Assembling RKHS with Pick kernels and assembling polyhedra in $\mathbb{CH}^n$

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Abstract. We study the geometry of Hilbert spaces with complete Pick kernels and the geometry of sets in complex hyperbolic space, taking advantage of the correspondence between the two topics. We focus on questions of assembling Hilbert spaces into larger spaces and of assembling sets into larger sets. Model questions include describing the possible three-dimensional subspaces of four-dimensional Hilbert spaces with Pick kernels and describing the possible triangular faces of a tetrahedron in $\mathbb{CH}^n$. A novel technical tool is a complex analog of the cosine of a vertex angle.

1 Introduction and summary

We begin with an informal overview; precise statements are in the later sections.

1.1 Prelude

There are close relationships between the analytic properties of certain Hilbert spaces and the geometric properties of certain sets in complex hyperbolic space. Here, we explore some aspects of those relationships in detail; other aspects are studied in [ARSW, Ro, Ro2, Ro3].

1.2 Introduction

A finite-dimensional reproducing kernel Hilbert space (RKHS) may have, or may fail to have, the complete Pick property (CPP), and substantial work has been done establishing which spaces have this property and describing its consequences, for instance, [AM, ARSW2, Ha, Sh]. Here, we study the rules governing building an RKHS+CPP from given pieces using specified amalgamation rules. Because of the correspondence between those Hilbert spaces and finite sets in complex hyperbolic space $\mathbb{CH}^n$, we obtain at the same time rules governing assembly of finite sets in $\mathbb{CH}^n$ into larger sets.

We have two essentially equivalent model questions: one geometric and the other functional analytic. The geometric version of question is more intuitive so we present it first.

Question 1: Given four triangles in complex hyperbolic space, $\mathbb{CH}^n$, is there a tetrahedron in $\mathbb{CH}^n$ with faces congruent to those triangles?

In Euclidean space of any dimension, the necessary and sufficient condition for three segments to be the sides of a triangle is that the lengths satisfy the triangle inequality and if that holds then those lengths determine the triangle up to congruence. On the other hand, given four Euclidean triangles with matching side lengths, there might not be a tetrahedron with faces congruent to those triangles, or, what is the same thing, edges the same lengths as the sides of those triangles. For instance, the numbers $\{4, 4, 4, 4, 4, 7\}$ are not the edge lengths of a tetrahedron. A necessary and sufficient condition for there to be such a tetrahedron is the nonnegativity of the determinant of the associated Cayley–Menger matrix, a matrix with entries constructed using the side lengths. This is an elegant condition, but it is an inequality for a sixth degree polynomial in the lengths, [WD].

In real or complex hyperbolic space, the necessary and sufficient condition for three lengths to be the side lengths of a triangle is that they satisfy the hyperbolic version of the triangle inequality (2.2) [Ro]. In $\mathbb{CH}^1$ and in $\mathbb{RH}^n$, $n \geq 1$, those lengths determine the congruence class of the triangle; but in $\mathbb{CH}^n$, $n > 1$, the side lengths do not determine the congruence class and an additional number is needed, for instance, the symplectic area of the triangle. Brehm [Br] presents the inequalities the four numbers must satisfy and shows that if they are satisfied, then there is a triangle in $\mathbb{CH}^n$, unique up to congruence, with that geometric data. We recall the details in Theorem 3.1.

If we are given four triangles to be the faces of a tetrahedron in $\mathbb{CH}^n$, then the starting data are determined by $4 \times 4 = 16$ parameters. If it is possible to form the tetrahedron, then various side lengths must be equal, reducing the number of parameters to 10. Further, it is not hard to show that if there is a tetrahedron, then the sum of the signed symplectic areas of the triangular faces must vanish, reducing the number of free parameters to 9. We see in Theorem 3.2 that the congruence class of a tetrahedron in $\mathbb{CH}^n$ is determined by nine numbers; Question 1 asks for the constraints those numbers must satisfy, and in Section 6, we give the explicit inequalities.

The relative complexity of the results for Euclidean space in [WD] is partially due to the choice of side lengths as data. In answering Question 1, in addition to working with side lengths, we also consider the geometry of the vertices. We describe them using a functional $\mathrm{kos}_a(b, c)$ of triples of points $\{a, b, c\}$ in $\mathbb{CH}^n$ which is a complex analog of the cosine of the angle at the vertex a formed by the segments ab and ac. If we can form a tetrahedron from given triangles, then at a vertex a of the tetrahedron there will be three edges and three bivalent vertices. The shape of each bivalent vertex is determined by a value of $\mathrm{kos}_a(\cdot, \cdot)$. The lengths of those three edges together with the three complex values of kos for the bivalent vertices give nine real parameters which suffice to determine if there is a tetrahedron and if so to determine its congruence class.

