , the function $\varphi'(s; F)$ has more than cT zeros in the rectangle*

$$\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_2, 0 < t < T\}.$$

In Theorem A the translation τ varies continuously in the interval $[0, T]$; therefore Theorem A can be called the continuous universality theorem. The aim of this note is to obtain discrete versions of Theorems A and B, when the translation τ takes values from a certain arithmetical progression with the first term zero and difference $h > 0$. Let, for $N \in \mathbb{N}$,

$$\mu_N(\dots) = \frac{1}{N + 1} \#\{0 \leq m \leq N : \dots\},$$

where in place of dots a condition satisfied by m is to be written.

THEOREM 1. *Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a function continuous on K which is analytic in the interior of K . Then, for every $\varepsilon > 0$ and arbitrary positive real number h ,*

$$\liminf_{N \rightarrow \infty} \mu_N \left(\sup_{s \in K} |\psi(s + imh; F) - f(s)| < \varepsilon \right) > 0.$$

The function $\psi(s, F)$ does not have an Euler product. Therefore, the function $f(s)$ in Theorem 1 is not necessarily non-vanishing.

THEOREM 2. *For arbitrary $\sigma_1, \sigma_2, \kappa/2 < \sigma_1 < \sigma_2 < (\kappa + 1)/2$, there exists a constant $c = c(\sigma_1, \sigma_2) > 0$ such that, for sufficiently large N , the function $\varphi'(s + imh; F)$ has a zero on the disc $|s - \hat{\sigma}| \leq (\sigma_2 - \sigma_1)/2, \hat{\sigma} = \frac{\sigma_1 + \sigma_2}{2}$, for more than cN numbers $m, 0 \leq m \leq N$.*

We note that in this paper there is no restriction on h as in former results [10].

We consider two cases. In the first case we assume the number $\exp\{\frac{2\pi k}{h}\}$ to be irrational for all $k \in \mathbb{Z} \setminus \{0\}$, while in the second case this number is supposed to be rational for some $k \neq 0$. In the first case, Theorem 1 with $\psi(s; F) = \varphi'(s; F)$ and Theorem 2 were proved in [10] if F is a newform of weight κ and level q . So, it remains to obtain Theorem 1 with $\psi(s; F) = \frac{\varphi'}{\varphi}(s, F)$ in the first case, and Theorems 1 and 2 in the second case.

The proof of Theorem 1 uses Bagchi's method based on limit theorems in the sense of weak convergence of probability measures in the spaces of analytic and meromorphic functions. An application of Theorem 1 together with Rouché's theorem yields Theorem 2.

2. A discrete limit theorem for $\frac{\varphi'}{\varphi}(s, F)$. Case 1. Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane. As usual, we define a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, where

$$\Omega = \prod_p \gamma_p$$

with $\gamma_p = \gamma$ for each prime p , $\mathcal{B}(S)$ denotes the class of Borel sets of the space S , and m_H is the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$. The measure m_H can be defined because Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p .

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with spheric metric d defined by the formulae

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1 + |s_1|^2}\sqrt{1 + |s_2|^2}}, \quad d(s, \infty) = \frac{2}{\sqrt{1 + |s|^2}}, \quad d(\infty, \infty) = 0,$$

$s, s_1, s_2 \in \mathbb{C}$. By $M(G)$ denote the space of meromorphic on G functions $g : G \rightarrow (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $g_n(s) \in M(G)$ converges to $g(s) \in M(G)$ if

$$d(g_n(s), g(s)) \rightarrow 0, \quad n \rightarrow \infty,$$

uniformly on compact subsets of G . The space $H(G)$ of analytic on G functions is a subspace of $M(G)$.

