BULL. AUSTRAL. MATH. SOC. 54C05
VOL. 31 (1985), 285-292. (54C35)

A NOTE ON SPACES RELATED TO NAMIOKA SPACES

J.P. LEE AND Z. PIOTROWSKI

Namioka proved that the following condition (*) given below holds, if X is Čech-complete and Y is a locally compact, σ-compact space.

(*) Let X and Y be spaces, Z be a metric space and let $f : X \times Y \to Z$ be separately continuous. Then there is a dense, G_δ set A in X such that $A \times Y \subset C(f)$.

Following Christensen a space X is called *Namioka* if (*) is true for any compact space Y . In this paper we introduce and study a new class of spaces which is closely related to Namioka spaces. Namely, we say that a space Y is *co-Namioka* if (*) holds for any Namioka space X .

I. Introduction

There are many papers which deal with the classical problem of determining the points of continuity for a separately continuous function;

Received 18 October 1984. The authors would like to thank their colleagues, R.A. McCoy and D.J. Lutzer for their helpful remarks; they also thank J.B. Brown who permitted them to use his unpublished example (see Example 2).

The second-named author wishes to extend his thanks to the Department of Mathematics of the State University of New York at Old Westbury for inviting him to the Summer Workshop on Topology, which made this paper possible.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/85 $A2.00 + 0.00.

see for example [2], [4], [10] or [11].

Throughout this paper, a space means a completely regular topological one. The set of points of continuity of f will be denoted by $C(f)$.

In what follows, a general condition given below and denoted by (*), will be called a continuity statement.

(*) Let X and Y be spaces, Z be a metric space and let $f : X \times Y \to Z$ be separately continuous. Then there is a dense, G_δ set A in X such that $A \times Y \subset C(f)$.

Namioka [10] proved that (*) holds if X is strongly countable complete (Čech-complete) and Y is locally compact and σ-compact.

A space X is *Namioka* [3] if (*) is true for any compact space Y.

So all Čech-complete spaces are Namioka [10] see also [2] and [11], where it is proved that some spaces defined by topological games are Namioka.

LEMMA 1 ([5]). *Let X be a Baire space.*

(a) If $A_1, A_2, \ldots, A_n, \ldots$ are dense, G_δ's of X, then so is $\bigcap_{i=1}^{\infty} A_i$ (Theorem 10.1).

(b) A subset of X which is the complement of a first category set contains a dense, G_δ subset of X (Exercise 6, p. 281).

LEMMA 2 ([11], Theorem 3, p. 501). *Namioka spaces are Baire.*

II. Co-Namioka spaces

In the main results of [2] and [11] (which are of type (*)), the space Y is assumed to be compact. So the following question arises.

What is a class $\mathcal{W}$ of spaces, strictly larger than the class $\mathcal{C}$ of all compact spaces, such that (*) is true for any Namioka space X and $Y \in \mathcal{W}$?

Let S be a "nice" subclass of Namioka spaces, for example LC - the class of all locally compact spaces.

A space Y will be called *co-Namioka* (respectively *co-Namioka* rel S)

https://doi.org/10.1017/S0004972700004755 Published online by Cambridge University Press

if (*) holds for any Namioka space X (respectively any space X from S).

Obviously, by this definition, compact spaces are co-Namioka; furthermore, co-Namioka spaces are co-Namioka spaces rel S , for any subclass of Namioka spaces N .

The following Proposition 1 was proved in [10] in case of strongly countably complete spaces X . Our proof uses the method of Namioka and relies heavily on Lemma 2.

PROPOSITION 1. *Every locally compact σ-compact space* is *co-Namioka.*

Proof. Since Y is locally compact and σ-compact there is a sequence $\{Y_i : i = 1, 2, \ldots\}$ of compact subsets of Y such that

$Y = \bigcup_{i=1}^{\infty} \operatorname{Int} Y_i$ ([5], Theorem 7.2, p. 241). Since X is Namioka, for every i , there is a dense G_δ set A_i in X such that $f|X \times Y$ is continuous on $A_i \times Y_i$. But then, clearly, f is continuous at each point of $\left(\bigcap_{i=1}^{\infty} A_i\right) \times Y$. Now by Lemma 1 *(a)*, $A = \bigcap_{i=1}^{\infty} A_i$ is a dense, G_δ subset of X , the latter being Namioka, and hence, by Lemma 2, Baire. So there is a dense, G_δ set $A \subset X$, with $A \times Y \subset C(f)$.

Proposition 1 suggests the question:

Must all co-Namioka spaces be Baire?

The following proposition that follows from [4], Theorem 2, p. 647, Lemma 1 *(b)* and Lemma 2, answers this question in the negative.