Here is an informal statement of our results on Question 1. The first part is a restatement of Brehm's result, and the second part is from Theorem 6.2. It includes the algebraic inequality that the values $\mathrm{kos}_a(\cdot, \cdot)$ must satisfy for there to be a tetrahedron.

Theorem 1.1   (1) *If a, x, y are vertices of a triangle in $\mathbb{CH}^n$, then the two real numbers* length (ax) *and* length(ay) *together with the complex number* $\text{kos}_a(x, y)$ *determine the congruence class of the triangle.*

The lengths must be in the interval $(0,1)$, and we must have $|\text{kos}_a(x, y)| \leq 1$. Given data that satisfy those conditions, there is a triangle with those data.

(2) *If a, x, y, z are the vertices of a tetrahedron $\mathbb{CH}^n$, then the three real numbers* length (ax), length(ay), *and* length (az) *together with the three complex numbers* $\text{kos}_a(y, z)$, $\text{kos}_a(z, x)$, *and* $\text{kos}_a(x, y)$ *determine the congruence class of the tetrahedron.*

The lengths must be in the interval $(0,1)$, and the values of kos *must be of modulus at most one and satisfy*

(1.1)
$$\left| \text{kos}_a(y, z) - \overline{\text{kos}_a(x, y)} \, \text{kos}_a(x, z) \right|^2 \leq \left(1 - |\text{kos}_a(x, y)|^2\right)\left(1 - |\text{kos}_a(x, z)|^2\right).$$

Given data that satisfy those inequalities, there is a tetrahedron with those data.

The functional kos and also the functional $\delta(\cdot, \cdot)$, the pseudohyperbolic distance between pairs of points, are defined for points in $\mathbb{CH}^n$. The "same" functionals are also defined for tuples of kernel functions in any RKHS. In an RKHS, $\delta(a, b)$ is related to the angle between kernel functions and $\text{kos}_a(b, c)$ describes the geometry of the projection of one kernel function onto the linear span two others. The functionals defined on points in $\mathbb{CH}^n$ are invariant under automorphisms of $\mathbb{CH}^n$, and the RKHS versions are invariant under rescalings of the Hilbert space. Accepting the guidance of Klein's Erlangen Program, these invariant quantities are geometric descriptors of sets in $\mathbb{CH}^n$ and of RKHS.

In Theorem 5.1, we use those functionals to give similar descriptions of finite sets in $\mathbb{CH}^n$ and of finite-dimensional RKHS with the CPP. The similarity extends to assembly questions in the two contexts, and in particular, Question 1 is essentially equivalent to the following.

Question 2: Given four three-dimensional RKHS+CPP, is there a four-dimensional space RKHS+CPP whose regular three-dimensional subspaces are rescalings of the given spaces?

A basic question is giving conditions which insure a finite-dimensional RKHS H has the CPP. The question is trivial if H is one-dimensional and elementary if the dimension is 2. Because having the CPP is equivalent to an associated geometric configuration being realizable in $\mathbb{CH}^n$, the three-dimensional case is settled by Brehm's result, the first statement in the previous theorem. If H is four-dimensional, then it is certainly necessary that each regular three-dimensional subspace satisfies Brehm's condition, but an example due to Quiggin [Q] shows that those conditions are not sufficient. This work arose as an attempt to place Quiggin's example in a general context and gives necessary and sufficient conditions for questions such as Question 2 to have a positive answer. The specific answer to Question 2, given in Theorem 6.4, is an inequality for values of the functional kos .

A theme of this paper is that certain geometric questions such as Question 1 and Hilbert space questions such as Question 2 are equivalent. Although we could have given a geometric analysis of Question 1 or a Hilbert space analysis of Question 2, we

have combined the two viewpoints to emphasize their unity. However, the geometric language often seems to carry more intuition, for instance, discussing congruence of sets rather than rescaling of Hilbert spaces, and so we often just use the geometric language and do not record the equivalent Hilbert space statements. However, the Hilbert space statements are always part of the story and, in fact, it is the author's view that they are more fundamental.