For $V > 0$, let $D_V = \{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}, |t| < V\}$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H(D_V)$ -valued random element $\frac{\varphi'}{\varphi}(s, \omega; F)$ by

$$\begin{aligned} \frac{\varphi'}{\varphi}(s, \omega; F) = & - \sum_p \frac{\alpha(p)\omega(p) \log p}{p^s} \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \\ & - \sum_p \frac{\beta(p)\omega(p) \log p}{p^s} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}. \end{aligned}$$

THEOREM 3. *Let $\exp\{\frac{2\pi k}{h}\}$ be irrational for all $k \in \mathbb{Z} \setminus \{0\}$. Then the probability measure*

$$\mu_N\left(\frac{\varphi'}{\varphi}(s + imh; F) \in A\right), \quad A \in \mathcal{B}(M(D_V)),$$

converges weakly to the distribution of the random element $\frac{\varphi'}{\varphi}(s, \omega; F)$ as $N \rightarrow \infty$.

For the proof of Theorem 3 we will use the following statement obtained in [12, Lemma 1], in the case of newforms.

LEMMA 4. *As $N \rightarrow \infty$, the probability measure*

$$\mu_N(\varphi(s + imh; F) \in A), \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to the distribution of the $H(D_V)$ -valued random element

$$\varphi(s, \omega; F) = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}$$

defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Proof of Theorem 3. Using the Cauchy integral formula we obtain easily that the function $u : H(D_V) \rightarrow H(D_V)$ given by the formula $u(g(s)) = g'(s)$, $g(s) \in H(D_V)$, is continuous. Therefore, [1, Theorem 5.1] and Lemma 4 imply, as $N \rightarrow \infty$, the weak convergence of the probability measure

$$\mu_N(\varphi'(s + imh, F) \in A), \quad A \in \mathcal{B}(H(D_V)),$$

to the distribution of the $H(D_V)$ -valued random element $\varphi'(s, \omega; F)$. From this and Lemma 4, using a modification of the Cramér-Wald criterion (see, for example, [4], [6], [7], [9]) we obtain that the probability measure

$$\mu_N((\varphi'(s + imh; F), \varphi(s + imh; F)) \in A), \quad A \in \mathcal{B}(H^2(D_V)), \tag{1}$$

where $H^2(D_V) = H(D_V) \times H(D_V)$, converges weakly to the distribution of the $H^2(D_V)$ -valued random element $(\varphi'(s, \omega; F), \varphi(s, \omega; F))$ as $N \rightarrow \infty$. The metric d defined above satisfies $d(\frac{1}{g_1}, \frac{1}{g_2}) = d(g_1, g_2)$, and therefore the function $u : H^2(D_V) \rightarrow M(D_V)$ defined by the formula $u(g_1, g_2) = \frac{g_1}{g_2}$, $g_1, g_2 \in H(D_V)$, is continuous, and the theorem is a consequence of the weak convergence of the measure (1) as well as of [1, Theorem 5.1] again.

In order to use functional limit theorems for the proof of the universality property, it is necessary to know the support of the limit measure. In our limit theorems the limit measures coincide with the distributions of some random elements. Thus, we have to find the supports of those random elements.

THEOREM 4. *The support of the random element $\frac{\varphi'}{\varphi}(s, \omega; F)$ is the whole of $H(D_V)$.*

See [5, Lemma 7] for a proof of this theorem.

3. Discrete limit theorems. Case 2. In this section we suppose that the number $\exp\{\frac{2\pi k}{h}\}$ is rational for some $k \neq 0$. Clearly, it suffices to consider only positive integers

k . As it was noted in [3], there exists $k_0 \in \mathbb{N}$ such that the numbers $k \in \mathbb{N}$ with the above property are the multiples of k_0 . Let $A_h = \{k \in \mathbb{N}; \exp\{\frac{2\pi k}{h}\} \text{ is rational}\}$, and let $k_0 = \min_{k \in A_h} k$. If $k \in A_h$, then $k = ak_0 + r$ with $0 \leq r < k_0$. Hence

$$\exp\left\{\frac{2\pi r}{h}\right\} = \exp\left\{\frac{2\pi k}{h}\right\} \exp\left\{-\frac{2\pi ak_0}{h}\right\}$$

is rational. Therefore, by the definition of k_0 we have that $r = 0$.