PROPOSITION 2. *Every second countable space* is *co-Namioka.*

This means, in particular, that if X is the unit interval and Y is the set Q of rational numbers, then (*) holds (!). Recall that Christensen [3], Theorem 1, p. 114, showed that if X is the set Q of rational numbers, Y is the unit interval and $Z = C_p(Q^2, [-1, 1])$, the space of continuous functions from Q^2 into $[-1, 1]$ equipped with the pointwise topology $\left(C_p(Q^2, [-1, 1])\right.$ is compact metric$\left.\right)$, then there is a

https://doi.org/10.1017/S0004972700004755 Published online by Cambridge University Press

separately continuous $f : X \times Y \to Z$ which does not satisfy (*).

Similarly, as in Proposition 2, it can be shown, using [13], Theorem 2, p. 438, Lemma 1 *(b)* and Lemma 2, that if we assume, additionally, that the range Z of separately continuous functions considered in (*) is a compact space, then every first countable space is co-Namioka.

In an attempt to generalize simultaneously both Propositions 1 and 2, say to all *Lindelöf spaces* (recall that in the class of locally compact spaces, a space is σ-compact if and only if it is Lindelöf [5], Theorem 7.2, p. 241) the following example ([12], Remark (b), p. 241) arises.

EXAMPLE 1. There is a hereditarily Lindelöf and hereditarily separable space which is *not* co-Namioka.

Proof. Let X and Z be the unit interval I and let Y be the space $C_p(I, I)$ of continuous functions from I into I equipped with the pointwise topology. Then $f(x, y) = y(x)$ is the required function.

The fact that Y is hereditarily Lindelöf and hereditarily separable, easily follows from the fact that Y has a countable network. It can be shown that $C_p(I, I)$ is of first category in itself [7], and is not a Frechét space and thus not first countable; this follows since $C_p(I, R)$ can be embedded in $C_p(I, I)$ and since $C_p(I, R)$ is not a k-space ([6] and [8]).

So, if not all Lindelöf, even hereditarily Lindelöf, spaces are co-Namioka, then perhaps either all *locally compact spaces that are also paracompact* or all *k-spaces* are co-Namioka? Again, the answer is no, even if we assume that such a space is both complete and metric.

The following unpublished example, due to Brown, was originally designed to answer Christensen's question [2] whether (*) holds for complete metric spaces X and Y.

EXAMPLE 2. There is a complete metric, locally compact space which is *not* co-Namioka.

Proof. Let $X = [0, 1]$, $Y = \dot{\bigcup}_{\alpha \in [0,1]} Y_\alpha$, $Y_\alpha = [0, 1]$ and let $Z = R$.

https://doi.org/10.1017/S0004972700004755 Published online by Cambridge University Press

The set $X \times Y$, equipped in "the open-page-book topology", is the free union of compact squares ("pages") $X \times Y_\alpha$, with $\alpha \in [0, 1]$.

Now let us order them in "a long line" and let us define, for every $\alpha \in [0, 1]$, a separately, but not jointly, continuous function $f : X \times Y_\alpha \to Z$, requiring though, that the point (or points) of discontinuity of f_α is (respectively are spread out) somewhere in $\{\alpha\} \times Y_\alpha$.

Now it is easy to see that if $f(x, y) \overset{\text{def}}{=\!=} f_\alpha(x, y)$ for $(x, y) \in X \times Y_\alpha$, then f does not satisfy (*). Hence Y is not co-Namioka, because X is Namioka.

We now arrive at the main problem of the paper.

PROBLEM 1. Characterize co-Namioka spaces.

Let us recall that a partial answer to this problem was obtained by Talagrand and is the main result of [12], Theorem 3.1, p. 241.

TALAGRAND'S THEOREM. *If X is compact and Y is a special K-analytic space, then* (*) *holds.*

(In our terminology: *Special K-analytic spaces are co-Namioka* rel $\mathcal{C}$, *where* $\mathcal{C}$ *stands for the class of compact spaces.*)

The following problem is closely related to Talagrand's result and our theorem.

PROBLEM 2. Do co-Namioka and co-Namioka rel($\mathcal{C}$) spaces coincide?

We shall prove the following:

THEOREM. *If X is locally compact and Y is a k_ω-space, then* (*) *holds.*

A space X is called a *k_ω-space* if $X = \bigcup_{n=1}^{\infty} X_n$ with X_n compact and increasing and if X has the weak topology of X_n's ; then this sequence $X_1, X_2, \ldots$ is called a k_ω-decomposition of X. k_ω-spaces, as σ-compact spaces are Lindelöf and paracompact; moreover, they are

https://doi.org/10.1017/S0004972700004755 Published online by Cambridge University Press

precisely hemicompact k-spaces.

The set Q of rational numbers, with the usual topology is not a k_ω-space, although the same set Q with the Sorgenfrey topology is k_ω. The first example of a non-Baire k_ω-space is due to Archangel'skiĭ and Franklin [1].

Both special K-analytic and k_ω-spaces contain, as Lindelöf spaces, those spaces that are both σ-compact and locally compact. However, the relation between special K-analytic and k_ω-spaces is not completely understood.