1.3　Contents

Here is an overview of the contents. In the next section, we give background information about geometry and about Hilbert spaces; in the section after that, we recall results connecting those topics. In Section 3, we formulate our basic assembly and coherence question and explain why Questions 1 and 2 are equivalent. Most of our analysis is based on the functional kos introduced in Section 2. That functional is new, and we take time in Section 4 to discuss the geometry associated with it. In Section 5, we describe finite sets X in $\mathbb{CH}^n$, and also an associated class of Hilbert spaces H. We use a version of spherical coordinates with values of kos taking the role of cosines of angles. That description involves the positive semidefiniteness of a matrix A, $A \geqslant 0$, which has the form $A = (\mathrm{kos}_1(i, j))$. Principal submatrices of A encode the geometry of subsets of X and of subspaces of H, and Sylvester's criterion lets us recast the fact that $A \geqslant 0$ as statements about those submatrices. We use that to relate the geometry of X to the geometry of its subsets and the geometry of H to that of its subspaces. In Section 6, we specialize the results from Section 5 to four-point sets and four-dimensional spaces, answer Questions 1 and 2, and analyze the examples of Quiggin [Q].

In Section 6.4, we consider what happens when a four-point set in $\mathbb{CH}^n$ is in fact inside a copy of $\mathbb{RH}^k$. Then our results specialize to results about sets in real hyperbolic space and there is then a fundamental simplification, the value of kos at a vertex equals the cosine of the vertex angle. The earlier results then simplify to the classical constraints on the vertex angles or the dihedral angles of a tetrahedron in real hyperbolic space or Euclidean space.

The brief final section contains a few remarks.

2　Background

2.1　Hyperbolic geometry

A background reference for complex hyperbolic geometry is [Go]. We will use the ball model of complex hyperbolic space $\mathbb{CH}^n$. In that model, the manifold for $\mathbb{CH}^n$ is the unit ball, $\mathbb{B}_n \subset \mathbb{C}^n$, and the geometry is the Poincare Bergman geometry determined by the transitive group of biholomorphic automorphisms of the ball, $\mathrm{Aut}\,\mathbb{B}_n$. For each $\alpha \in \mathbb{B}_n$, there is a unique involution $\phi_a \in \mathrm{Aut}\,\mathbb{B}_n$, which satisfies $\phi_a(a) = 0$. The group $\mathrm{Aut}\,\mathbb{B}_n$ is generated by those involutions together with the unitary maps. We will say two sets $Z, W \subset \mathbb{CH}^n$ are congruent, $Z \sim W$, if there is a $\phi \in \mathrm{Aut}\,\mathbb{B}_n$ with $\phi(Z) = W$. If the sets are ordered then, absent other comment, we suppose that ϕ respects the ordering. Congruence is an equivalence relation, and we are particularly interested in congruence equivalence classes.

The pseudohyperbolic metric δ on $\mathbb{CH}^n$ is defined by $\forall \alpha, \beta \in \mathbb{CH}^n$

$$(2.1) \qquad \delta(\alpha, \beta) = |\phi_\alpha(\beta)| = |\phi_\beta(\alpha)|.$$

Equivalently, δ is the distance on $\mathbb{B}_n$, which satisfies $\delta(0, z) = |z|$ for $z \in \mathbb{B}_n$ and is Aut $\mathbb{B}_n$ invariant [DW]. δ is not a length metric; the length metric it generates is the Poincare Bergman metric for the ball normalized to agree with the Euclidean metric to second order at the origin. For $n = 1$, the formula is $\delta(\alpha, \beta) = \left| \frac{\alpha - \beta}{1 - \bar{\beta}\alpha} \right|$. The formula for general n is given in (2.9). δ satisfies a strengthened version of the triangle inequality, [DW]; for $x, y, z \in \mathbb{B}_n$,

$$(2.2) \qquad \frac{|\delta(x, z) - \delta(z, y)|}{1 - \delta(x, z)\delta(z, y)} \le \delta(x, y) \le \frac{\delta(x, z) + \delta(z, y)}{1 + \delta(x, z)\delta(z, y)}.$$