For $m \in \mathbb{N}$, we define

$$\omega(m) = \sum_{p^\alpha \parallel m} \omega^\alpha(p), \tag{2}$$

where $p^\alpha \parallel m$ means that $p^\alpha | m$ but $p^{\alpha+1} \nmid m$. Let

$$\exp\left\{\frac{2\pi k_0}{h}\right\} = \frac{m_0}{n_0}, \quad m_0, n_0 \in \mathbb{N}, \quad (m_0, n_0) = 1,$$

and

$$\Omega_0 = \{\omega \in \Omega : \omega(m_0) = \omega(n_0)\}.$$

Then Ω_0 is a closed subgroup of Ω , and therefore it is also a compact topological Abelian group. Hence we can define a probability space $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$, where m_{0H} is the probability Haar measure on $(\Omega_0, \mathcal{B}(\Omega_0))$. Denote the elements of Ω_0 by ω_0 and define

$$\varphi(s, \omega_0; F) = \prod_p \left(1 - \frac{\alpha(p)\omega_0(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega_0(p)}{p^s}\right)^{-1}.$$

By the same method as in the case of $\varphi(s, \omega; F)$ it can be proved that $\varphi(s, \omega_0; F)$ is an $H(D_V)$ -valued random element defined on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$.

THEOREM 5. *Suppose that $\exp\{\frac{2\pi k}{h}\}$ is rational for some $k \neq 0$. Then the probability measure*

$$P_N(A) \stackrel{\text{def}}{=} \mu_N(\varphi(s + imh; F) \in A), \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to the distribution of the random element $\varphi(s, \omega_0; F)$ as $N \rightarrow \infty$.

Proof. The structure of the proof is similar to that in the case of h when $\exp\{\frac{2\pi k}{h}\}$ is irrational for all $k \in \mathbb{Z} \setminus \{0\}$. Therefore we will give a sketch of the proof only. \square

We start with a limit theorem on a finite-dimensional torus. Without loss of generality we can suppose that the prime numbers p_1, \dots, p_l occur in the factorization of the numbers m_0 and n_0 , i.e., we have that

$$\exp\left\{\frac{2\pi k_0}{h}\right\} = \frac{m_0}{n_0} = p_1^{\alpha_1} \cdots p_l^{\alpha_l} \tag{3}$$

with some $\alpha_l \in \mathbb{Z} \setminus \{0\}, j = 1, \dots, l$. Let p_1, \dots, p_r be all prime divisors of $n!$, $r > l$. Define a finite-dimensional torus

$$\Omega_r = \prod_{j=1}^r \gamma_{p_j},$$

where $\gamma_{p_j} = \gamma$ for $j = 1, \dots, r$, and let

$$\Omega_{0r} = \{\omega \in \Omega_r : \omega(m_0) = \omega(n_0)\},$$

where for the extension of $\omega(p)$ to the set \mathbb{N} , $\omega \in \Omega_r$, formula (2) is used. Then in [3] it was proved that the probability measure

$$Q_{hN}(A) = \mu_N((p_1^{imh}, \dots, p_r^{imh}) \in A), \quad A \in \mathcal{B}(\Omega_r),$$

converges weakly to the Haar measure m_{0r} on $(\Omega_{0r}, \mathcal{B}(\Omega_{0r}))$ as $N \rightarrow \infty$. Note that in this place the property of h that $\exp\{\frac{2\pi k}{h}\}$ is rational for some $k \neq 0$ is used. To prove the weak convergence of the measure Q_{hN} , we use the Fourier transform method. Denoting by $g_N(k_1, \dots, k_r), k_j \in \mathbb{Z}, j = 1, \dots, r$, the Fourier transform of the measure Q_{hN} , we have that

$$\begin{aligned} g_N(k_1, \dots, k_r) &= \frac{1}{N+1} \sum_{m=0}^N \prod_{j=1}^r p_j^{imhk_j} = \frac{1}{N+1} \sum_{m=0}^N \exp\left(imh \sum_{j=1}^r k_j \log p_j\right) \\ &= \begin{cases} 1, & \text{if } k_1 = k\alpha_1, \dots, k\alpha_l, k_{l+1} = \dots = k_r = 0, \\ \frac{1}{N+1} \frac{1 - \exp\{i(N+1)h \sum_{j=1}^r k_j \log p_j\}}{1 - \exp\{ih \sum_{j=1}^r k_j \log p_j\}} & \text{otherwise.} \end{cases} \end{aligned}$$

