Canad. Math. Bull. Vol. **32** (4), 1989

A GROUP THEORETIC CHARACTERIZATION OF THE 2-DIMENSIONAL SPHERICAL GROUPS

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ABSTRACT. It is shown that for a finite group G to be isomorphic to a subgroup of $SO(3)$ (or, equivalently, of $PSL(2, \mathbb{C})$) it is necessary and sufficient that G satisfies the property that the normalizer of every cyclic subgroup is either cyclic or dihedral.

Let $SO(3)$ denote the group of orthogonal linear transformations with determinant 1 on 3-dimensional Euclidean space. We refer to the finite subgroups of $SO(3)$ as *spherical groups*. This class of groups plays an important role in 2- and 3-dimensional geometry, and in this context many characterizations have arisen. For example it is well-known (see [9], [2: 33] or [10: 2.6]) that the spherical groups occur as the rotational symmetry groups of the regular polyhedra and that they may be classified as: $\mathbb{Z}_r$ (cyclic), $\mathbb{D}_r$ (dihedral), $\mathbb{A}_4$ (tetrahedral), $\mathbb{S}_4$ (octahedral), and $\mathbb{A}_5$ (icosahedral). These groups have also been described in various other ways: as the finite groups of sense-preserving isometries of Euclidean 3-space ([10]); as the finite subgroups of the group of orientation-preserving (topological) homeomorphisms of the 2-sphere S^2 (see [4]); as the class of groups of genus zero ([1]); and as the finite groups having a presentation of the form

$$(p, q, r) = \langle R, S \mid R^p = S^q = (RS)^r = 1\rangle$$

for some positive integers $p \leqq q \leqq r$([3]). In this latter characterization, the group (p, q, r) is sometimes referred to as a *triangle group* and it may be shown to be finite precisely when the inequality

$$\left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{r}\right) < 2$$

holds. An enumeration of the positive integer solutions of this inequality leads to a determination of the finite triangle groups as:

$$(1, q, r) \cong \mathbb{Z}_{gcd\{q,r\}}, (2, 2, r) \cong \mathbb{D}_r, (2, 3, 3) \cong \mathbb{A}_4, (2, 3, 4) \cong \mathbb{S}_4, (2, 3, 5) \cong \mathbb{A}_5.$$

Received by the editors February 17, 1988 and, in revised form, June 8, 1988.
1980 *Mathematics Subject Classification*. Primary 20D99; Secondary 20F38, 30F40, 57S30
Keywords. finite group, Kleinian group, spherical group, triangle group.

In this note we shall derive an algebraic description of the spherical groups to add to the list of characterizations given above.

THEOREM . *A finite group G is isomorphic to a spherical group if and only if the normalizer of every nontrivial cyclic subgroup of G is either cyclic or dihedral.*

Our interest in this characterization of the spherical groups stems from the important role it plays in the more general *recognition problem for Kleinian groups.* A Kleinian group is a discrete subgroup of the group $Iso^+(\mathbb{H}^3)$ of orientation preserving isometries of 3-dimensional hyperbolic space $\mathbb{H}^3$; equivalently – by the standard correspondence between linear fractional transformations and 2×2 matrices – it is a discrete subgroup of $PSL(2, \mathbb{C})$. The recognition problem for Kleinian groups is the problem of finding necessary and sufficient intrinsic conditions on an abstract group for it to be realizable as a Kleinian group. In the action of $Iso^+(\mathbb{H}^3)$ on $\mathbb{H}^3$ the point stabilizers are isomorphic to $SO(3)$ and each finite subgroup of $Iso^+(\mathbb{H}^3)$ stabilizes some point of $\mathbb{H}^3$. Therefore the finite Kleinian groups correspond to the spherical groups and the Theorem then solves the recognition problem for the class of finite Kleinian groups. This result is a key step in the solution of the recognition problem for finitely generated virtually free Kleinian groups which was given in section 4 of [5]. In related work in [6] we have also solved the recognition problem for the class of non-cocompact 2-dimensional non-Euclidean crystallographic groups.

We now turn to the proof of the Theorem. Our proof will be elementary, employing only some well-known concepts of linear algebra and of finite group theory. In particular, the essential linear algebra facts may be found in section 33 of [2], and the class equation from finite group theory provides a good analogy for the main technique of our proof.