The space $\mathbb{CH}^n$ contains various totally geodesic submanifolds of interest here. These include the classical geodesics which are totally geodesic copies of $\mathbb{RH}^1$, and also include totally geodesic copies of $\mathbb{CH}^1$, sometimes called *complex geodesics*. Every pair of points is contained in a unique classical geodesic which is in turn contained in a unique complex geodesic. There are also totally geodesic copies of the real hyperbolic plane $\mathbb{RH}^2$ inside $\mathbb{CH}^n$. In particular, for $\mathbb{R}^2 = \{(x, y, 0, ..., 0) : x, y \in \mathbb{R}\} \subset \mathbb{C}^n$, consider the two-dimensional real ball, $RB_2 = \mathbb{R}^2 \cap \mathbb{CH}^n$. That intersection is a totally geodesically embedded submanifold whose induced geometry is that of the Beltrami–Klein model of $\mathbb{RH}^2$ with constant curvature $-1/4$. All of the automorphic images of RB_2 are also totally geodesically embedded submanifolds of real dimension 2, and they, together with the complex geodesics, are the only such.

The geometry of RB_2 is not conformal with the Euclidean geometry of the containing $\mathbb{R}^2$. However, the two geometries are conformally equivalent at the origin, in particular angles with vertex at the origin have the same size in both geometries. There is a useful picture of RB_2 in [Go, p. 83], and there is a discussion of the geometry of that model (although the version with curvature -1) as well as the more familiar Poincaré disk model in Appendices B and C of [J].

If G is a complex geodesic and $x \in \mathbb{CH}^n$, we define the metric projection of x onto G, $P_G x$, to be that point in G which is closest to x in the pseudohyperbolic metric.

For $z, w \in \mathbb{B}_r$, we define the *kernel function k* by

$$(2.3) \qquad k_z(w) = k(w, z) = \frac{1}{1 - \langle\langle w, z \rangle\rangle}.$$

(We write $\langle\langle \cdot, \cdot \rangle\rangle$ for the inner product on $\mathbb{C}^n$ to distinguish from the inner products on the general Hilbert spaces we consider.)

There is a fundamental identity which describes the interaction of the involutive automorphisms with the kernel functions [Ru]: for any $y, z, w \in \mathbb{B}_r$,

$$(2.4) \qquad k(\phi_y(z), \phi_y(w)) = \frac{k(y, y)^{1/2}}{k(z, y)} \frac{k(y, y)^{1/2}}{k(y, w)} k(z, w).$$

By a triangle in $\mathbb{CH}^n$, we mean an ordered set of three distinct points called vertices, or those vertices together with the sides, the geodesic segments connecting

the vertices. We allow the degenerate case of three points in a single geodesic. The length of a side is the δ distance between the corresponding vertices. Triangles in $\mathbb{CH}^n$ do not necessarily have "faces"; three points in $\mathbb{CH}^n$ are not generally contained in a totally geodesic submanifold of real dimension 2.

Similarly, a tetrahedron in $\mathbb{CH}^n$ is an ordered set of four vertices. Its four subsets of three vertices are its triangular faces, even though they may not be faces in the geometric sense.

2.2   Hilbert spaces with reproducing kernels

Our background references for Hilbert spaces are [AM, ARSW2, PR, Ro].

Except for the spaces DA_r discussed this section, all the Hilbert spaces in this paper are finite-dimensional.

An n-dimensional RKHS is an n-dimensional Hilbert space H together with a distinguished basis of vectors called *reproducing kernels*, $RK(H) = \{h_i\}_{i=1}^n$. For any $v \in H$, we write $\hat{v}$ for its normalized version; $\hat{v} = v / \|v\|$. For $h_i, h_j \in RK(H)$, we write h_{ij} for their inner product and $\widehat{h_{ij}}$ for the inner product of their normalizations:

$$h_{ij} = \langle h_i, h_j \rangle, \quad \widehat{h_{ij}} = \langle \hat{h_i}, \hat{h_j} \rangle = \left\langle \frac{h_i}{\|h_i\|}, \frac{h_j}{\|h_j\|} \right\rangle.$$

The Gram matrix of H is the matrix $\mathrm{Gr}(H) = \left(h_{ij} \right)_{i,j=1}^n$.

Any v in H is regarded as a function on the index set of $RK(H)$ by setting $v(i) = \langle v, h_i \rangle$. If m is a function on $RK(H)$, then the associated multiplier operator M_m acting on H is the linear operator which satisfies $M_m v(i) = m(i) v(i)$.

A *regular subspace* of H is a subspace J spanned by a subset S of $RK(H)$ and regarded as an RKHS by setting $RK(J) = S$.