Note that in view of the linear independence over the field of rational numbers of logarithms of prime numbers and (3),

$$1 - \exp\left(hi \sum_{j=1}^r k_j \log p_j\right) \neq 0$$

for $(k_1, \dots, k_l, k_{l+1} = \dots = k_r) \neq (k\alpha_1, \dots, k\alpha_l, 0, \dots, 0), k \in \mathbb{Z}$. Therefore,

$$\lim_{N \rightarrow \infty} g_N(k_1, \dots, k_r) = \begin{cases} 1, & \text{if } (k_1, \dots, k_l, k_{l+1} = \dots = k_r) \neq (k\alpha_1, \dots, k\alpha_l, 0, \dots, 0), \\ 0, & \text{otherwise.} \end{cases}$$

This implies the weak convergence of Q_{hN} to m_{0r} as $N \rightarrow \infty$.

The proof thereafter contains the following sequence of statements: limit theorems in the space of analytic functions for Dirichlet polynomials and for absolutely convergent Dirichlet series, approximations in the mean by absolutely convergent series, limit theorems in the space of analytic functions for $\varphi(s; F)$ and $\varphi(s, \omega_0; F)$, and the identification of the limit measure. For Matsumoto zeta-functions, all these statements for the complex plane were proved in [3]. Here we have to generalize the mentioned statements for the space of analytic functions.

Let

$$p_n(s) = \sum_{k=1}^n a_k k^{-s}, \quad a_k \in \mathbb{C},$$

and

$$p_n(s, \omega_0) = \sum_{k=1}^n a_k \omega_0(k) k^{-s},$$

where ω_0 is an element of Ω_0 , be two Dirichlet polynomials. Then, taking into account the weak convergence of the measure Q_{hN} and [1, Theorem 5.1], similar to the proofs of [3, Theorems 4 and 5] we deduce that the probability measures

$$\mu_N(p_n(s + imh) \in A), \quad A \in \mathcal{B}(H(D_V)), \tag{4}$$

and

$$\mu_N(p_n(s + imh, \omega_0) \in A), \quad A \in \mathcal{B}(H(D_V)), \tag{5}$$

both converge weakly to the same probability measure on $(H(D_V), \mathcal{B}(H(D_V)))$ as $N \rightarrow \infty$.

Now let, for $\sigma_1 > 1/2$ and $n \in \mathbb{N}$,

$$\varphi_n(s; F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}$$

and

$$\varphi_n(s, \omega_0; F) = \sum_{m=1}^{\infty} \frac{c(m)\omega_0(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

The latter series both converge absolutely for $\sigma > \kappa/2$. Then the analogue of [3, Theorem 6] says that there exists a probability measure P_n on $(H(D_V), \mathcal{B}(H(D_V)))$ such that both the measures

$$\mu_N(\varphi_n(s + imh; F) \in A), \quad A \in \mathcal{B}(H(D_V)), \tag{6}$$

and

$$\mu_N(\varphi_n(s + imh, \omega_0; F) \in A), \quad A \in \mathcal{B}(H(D_V)), \tag{7}$$

converge weakly to P_n as $N \rightarrow \infty$. To prove this, instead of the Euclidean metric used in [3] we have to deal with a metric ρ in $H(D_V)$ which induces its topology. There exists a sequence $\{K_l\}$ of compact subsets of D_V such that

$$D_V = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, and if K is a compact subset of D_V , then $K \subseteq K_l$ for some l . Then the metric ρ can be defined by

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\rho_l(g_1, g_2)}{1 + \rho_l(g_1, g_2)}, \quad g_1, g_2 \in H(D_V),$$

where

$$\rho_l(g_1, g_2) = \sup_{s \in K_l} |g_1(s) - g_2(s)|.$$

Moreover, we use the weak convergence of probability measures (4) and (5) as well as [1, Theorem 4.2].