First let us suppose that G is isomorphic to a spherical group. By a somewhat tedious direct consideration of the various isomorphism types for G (cyclic, dihedral, tetrahedral, octahedral, or icosahedral) it is not hard to show that G must satisfy the normalizer hypothesis of the Theorem. A more compelling argument is suggested by the geometry and goes as follows: Each orthogonal transformation of R^3 with determinant 1 fixes every point in some 1-dimensional subspace. If the transformation is nontrivial then this subspace is unique and it is called the *axis* of the transformation. Now assume that G is a finite subgroup of $SO(3)$ and let C be a nontrivial cyclic subgroup of G. There is a common axis, call it W, for all nontrivial elements of C. Each element y of the normalizer of C in G must leave W invariant. (Indeed if x is a nontrivial element of C then $yxy^{-1} \in C$ and yxy^{-1} fixes every element of W and of $y(W)$. The uniqueness of the axis of yxy^{-1} implies that $y(W) = W$.) Using orthogonality it follows that each element of $N(C)$ leaves the 2-dimensional subspace $W^\perp$ invariant. In this way we see that $N(C)$ corresponds to a finite group of orthogonal transformations of 2-dimensional Euclidean space. Such a finite group is either cyclic or dihedral ([2: 14]) and so the hypothesis of the Theorem holds.

To prove the converse we will use a counting argument involving the maximal cyclic subgroups of G. An argument similar to that given here was used by Burnside in describing the subgroups of $PSL(2, q)$ ([1: 325–6], see also [8]). However our Theorem is not a consequence of Burnside's theorem as we must start with an arbitrary finite group G which satisfies the normalizer hypothesis and we cannot assume a priori that this is a 2×2 matrix group. In the next paragraph we show that the hypothesis of the theorem reduces to two conditions ((B) and (C)) concerning the maximal cyclic subgroups of G.

Let G be a finite group satisfying the normalizer hypothesis of the Theorem:

(A) *The normalizer of any nontrivial cyclic subgroup of G is either cyclic or dihedral.*

Suppose that M_1 and M_2 are distinct maximal cyclic subgroups of G and that $x \neq 1$ is an element of $M_1 \cap M_2$. If $\langle x \rangle$ denotes the cyclic subgroup generated by x then, by hypothesis (A), its normalizer $N(\langle x \rangle)$ is either cyclic or dihedral. Since $N(\langle x \rangle)$ contains both M_1 and M_2 it can't be cyclic as that would contradict maximality of M_1. Therefore $N(\langle x \rangle)$ is a dihedral group which contains distinct maximal cyclic subgroups M_1 and M_2. From this it follows that $M_1 \cap M_2 = 1$. For suppose that $N(\langle x \rangle)$ is dihedral of order $2r$. Then it is isomorphic to the triangle group $(2, 2, r) = \langle R, S | R^2 = S^2 = (RS)^r = 1 \rangle$ where each element can be uniquely expressed in the form $R^\epsilon (RS)^i$ where $0 \leqq \epsilon \leqq 1$ and $0 \leqq i < r$. Using this the maximal cyclic subroups are readily seen to be $\langle RS \rangle$ (of order r) and $\langle R(RS)^i \rangle, 0 \leqq i < r$ (of order 2) – and these subgroups have pairwise trivial intersection as claimed. To summarize, we have shown:

(B) *Distinct maximal cyclic subgroups of G intersect trivially.*

Consider further the maximal cyclic subgroup M_1. By condition (A), $N(M_1)$ is either cyclic or dihedral. If $N(M_1)$ is cyclic then $N(M_1) = M_1$, whereas if $N(M_1)$ is dihedral then M_1 has index 2 in $N(M_1)$ since a normal maximal cyclic subgroup of a dihedral group must have index 2. (To see this consider the description of $\mathbb{D}_r \cong (2, 2, r)$ as above. If $r = 2$ then every nontrivial proper subgroup of $\mathbb{D}_r \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ has index two. If $r > 2$ then $\langle RS \rangle$ is the only normal maximal cyclic subgroup of $\mathbb{D}_r$ and it has index two.) Thus, if $[N(M_1) : M_1]$ denotes the index of M_1 in $N(M_1)$, we have shown:

(C) $[N(M_1) : M_1] \leqq 2$ *for each maximal cyclic subgroup M_1 of G.*

To complete the proof of the Theorem we will count the maximal cyclic subgroups of G using conditions (B) and (C). This will lead to seven possibilities for G in which the number of maximal cyclic subgroups and the orders of their normalizers are specified. One of the possibilities is shown to be vacuous. In each of the remaining possible cases, an analysis of the group structure will show that G is ismorphic to $\mathbb{Z}_r, \mathbb{D}_r, \mathbb{A}_4, \mathbb{S}_4$, or $\mathbb{A}_5$ – and these are in turn isomorphic to spherical groups as remarked above.