We now recall the Drury–Arveson spaces, DA_r; some basic references are [AM, Ar, Sh]. With $k_z(\cdot)$ the functions from (2.3), DA_r is the infinite-dimensional RKHS with kernel functions $\{k_z : z \in \mathbb{B}_r = \mathbb{CH}^r\}$ and inner product given by $\langle k_s, k_t \rangle = k_s(t)$. DA_1 is the classical Hardy space H^2 of the unit circle, a space which can also be described as the space of holomorphic functions on the unit disk with square summable sequences of Taylor coefficients. For $r > 1$, DA_r is a proper subspace of the Hardy space of the sphere. It is in the scale of Besov–Sobolev spaces, which also contains the Hardy space of the sphere and the Bergman space of the ball [ARSW2].

We are particularly interested in finite-dimensional regular subspaces of DA_r. For $Z = \left\{ z_j \right\}_{j=1}^n \subset \mathbb{CH}^r$, let $DA_r(Z)$ be the regular subspace of DA_r spanned by the kernel functions $\{k_{z_i}\}_{i=1}^n$. We abbreviate them by $\{k_i\}$ and set $k_{ij} = \langle k_i, k_j \rangle$.

If $r' > r$, there are natural inclusions of DA_r, $\mathbb{C}^r$, $\mathbb{B}_r$, and $\mathbb{CH}^r$ into the corresponding objects with r'. These inclusions interact in harmless ways with the constructions we are using. For instance, an inclusion of $\mathbb{B}_r$ into $\mathbb{B}_{r'}$ takes $Z \subset \mathbb{B}_r$ to a $Z' \subset \mathbb{B}_{r'}$. There is then an obvious natural map of $DA_r(Z)$ onto $DA_{r'}(Z')$ which preserves all the structure of interest here. Going forward, we will suppose r is sufficiently large, identify such pairs of sets and spaces, and drop the subscript r; thus, $DA(Z)$.

2.2.1 Rescaling

Rescaling is a fundamental equivalence relation between RKHSs. Given a finite-dimensional RKHS, G, H with $RK(H) = \{h_\alpha\}_{\alpha \in A}$, $RK(G) = \{g_\beta\}_{\beta \in B}$, we say G is a *rescaling* of H and write $G \sim H$ if there is a one-to-one map θ of A onto B and a nonvanishing complex valued function γ defined on A such that for all α_1, α_2 in A,

$$(2.5) \qquad \langle h_{\alpha_1}, h_{\alpha_2} \rangle = \left\langle \gamma(\alpha_1) g_{\theta(\alpha_1)}, \gamma(\alpha_2) g_{\theta(\alpha_2)} \right\rangle = \gamma(\alpha_1) \overline{\gamma(\alpha_2)} \left\langle g_{\theta(\alpha_1)}, g_{\theta(\alpha_2)} \right\rangle.$$

For instance, the linear map which sends each kernel function h_α to its normalized version, $\widehat{h_\alpha} = h_\alpha / \|h_\alpha\|$ is a rescaling. If A and B are ordered, then, unless we specify otherwise, we suppose that θ respects the ordering.

A basic class of rescalings in our discussion are those induced on spaces $DA(Z)$ by ball automorphisms. It is a consequence of (2.4) that given any $Z \subset \mathbb{B}_r$ and any $y \in \mathbb{B}$, the automorphism ϕ_y induces a rescaling of $DA(Z)$ to $DA(\phi_y(Z))$.

If G_H is a regular subspace of G and $H \sim G_H$, we will write $H \rightsquigarrow G$ or $H \rightsquigarrow G_H \subset G$ and will say that H has a rescaling into G.

2.2.2 Assuming irreducibility

In [AM, p. 79], an RKHS H is called *irreducible* if no two elements of $RK(H)$ are parallel and no two are orthogonal. We denote the class of all finite-dimensional irreducible RKHSs by $\mathcal{RK}$. If $H \in \mathcal{RK}$, then the entries of the Gram matrix of H are nonzero. That is convenient when we define the invariants δ and kos later.

The $DA(X)$ spaces we work with here all irreducible. The first condition is a consequence of their definitions; the second holds because the kernel functions in (2.3) are nonvanishing.

2.2.3 The complete Pick property

The CPP is a property which some $H \in \mathcal{RK}$ have and some do not. The class of H with that property is of fundamental importance in function-theoretic operator theory and is actively studied. Some background references are [AM, Ar, ARSW2, Sh]. A sample recent reference is [Ha].

We will use the following definition of the CPP.

Definition 2.1 We will say a finite-dimensional RKHS H has the CPP if there is finite $X \subset \mathbb{CH}^r$ so that H is a rescaling of $DA_r(X)$.