To pass from $\varphi_n(s; F)$ to $\varphi(s; F)$ an approximation of $\varphi(s; F)$ by $\varphi_n(s; F)$ in the mean is applied. Using the estimate ([11], [12])

$$\int_0^T |\varphi(\sigma + it; F)|^2 dt \ll T, \quad \sigma > \frac{\kappa}{2},$$

we obtain that, for every compact subset $K \subset D_V$,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \sup_{s \in K} |\varphi(s + imh; F) - \varphi_n(s + imh; F)| = 0. \tag{8}$$

The proof of a similar result for $\varphi(s, \omega_0; F)$ is more complicated. Let $a_h = \{p^{-ih} : p \text{ is prime}\}$. Define a transformation f_h of Ω_0 taking $f_h(\omega_0) = a_h \omega_0, \omega_0 \in \Omega_0$. Then f_h is a measurable measure preserving transformation on $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$. In [3] it was proved that the transformation f_h is ergodic. This together with the classical Birkhoff theorem implies, for $\sigma > \kappa/2$ and almost all $\omega_0 \in \Omega$, the estimate

$$\sum_{m=0}^N |\varphi(\sigma + imh, \omega_0; F)| \ll N.$$

From this we find that, for every compact subset $K \subset D_V$,

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \sup_{s \in K} |\varphi(s + imh, \omega_0; F) - \varphi_n(s + imh, \omega_0; F)| = 0 \tag{9}$$

for almost all $\omega_0 \in \Omega_0$.

Now the weak convergence of probability measures (6) and (7), the relations (8) and (9), and [1, Theorem 4.2] show that on $(H(D_V), \mathcal{B}(H(D_V)))$ there exists a probability measure P such that the probability measures P_N and

$$\mu_N(\varphi(s + imh, \omega_0; F) \in A), \quad A \in (H(D_V)), \tag{10}$$

both converge weakly to P as $N \rightarrow \infty$.

It remains to prove that the measure P is the distribution of the random element $\varphi(s, \omega_0; F)$. This is obtained by a standard way. Let $A \in \mathcal{B}(H(D_V))$ be a continuity set of the measure P . Then the weak convergence of the measure (10) yields

$$\lim_{N \rightarrow \infty} \mu_N(\varphi(s + imh, \omega_0; F) \in A) = P(A) \tag{11}$$

for almost all $\omega_0 \in \Omega_0$. For a fixed set A define on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$ the random variable θ by

$$\theta(\omega_0) = \begin{cases} 1 & \text{if } \varphi(s, \omega_0; F) \in A, \\ 0 & \text{if } \varphi(s, \omega_0; F) \notin A. \end{cases}$$

Denote by $\mathbb{E}\xi$ the expectation of the random element ξ . Then by the definition of θ

$$\mathbb{E}\theta = \int_{\Omega_0} \theta dm_{0H} = m_{0H}(\omega_0 \in \Omega_0 : \varphi(s, \omega_0; F) \in A) = P_\varphi(A) \tag{12}$$

is the distribution of $\varphi(s, \omega_0; F)$. Since the transformation f_h is ergodic, the Birkhoff theorem shows that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{m=0}^N \theta(f_h^m(\omega_0)) = \mathbb{E}\theta \tag{13}$$

for almost all $\omega_0 \in \Omega$. However, by the definitions of f_h and θ ,

$$\frac{1}{N+1} \sum_{m=0}^N \theta(f_h^m(\omega_0)) = \mu_N(\varphi(s + imh, \omega_0; F)).$$

Therefore, in view of (12) and (13),

$$\lim_{N \rightarrow \infty} \mu_N(\varphi(s + imh, \omega_0; F)) = P_\varphi(A)$$

for almost all $\omega_0 \in \Omega_0$. Hence and from (11) we have that $P(A) = P_\varphi(A)$ for all continuity sets A of the measure P . Therefore, $P(A) = P_\varphi(A)$ for all $A \in \mathcal{B}(H(D_V))$. Theorem 5 is proved.