In the proof of our theorem we rely on the following Lemma 3; the result shown in the lemma seems to be a part of folklore, however, we decided to attach a short, hopefully, new proof; compare [5], Proof of Theorem 4.4, p. 263.

LEMMA 3. *Let X be locally compact and let Y be a k_ω-space. Then the product topology of $X \times Y$ coincides with the weak topology of the sets $X \times Y_n$, where the Y_n's are elements of some k_ω decomposition of Y.*

Proof. The space X, a locally compact space, admits the Alexandroff compactification αX, and X is open in αX. Therefore, every $X \times Y_n$ is open in $\alpha X \times Y_n$, for $n = 1, 2, \ldots$.

Next, let U be a subset of $X \times Y$ that intersects every $X \times Y_n$ as an open set.

Then, for every n, the set $(\alpha X \times Y_n) \cap U$ is open in $\alpha X \times Y_n$. Hence we get that U is open in $\alpha X \times Y$. Obviously, U is then open in $X \times Y$. This finishes the proof of Lemma 3.

REMARK. Since locally compact spaces are *characterized* as open subspaces of their Stone-Čech compactifications, [5], Theorem 8.3, p. 245, we see no way of extending Lemma 3 with its present proof.

Proof of the theorem. Clearly, we are able to determine the points of continuity of f restricted to (countably many) "layers" $X \times Y_n$,

$n = 1, 2, \ldots$. Now because the product topology of $X \times Y$ coincides with the weak topology of $X \times Y_n$ (Lemma 3) we can find such a *simultaneous* set, in X , whose Cartesian product with the space Y is contained in $C(f)$. In other words, we apply Lemma 3 to any k_ω-decomposition, which is a *countable* family of compact subsets of Y .

In fact, f is continuous on $A_n \times Y_n$, where A_n is a dense, G_δ (in X) and Y_n is a (compact) element of k_ω-decomposition. Obviously, f is continuous on $\left(\bigcap_{n=1}^{\infty} A_n\right) \times Y$ and $A = \bigcap_{n=1}^{\infty} A_n$ is a dense, G_δ , X being locally compact. This proves our theorem.

Obviously, the countability of the family $\{A_n\}$ is needed to get a simultaneous dense, G_δ set of points of continuity.

Similar arguments to those given in the proof of our theorem show the following result that is closely related to Mirzoian's theorem [9].

PROPOSITION 3. *Let X be locally compact, Y be a metric, k_ω-space, Z be compact metric and let a function $f : X \times Y \to Z$ have all its x-sections f_x continuous and its y-sections f_y continuous, for the y's belonging to a dense subset D of Y . Then ere is a residual set $A \subset X$ such that $A \times Y \subset C(f)$.*

References

[1] A.V. Archangel'skiĭ and S.P. Franklin, "Ordinal invariants for topological spaces", *Michigan Math. J.* 15 (1968), 313-320.

[2] J.P.R. Christensen, "Joint continuity of separately continuous functions", *Proc. Amer. Math. Soc.* 82 (1981), 455-461.

[3] J.P.R. Christensen, "Remarks on Namioka spaces and R.E. Johnson's theorem on the norm separability of the range of certain mappings", *Math. Scand.* 52 (1983), 112-116.

[4] J. Calbrix et J.P. Troallic, "Applications séparément continues", *C. R. Acad. Sci. Paris, Sér. A* 288 (1979), 647-648.

https://doi.org/10.1017/S0004972700004755 Published online by Cambridge University Press

[5] James Dugundji, *Topology* (Allyn and Bacon, Boston, 1978).

[6] J. Gerlits, "Some properties of $C(X)$, II", *Topology Appl.* **15** (1983), 255-262.

[7] D.J. Lutzer and R.A. McCoy, "Category in function spaces", *Pacific J. Math.* **90** (1980), 145-168.

[8] R.A. McCoy, "k-space function spaces", *Internat. J. Math. Math. Sci.* **3** (1980), 701-711.

[9] M.M. Mirzoian, "On the cluster sets of mappings of topological spaces", *Soviet Math. Dokl.* **19** (1978), 1326-1329.

[10] I. Namioka, "Separate and joint continuity", *Pacific J. Math.* **51** (1974), 515-531.

[11] J. Saint Raymond, "Jeux topologiques et espaces de Namioka", *Proc. Amer. Math. Soc.* **87** (1983), 499-504.

[12] M. Talagrand, "Deux generalisations d'un theoreme de I. Namioka", *Pacific J. Math.* **81** (1979), 239-251.

[13] J.D. Weston, "Some theorems on cluster sets", *J. London Math. Soc.* **33** (1958), 435-441.

Department of Mathematics,
State University of New York/College at Old Westbury,
Box 210,
Old Westbury,
Long Island,
New York 11568,
USA;

Department of Mathematical and Computer Sciences,
Youngstown State University,
Youngstown,
Ohio 44555,
USA.