Let s be the number of conjugacy classes of maximal cyclic subgroups of G and let $M_1, \ldots, M_s$ be maximal cyclic subgroups representing these conjugacy classes. Denote the order of M_k by $m_k = |M_k|$, and let $i_k = [N(M_k) : M_k]$. By reindexing, if necessary, we assume that $i_k \leqq i_{k+1}$ for each k. With this notation the number of subgroups in the conjugacy class of M_k is

$$[G : N(M_k)] = \frac{|G|}{i_k m_k}.$$

By condition (B) each nontrivial element of G is in a unique conjugate of M_k for some k. Counting the number of nontrivial elements of G gives:

$$|G| - 1 = \sum_{k=1}^{s} \frac{|G|}{i_k m_k} (m_k - 1)$$

which we rewrite in the more convenient expression:

$$\text{(D)} \qquad 1 - \frac{1}{|G|} = \sum_{k=1}^{s} \frac{1}{i_k} \left(1 - \frac{1}{m_k}\right).$$

Now we observe that $m_k \geqq 2$ for each k (since M_k is nontrivial). Hence it follows that

$$1 - \frac{1}{|G|} \geqq \sum_{k=1}^{s} \frac{1}{i_k} \left(\frac{1}{2}\right)$$

and therefore

$$\sum_{k=1}^{s} \frac{1}{i_k} < 2.$$

This latter inequality has very few solutions. In fact, since $i_k \leqq 2$ by condition (C), we have $s \leqq 3$ and the s-tuple $(i_1, \ldots, i_s)$ must equal either (1), (2), (1, 2), (2, 2), or (2, 2, 2). Since $i_k m_k = |N(M_k)| \leqq |G|$, formula (D) implies that

$$1 - \frac{1}{|G|} \leqq \sum_{k=1}^{s} \left(\frac{1}{i_k} - \frac{1}{|G|}\right)$$

and this in turn yields

$$\sum_{k=1}^{s} \frac{1}{i_k} \geqq 1 + \frac{s-1}{|G|}.$$

Using this we see that $(i_1, \ldots, i_s)$ must equal either (1), (1, 2), or (2, 2, 2). We will examine these as three separate cases.

CASE 1. $s = 1$ and $i_1 = 1$. In this case formula (D) shows that

$$1 - \frac{1}{|G|} = 1 - \frac{1}{m_1}$$

and so G has order m_1. Therefore G equals its cyclic subgroup M_1, and so $G \cong \mathbb{Z}_{m_1}$.

CASE 2. $s = 2, i_1 = 1$ and $i_2 = 2$. By formula (D) we have

$$1 - \frac{1}{|G|} = \left(1 - \frac{1}{m_1}\right) + \frac{1}{2}\left(1 - \frac{1}{m_2}\right).$$

This leads to the inequality

$$\frac{1}{m_1} + \frac{1}{2m_2} = \frac{1}{2} + \frac{1}{|G|} > \frac{1}{2},$$

and, since $m_2 \geqq 2$, it follows that

$$\frac{1}{m_1} > \frac{1}{4}.$$

There are two possibilities, $m_1 = 2$ and $m_1 = 3$.

SUBCASE A. $m_1 = 2$. In this subcase we have

$$1 - \frac{1}{|G|} = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{m_2}\right) = 1 - \frac{1}{2m_2},$$

and consequently $|G| = 2m_2$. Thus M_2 has index two in G and so G is the normalizer of this cyclic subgroup. By hypothesis (A) we conclude that $G \cong \mathbb{D}_{m_2}$.