On the probability space $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$ define an $H(D_V)$ -valued random element $\varphi'(s, \omega_0; F)$ by the formula

$$\begin{aligned} \varphi'(s, \omega_0; F) &= \prod_p \left(1 - \frac{\alpha(p)\omega_0(p)}{p^s} \right)^{-1} \left(1 - \frac{\beta(p)\omega_0(p)}{p^s} \right)^{-1} \\ &\times \left(-\sum_p \frac{\alpha(p)\omega_0(p) \log p}{p^s} \left(1 - \frac{\alpha(p)\omega_0(p)}{p^s} \right)^{-1} \right. \\ &\left. - \sum_p \frac{\beta(p)\omega_0(p) \log p}{p^s} \left(1 - \frac{\beta(p)\omega_0(p)}{p^s} \right)^{-1} \right). \end{aligned}$$

THEOREM 6. *Suppose that $\exp\{\frac{2\pi k}{h}\}$ is rational for some $k \neq 0$. Then the probability measure*

$$\mu_N(\varphi'(s + imh; F) \in A), \quad A \in \mathcal{B}(H(D_V)),$$

converges weakly to the distribution of the random element $\varphi'(s, \omega_0; F)$ as $N \rightarrow \infty$.

Proof. Since the function $u : H(D_V) \rightarrow H(D_V)$ defined by $u(g(s)) = g'(s)$, $g(s) \in H(D_V)$, is continuous, the theorem is a consequence of Theorem 5 and Theorem 5.1 of [1].

Now on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$ define an $H(D_V)$ -valued random element $\frac{\varphi'}{\varphi}(s, \omega_0; F)$ by the formula

$$\begin{aligned} \frac{\varphi'}{\varphi}(s, \omega_0; F) = & - \sum_p \frac{\alpha(p)\omega_0(p) \log p}{p^s} \left(1 - \frac{\alpha(p)\omega_0(p)}{p^s}\right)^{-1} \\ & - \sum_p \frac{\beta(p)\omega_0(p) \log p}{p^s} \left(1 - \frac{\beta(p)\omega_0(p)}{p^s}\right)^{-1}. \end{aligned}$$

□

THEOREM 7. *Suppose that $\exp\{\frac{2\pi k}{h}\}$ is rational for some $k \neq 0$. Then the probability measure*

$$\mu_N \left(\frac{\varphi'}{\varphi}(s + imh; F) \in A \right), \quad A \in \mathcal{B}(M(D_V)),$$

converges weakly to the distribution of the random element $\frac{\varphi'}{\varphi}(s, \omega_0; F)$ as $N \rightarrow \infty$.

Proof. The theorem is derived from Theorem 6 in the same way as the second part of the proof of Theorem 3. □

3. The supports of the random elements $\varphi'(s, \omega_0; F)$ and $\frac{\varphi'}{\varphi}(s, \omega_0; F)$. We start with a statement on the support of the random element $\varphi(s, \omega_0; F)$. Let

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

LEMMA 8. *The support of the random element $\varphi(s, \omega_0; F)$ is the set S_V .*

Proof. By the construction of the torus Ω , $\{\omega(p)\}$ is a sequence of independent random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Hence it follows easily that $\{\omega_0(p)\}$ is also a sequence of independent random variables defined on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$. Therefore, the proof of the lemma coincides with that of [8, Lemma 8]. □

THEOREM 9. *The support of the random element $\varphi'(s, \omega_0; F)$ is the whole of $H(D_V)$.*