This is very much a definition of convenience. The classical definition of the CPP is in terms of extension properties of multipliers. Using that classical definition neither the fact that for any X, we have $DA_r(X) \in \mathcal{CPP}$ nor the fact that any $H \in \mathcal{CPP}$ is a rescaling of a $DA_r(X)$ is straightforward. The full story is in [AM] where one can also find a characterization of $H \in \mathcal{CPP}$ using the Gram matrix entries of H.

If $H \in \mathcal{RK}$ has the CPP, then we write $H \in \mathcal{CPP}$. It is immediate that every $DA_r(X) \in \mathcal{CPP}$, and that the CPP is preserved by rescaling and inherited by regular subspaces.

2.3 Invariants

Given $H \in \mathcal{RK}$, we now define several numerical functions of tuples of elements of $RK(H)$ the set of reproducing kernels of H. The definitions involve algebraic combinations of Gram matrix entries which by (2.5) are seen to be unchanged when H is replaced by a rescaling of H. Thus, the functions are rescaling invariants, well defined on the rescaling equivalence classes.

These functionals can also be regarded as defined for tuples of points in $\mathbb{CH}^n$ as follows. Given a functional F defined on, for instance, pairs of kernel functions, and given $x, y \in \mathbb{CH}^n$ select by $X \subset \mathbb{CH}^n$ with $x, y \in X$ and set $F(x, y) = F(k_x, k_y)$ where k_x and k_y are the kernel functions in $RK(DA(X))$. It is straightforward to check that this value is independent of the choice of X (and we will not mention X again) and that if F acting on Hilbert spaces is rescaling invariant, then the "same" functional acting on $\mathbb{CH}^n$ is automorphism invariant. We will use the same names and notation for the functionals acting on an $RK(H)$ and for the induced functionals acting on points in $\mathbb{CH}^n$.

2.3.1 Distance

For $H \in \mathcal{RK}$ and $h_1, h_2 \in RK(H)$, we define δ_H by

$$(2.6) \qquad \delta_H(i, j) = \delta_H(h_i, h_j) = \sqrt{1 - \frac{\left|h_{ij}\right|^2}{h_{ii} h_{jj}}} = \sqrt{1 - |\widehat{h_{ij}}|^2}.$$

This function is a metric on H ([AM, Lemma 9.9], [ARSW]) and is clearly invariant under rescaling. Using the scheme of the previous paragraphs, we can extend this definition to a functional, call it δ' for the moment, acting on pairs of points in $\mathbb{CH}^n$. One of the main links between the Hilbert space theory and hyperbolic geometry is that δ' equals the pseudohyperbolic distance, δ, between points, already defined in (2.1). To see this, use (2.3) and rewrite (2.4) as, for $w, x, y, z \in \mathbb{CH}^n$,

$$(2.7) \qquad 1 - \frac{k(w, y)k(y, z)}{k(y, y)k(w, z)} = \left\langle \left\langle \phi_y(z), \phi_y(w) \right\rangle \right\rangle.$$

Taking $z = w$, we have

$$(2.8) \qquad \delta^2(y, w) = \left|\phi_y(w)\right|^2 = 1 - \frac{\left|k(y, w)\right|^2}{k(y, y)k(w, w)} = \delta'^2(y, w).$$

The first equality in (2.8) is the definition of δ, the second is a special case of (2.7), and the third is the definition of δ'^2. Thus, $\delta = \delta'$. Alternatively, note that for any $z \in \mathbb{B}_n$, we have $\delta(0, z) = |z| = \delta'(0, z)$ and both δ and $\acute{\delta}'$ are invariant under automorphisms; hence, the two metrics are equal. We now drop the notation δ' and simply write δ for the functional defined on RKHS and for the pseudohyperbolic metric. Using (2.7), we can express δ in terms of coordinates; for $y, w \in \mathbb{B}_n$,

$$(2.9) \qquad \delta^2(y, w) = 1 - \frac{(1 - \langle\langle y, y \rangle\rangle)(1 - \langle\langle w, w \rangle\rangle)}{\left|1 - \langle\langle y, w \rangle\rangle\right|^2}.$$

2.3.2 Angular invariant

For $H \in \mathfrak{RK}$, we define the *angular invariant* α by, for $k_1, k_2, k_3 \in RK(H)$,

$$(2.10) \quad \alpha(1,2,3) = \alpha(k_1, k_2, k_3) = -\arg \langle k_1, k_2 \rangle \langle k_2, k_3 \rangle \langle k_3, k_1 \rangle = -\arg k_{12} k_{23} k_{31}.$$