SUBCASE B. $m_1 = 3$. In this subcase we have

$$1 - \frac{1}{|G|} = \left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(1 - \frac{1}{m_2}\right) = \frac{7}{6} - \frac{1}{2m_2}.$$

This implies that

$$\frac{7}{6} - \frac{1}{2m_2} < 1$$

and so m_2 must equal 2. Thus

$$1 - \frac{1}{|G|} = \frac{7}{6} - \frac{1}{4} = \frac{11}{12}$$

which shows that G has order 12. Let x be a generator of M_1 and let y be a generator of M_2. The element yx is contained in a maximal cyclic subgroup whose order is either $m_1 = 3$ or $m_2 = 2$. Therefore, being nontrivial, yx has order 2 or 3. If yx has order two then $yxy^{-1} = (yx)^2x^{-1} = x^{-1}$ which implies that $y \in N(M_1) - M_1$; but this contradicts the assumption that $i_1 = 1$. It follows that yx has order three. Let H be the subgroup of G generated by x and y. There is an epimorphism from the triangle group $(2, 3, 3) = \langle R, S | R^2 = S^3 = (RS)^3 = 1\rangle (\cong \mathbb{A}_4)$ to H given by $R \longmapsto y, S \longmapsto x$. Since $|G| = 12$ and H contains elements of orders 2 and 3, the order of H is either 6 or 12; however $\mathbb{A}_4$ does not have a quotient group of order 6 so we must have $|H| = 12$. Thus G equals H, and G is isomorphic to $\mathbb{A}_4$.

CASE 3. $s = 3$ and $i_1 = i_2 = i_3 = 2$. By changing the indexing if necessary we may assume in this case that $m_1 \leqq m_2 \leqq m_3$. Now formula (D) yields:

$$\sum_{k=1}^{3}\left(1-\frac{1}{m_k}\right) = 2\left(1-\frac{1}{|G|}\right) < 2$$

which implies that (m_1, m_2, m_3) equals one of $(2, 2, m_3)$, (2, 3, 3), (2, 3, 4), or (2, 3, 5).

SUBCASE A. $m_1 = m_2 = 2$. Here we have

$$2\left(1-\frac{1}{|G|}\right) = \frac{1}{2}+\frac{1}{2}+\left(1-\frac{1}{m_3}\right) = 2-\frac{1}{m_3}$$

and it follows that $|G| = 2m_3$. As in case 2(a), hypothesis (A) now implies that $G \cong \mathbb{D}_{m_3}$.

SUBCASE B. $m_1 = 2, m_2 = m_3 = 3$. Here we have

$$2\left(1-\frac{1}{|G|}\right) = \frac{1}{2}+\frac{2}{3}+\frac{2}{3} = \frac{11}{6}$$

which shows that $|G| = 12$. Then M_2 and M_3 are Sylow 3-subgroups of G which is impossible as Sylow 3-subgroups must be conjugate. Therefore this subcase cannot arise.

SUBCASE C. $m_1 = 2, m_2 = 3$, and $m_3 = 4$. We have

$$2\left(1-\frac{1}{|G|}\right) = \frac{1}{2}+\frac{2}{3}+\frac{3}{4} = \frac{23}{12}$$

which shows that $|G| = 24$. Let y be an element of order 3 in M_2 and let x be an element of order 2 which is conjugate to the generator of M_1 but not contained in $N(M_2)$. (There is such an element since G has

$$\frac{|G|}{i_1 m_1} = 6$$

distinct elements of order 2 which are conjugate to the generator of M_1 but only 3 of these are in the subgroup $N(M_2)$ which has order 6.) Consider the element xy whose order is either 2, 3, or 4. Its order cannot be 2 because then $xyx^{-1} = y^{-1}$ which implies that $x \in N(M_2)$ in contradiction of the choice of x. Next suppose that xy has order 3 and let K be the subgroup of G generated by x and y. There is an epimorphism from the triangle group $(2, 3, 3) = \langle R, S \mid R^2 = S^3 = (RS)^3 = 1\rangle (\cong \mathbb{A}_4)$ to K given by $R \longmapsto x, S \longmapsto y$. Observe that K has order 12 since $\mathbb{A}_4$ has no quotient group of order 6 and $|K| \geqq 6$. There are