Proof. We repeat the arguments of the proof of Lemma 6. The function $u : S_V \rightarrow H(D_V)$ defined by $u(g(s)) = g'(s), g(s) \in S_V$, is continuous. Hence the set $u^{-1}G$ is open for any open set $G \subset H(D_V)$. Using the metric ρ defined in the proof of Theorem 5 and the Mergelyan theorem on uniform approximation of analytic functions by polynomials (see, for example, [14]), we ensure that the set $u^{-1}G \subset S_V$ is non-empty: there exists a polynomial $g(s) \in u^{-1}G$, and $g(s) \neq 0$ on D_V . Now we have by Lemma 8 that

$$m_{0H}(\omega_0 \in \Omega_0 : (\varphi(s, \omega_0; F))' \in G) = m_{0H}(\omega_0 \in \Omega : \varphi(s, \omega_0; F) \in u^{-1}G) > 0.$$

Since the support of $\varphi'(s, \omega_0; F)$ consists of all $g \in H(D_V)$ such that for every neighborhood G of g the distribution $P_{\varphi'}$ of $\varphi'(s, \omega_0; F)$ satisfies the inequality $P_{\varphi'}(G) > 0$, hence the theorem follows. □

THEOREM 10. *The support of the random element $\frac{\varphi'}{\varphi}(s, \omega_0; F)$ is the whole of $H(D_V)$.*

Proof. As it was noted above, $\{\omega_0(p)\}$ is a sequence of independent random variables on $(\Omega_0, \mathcal{B}(\Omega_0), m_{0H})$. Therefore, the proof of the theorem runs in the same way as that of [5, Lemma 7]: everywhere in place of $\omega(p)$ we write $\omega_0(p)$, only. \square

Proof of Theorems 1 and 2.

Proof of Theorem 1. We will consider only $\psi(s; F) = \varphi'(s; F)$ in the second case. The remaining cases can be treated similarly.

First of all we observe that there exists a number $V > 0$ such that $K \subset D_V$, and suppose that the function $f(s)$ is analytically continuable to the region D_V . Denote by G the set of functions $g \in H(D_V)$ such that

$$\sup_{s \in K} |g(s) - f(s)| < \varepsilon.$$

By Theorem 9 the function $f(s)$ belongs to the support of the random element $\varphi'(s, \omega_0; F)$, moreover, the set G is open. Therefore, by Theorem 6, taking into account [1, Theorem 2.1, Statement 4] and the above-mentioned property of the support, we have that

$$\liminf_{N \rightarrow \infty} \mu_N \left(\sup_{s \in K} |\varphi'(s + i\tau; F) - f(s)| < \varepsilon \right) \geq m_{0H}(\omega_0 \in \Omega_0 : \varphi'(s, \omega_0; F) \in G) > 0.$$

Now let $f(s)$ satisfy the hypothesis of Theorem 1. By the Mergelyan theorem, (see, for example, [15]), there exists a polynomial $p_n(s)$ such that

$$\sup_{s \in K} |f(s) - p_n(s)| < \frac{\varepsilon}{2}. \tag{14}$$

Since $p_n(s)$ is analytic in D_V , by the beginning of the proof it follows that

$$\liminf_{N \rightarrow \infty} \mu_N \left(\sup_{s \in K} |\varphi'(s + i\tau; F) - p_n(s)| < \frac{\varepsilon}{2} \right) > 0.$$

This and (14) yield the assertion of the theorem.

If $\psi(s; F) = \frac{\varphi'}{\varphi}(s; F)$, then for the proof of Theorem 1 in the first and the second cases Theorems 3 and 4, and Theorems 7 and 10 are applied.

Proof of Theorem 2. As it was noted above, we have to consider only the second case. The investigation of the latter case is similar to that of the first one and we shall only sketch the arguments. \square

Define

$$\widehat{\sigma} = \frac{\sigma_1 + \sigma_2}{2}, \quad \sigma = \max \left\{ \left| \sigma_1 - \frac{2\kappa + 1}{4} \right|, \left| \sigma_2 - \frac{2\kappa + 1}{4} \right| \right\},$$

and we take $f(s) = s - \widehat{\sigma}$ and $0 < \varepsilon < \frac{\sigma_2 - \sigma_1}{20}$. Then in view of Theorem 1 there exists a constant $c = c(\sigma_1, \sigma_2) > 0$ such that, for sufficiently large N ,

$$\mu_N \left(\max_{|s - \frac{2\kappa + 1}{4}| \leq \sigma_0} |\varphi'(s + imh; F) - f(s)| < \varepsilon \right) > c. \tag{15}$$