(In general situations, care is needed in selecting a branch of $\arg$. However, for $k \in RK(DA(Z))$, $\operatorname{Re} k > 0$ and that lets us avoid problems.) Notice that α satisfies a cocycle identity; if k_4 is a fourth kernel function, then

$$(2.11) \quad \alpha(1,2,3) - \alpha(2,3,4) + \alpha(3,4,1) - \alpha(4,1,2) = 0.$$

We discuss some of the geometry associated with α in Section 4.4. More about this invariant is in [BIM, C, CO, Go, HM, M, Ro].

2.3.3 kos

For $H \in \mathfrak{RK}$, we define kos, a functional of triples of kernel functions. For $k_1, k_2, k_3 \in RK(H)$, $k_1 \neq k_2, k_3$, set

$$(2.12) \quad \operatorname{kos}_{k_1}(k_2, k_3) = \operatorname{kos}_1(2,3) = \frac{1}{\delta_{12}\delta_{13}}\left(1 - \frac{k_{21}k_{13}}{k_{11}k_{23}}\right),$$

and note the symmetry $\operatorname{kos}_1(2,3) = \overline{\operatorname{kos}_1(3,2)}$.

If $H = DA(X)$ for some $X \subset \mathbb{CH}^n$, then kos is related to the geometry of X. Recall that if $x \in \mathbb{CH}^n$, then ϕ_x is the ball involution which interchanges x and 0. We can use (2.7) and obtain

$$(2.13) \quad \operatorname{kos}_x(y,z) = \frac{1}{\delta(x,y)\delta(x,z)} \langle\langle \varphi_x(y), \varphi_x(z) \rangle\rangle.$$

In particular, if ξ is at the origin, $\xi = o$, then ϕ_ξ is the identity, for any w $\phi_\xi(w) = w$, and $\delta(\xi, w) = \delta(o, w) = |w|$. In that case,

$$(2.14) \quad \operatorname{kos}_o(y,z) = \left\langle\left\langle \frac{y}{|y|}, \frac{z}{|z|} \right\rangle\right\rangle = \langle\langle \widehat{y}, \widehat{z} \rangle\rangle.$$

(Although kos is invariant under automorphisms of $\mathbb{B}_n$, this formula is an inhomogenous representation and is not invariant.)

If the vectors y and z were in $\mathbb{R}^n$, this would be the inner product of unit vectors in $\mathbb{R}^n$ and would equal the cosine of the angle between the segments oy and oz. That is the source of the name kos.

If $\dim H = n$, then for $1 \leq s \leq n$, we define the $(n-1) \times (n-1)$ matrices

$$(2.15) \quad \operatorname{KOS}(H,s) = \operatorname{KOS}(\operatorname{Gr}(H),s) = \left(\operatorname{kos}_s(i,j)\right)_{\substack{i,j=1 \\ i,j\neq s}}^{n},$$

$$(2.16) \quad \operatorname{MQ}(H,s) = \left(\delta_{si}\delta_{sj}\operatorname{kos}_s(i,j)\right)_{\substack{i,j=1 \\ i,j\neq s}}^{n} = \left(1 - \frac{k_{is}k_{sj}}{k_{ij}k_{ss}}\right)_{\substack{i,j=1 \\ i,j\neq s}}^{n}.$$

We also write $\operatorname{KOS}(X,s)$ for $\operatorname{KOS}(DA(X),s)$.

2.4 Matrix notation

For an $n \times n$ matrix A, we write $A > 0$ if A is positive definite and $A \geqslant 0$ if it is positive semidefinite. We say B is a *principal submatrix* of A if it is obtained from A by removing certain rows and also the corresponding columns. Note that if $H \in \mathcal{RK}$, then J is a regular subspace of H if and only if $\mathrm{Gr}(J)$ is a principal submatrix of $\mathrm{Gr}(H)$. We denote the set of all principal submatrices of A by $\mathcal{PS}(A)$. If $B \in \mathcal{PS}(A)$ and the indices of the rows and columns retained in B are an initial segment, indices j with $1 \leq j \leq k$ for some $k \leq n$, then B is said to be a *leading* principal submatrix. The determinants of those matrices are called *principal minors* and *leading principal minors*, respectively.