$$\frac{|G|}{i_3 m_3} = 3$$

distinct conjugates of M_3 in G and these carry three distinct elements of order 2 (by condition (B)). Since K has index 2 in G each of these three elements (which are squares of elements of order 4 in G) are in K, and these are the only elements of order 2 in K (since $K \cong \mathbb{A}_4$ and $\mathbb{A}_4$ has three elements of order 2). Therefore no maximal cyclic subgroup in G of order 2 is contained in K. In particular, $x \notin K$ which contradicts our choice of K. We conclude that xy must have order 4. Let H be the subgroup generated by x and y. There is an epimorphism from the triangle group $(2,3,4) = \langle R, S | R^2 = S^3 = (RS)^4 = 1 \langle (\cong \mathbb{S}_4)$ to K given by $R \longmapsto x, S \longmapsto y$. Notice that a Sylow 2-subgroup of H contains maximal cyclic subgroups of orders 2 and 4, thus it cannot be cyclic and its order must be 8. This implies that $|H| = 24$, and so $G = H$ and $G \cong \mathbb{S}_4$.

SUBCASE D. $m_1 = 2, m_2 = 3$, and $m_3 = 5$. We have

$$2\left(1 - \frac{1}{|G|}\right) = \frac{1}{2} + \frac{2}{3} + \frac{4}{5} = \frac{59}{30}$$

which shows that G has order 60. Choose elements: $y \in M_2$ of order 3; $z \in N(M_2)$ of order 2; $x \in N(\langle z \rangle) - \langle z \rangle$ of order 2 (this choice of x being possible since $\langle z \rangle$ is conjugate to M_1 so that $N(\langle z \rangle)$ is conjugate to $N(M_1) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). Consider the element xy, whose order must be either 2, 3, or 5. Its order cannot be 2 because then x would be in the normalizer of M_2 which gives a contradiction since $x \in N(\langle z \rangle) - \langle z \rangle$ and $N(\langle z \rangle) \cap N(M_2) = \langle z \rangle$. Suppose that xy has order 3. Let K be the subgroup generated by x and y, and consider the epimorphism from the triangle group $(2,3,3) \cong \mathbb{A}_4$ to K given by $R \longmapsto x, S \longmapsto y$. As before this must be an isomorphism since $|K| \geqq 6$. As a result, $N_K(\langle x \rangle)$ has order 4 so that $N_K(\langle x \rangle) = N(\langle x \rangle)$ and, in particular, $z \in N(\langle x \rangle) \subset K$. Therefore $z \in N_K(M_2)$ which is impossible – in $\mathbb{A}_4$ each subgroup of order 3 is its own normalizer. We conclude that xy must have order 5. The subgroup H generated by x and y is an epimorphic image of the triangle group $(2,3,5) \cong \mathbb{A}_5$. As H contains elements of orders 2, 3, and 5 its order is either 30 or 60. Since $\mathbb{A}_5$ has no quotient group of order 30, the order of H is 60 so that $G = H$ and $G \cong \mathbb{A}_5$.

We have now examined all possible cases and found that each nonvacuous one leads to G being isomorphic to a spherical group. This completes the proof of the Theorem.

In the proof of the Theorem we have shown that hypothesis (A) implies the conditions (B) and (C). We will now show that the converse also holds. To this end suppose that the finite group G satisfies conditions (B) and (C), and let C be a nontrivial cyclic subgroup. Let M be a maximal cyclic subgroup containing C. By condition (B) M is unique, and it follows that $N(C) = N(M)$. By condition (C), either $N(M) = M$, which is cyclic, or else M has index two in $N(M)$. In the latter case if $a \in N(M) - M$ then the maximal cyclic subgroup containing a intersects M trivially by condition (B) and therefore a must have order two (since $a^2 \in M$); in particular, if x generates M, then a and ax are involutions which generate $N(M)$ and $N(M)$ is dihedral. We conclude that $N(C)$ is either cyclic or dihedral. This shows that conditions (B) and (C) together are equivalent to hypothesis (A) (and to G being isomorphic to a spherical group) as claimed. On the other hand, it is easily seen that conditions (B) and (C) are independent of each other: an elementary abelian group, $G = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (p prime), with order larger than four, satisfies condition (B) but not condition (C); whereas the group $G = \langle x, y | x^8 = y^2 = 1, yxy = x^3 \rangle$ of order 16 can be seen to satisfy condition (C) but not condition (B). In general much is known about finite groups satisfying condition (B). For instance the nonsolvable finite groups with this property are classified in [7]. At the other end of the spectrum, each group with prime exponent satisfies condition (B); so the solvable groups with this property are more complicated.

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