Since

$$\left\{ s \in \mathbb{C} : |s - \widehat{\sigma}| = \frac{\sigma_2 - \sigma_1}{2} \right\} \subset \left\{ s \in \mathbb{C} : \left| s - \frac{2\kappa + 1}{4} \right| \leq \sigma_0 \right\},$$

we have that, for m satisfying (15),

$$\max_{|s - \widehat{\sigma}| = \frac{\sigma_2 - \sigma_1}{2}} |\varphi'(s + imh; F) - (s - \widehat{\sigma})| < \frac{\sigma_2 - \sigma_1}{20}.$$

Therefore, the functions $s - \widehat{\sigma}$ and $\varphi'(s + imh) - (s - \widehat{\sigma})$ on the disc $\{s \in \mathbb{C} : |s - \widehat{\sigma}| \leq \frac{\sigma_2 - \sigma_1}{2}\}$ satisfy the hypotheses of Rouché's theorem. Since the function $s - \widehat{\sigma}$ on this disc has exactly one zero, by Rouché's theorem the function $\varphi'(s + imh; F)$ on this disc also has one zero. By (15) the number of such m , $0 \leq m \leq N$, for sufficiently large N , is greater than cN . This proves the theorem.

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REFERENCES

1. P. Billingsley, *Convergence of Probability Measures* (John Wiley and Sons, New York, 1968).
2. P. Deligne, La conjecture de Weil I, *Publ. I. H. E. S.* **43** (1974), 273–307.
3. R. Kačinskaitė and A. Laurinčikas, On the value distribution of the Matsumoto zeta-function, *Acta Math. Hungar.* **105** (2004), 339–359.
4. A. Laurinčikas, A joint limit theorem for zeta-functions attached to certain cusp forms, *Publ. Math. Debrecen* **59** (2001), 175–186.
5. A. Laurinčikas, On the derivatives of zeta-functions of certain cusp forms, *Glasgow Math. J.* **47** (2005), 87–96.
6. A. Laurinčikas and R. Garunkštis, *The Lerch Zeta-Function* (Kluwer Academic Publishers, Dordrecht, Boston, London, 2002).
7. A. Laurinčikas and K. Matsumoto, Joint value-distribution theorems on Lerch zeta-functions, *Liet. Matem. Rink.* **38** (1998), 312–326 (= *Lith. Math. J.* **38** (1998), 238–249).
8. A. Laurinčikas and K. Matsumoto, The universality of zeta-functions attached to certain cusp forms, *Acta Arith.* **98** (2001), 345–359.
9. A. Laurinčikas and K. Matsumoto, The joint universality of zeta-functions attached to certain cusp forms, *Fiz. Mat. Fak. Moksl. semin. Darb.* **5** (2002), 58–75.
10. A. Laurinčikas, K. Matsumoto and J. Steuding, The discrete universality of L -functions of newforms, *Math. Notes* (submitted).
11. K. Matsumoto, A probabilistic study on the value-distribution of Dirichlet series attached to certain cusp forms, *Nagoya Math. J.* **116** (1989), 123–138.
12. K. Matsumoto, Recent developments in the mean square of the Riemann zeta and other zeta-functions, in *Number Theory* (ed. R. P. Bambah, V. C. Dumir and R. J. Hans-Gill), (Birkhäuser, 2000), 241–286.
13. A. Reich, Werteverteilung von Zetafunktionen, *Arch. Math. (Basel)* **34** (1980), 440–451.
14. S. M. Voronin, Theorem on the “universality” of the Riemann zeta-function, *Izv. Akad. Nauk. SSSR, Ser. matem.* **39** (1975), 475–486 (in Russian) = *Math. USSR Izv.* **9** (1975), 443–453.
15. J. L. Walsh, Interpolation and approximation by rational functions in the complex domain, *Amer. Math. Soc. Coll. Pub.*, 1960.