3 The $\mathcal{CPP}$ and point sets in $\mathbb{CH}^n$

We will sometimes use a model triangle Γ or a model tetrahedron Δ which have convenient coordinates. Our model triangle is $\Gamma \subset \mathbb{CH}^2$:

(3.1)
$$\Gamma = \{x_1, x_2, x_3\} = \{(0,0), (a,0), (x,b)\},$$
$$a > 0, b \geq 0, \ x \in \mathbb{C},$$
$$0 < a, \ |x|^2 + b^2 < 1.$$

Our model tetrahedron is $\Delta \subset \mathbb{CH}^3$:

(3.2)
$$\Delta = \{x_1, x_2, x_3, x_4\} = \{(0,0,0), (a,0,0), (x,b,0), (y,z,c)\},$$
$$a > 0, \ b, c \geq 0, \ x, y, z \in \mathbb{C},$$
$$0 < |a|^2, \ |x|^2 + |b|^2, \ |y|^2 + |z|^2 + |c|^2 < 1.$$

It is easy to check that any triangle in $\mathbb{CH}^n$, once the numbering of the vertices is fixed, is congruent to a unique Γ; similarly for tetrahedra and Δ.

To a three-dimensional $H \in \mathcal{RK}$ with $RK(H) = \{h_i\}_{i=1}^3$, we associate the following sets of invariant quantities:

$$S = \left\{ |\widehat{h_{12}}|, |\widehat{h_{23}}|, |\widehat{h_{13}}|, \alpha_{123} \right\},$$
$$S' = \{\delta_{12}, \delta_{13}, \delta_{23}, \alpha_{123}\},$$
(3.3)
$$S'' = \{\delta_{12}, \delta_{13}, \mathrm{kos}_1(2,3)\}.$$

And, for convenience, we set

(3.4)
$$\Gamma_{abc} = \left| \frac{\widehat{h_{ab}} \widehat{h_{bc}}}{\widehat{h_{ca}}} \right| = \sqrt{ \frac{(1 - \delta_H^2(a,b))(1 - \delta_H^2(b,c))}{(1 - \delta_H^2(c,a))} }.$$

The following describes three-point sets X in $\mathbb{CH}^2$ and the associated $DA(X)$ spaces.

Theorem 3.1 [AM, Br, Ro]   *Given a three-dimensional $H \in \mathcal{RK}$, the following are equivalent:*

(1) $H \in \mathcal{CPP}$.
(2) *There is a three-point set X in $\mathbb{CH}^2$ with $H \sim DA(X)$.*

(3) *There is a Γ as in (3.1) with $H \sim DA(\Gamma)$.*

(4) *Let J be the regular subspace of H spanned by $\{h_1, h_2\}$. Let M be the multiplier on J of norm one specified by the following action of its adjoint:*

$$M^* h_1 = 0,$$
$$M^* h_2 = \delta_H(h_1, h_2)h,$$

then M extends to a multiplier of norm one on H.

(5) $\mathrm{KOS}(H, 1) \geqslant 0$.

(6)

$$(3.5) \qquad\qquad |\mathrm{kos}_1(2, 3)| \leq 1.$$

(7) *S and the Γ's defined from S using (3.4) satisfy*

$$(3.6) \qquad\qquad \Gamma_{123} + \Gamma_{231} + \Gamma_{312} \leq 2\cos\alpha_{123}.$$

(8) *S' and the Γ's defined from S' using (3.4) satisfy (3.6).*

Furthermore, X sits inside a complex geodesic if and only if $\det \mathrm{KOS}(H, 1) = 0$, or equivalently $|\mathrm{kos}_1(2, 3)| = 1$, or the b coordinate of Γ in (3.1) is 0. It sits inside a totally real geodesic submanifold of real dimension 2 if and only if $\mathrm{kos}_1(2, 3)$ is real.

Conversely, given

data S and Γ's defined from S using (3.4) such that (3.6) holds,

or

data S' and Γ's defined from S' using (3.4) such that (3.6) holds,

or

data S'' f or which (3.5) holds,

there is triangle X in $\mathbb{CH}^2$, unique up to congruence, which has those parameters.

Proof The equivalence of the first two statements is by definition. The equivalence of statements (2)–(6) is in [Ro]. Statement (7) and the parts of (8) about existence of the triangle are due to Brehm [Br]. The statement about X being in a complex geodesic is implicit in the proof of Theorem 16 in [Ro]. ∎

Thus, each of S, S', or S'' could be used to describe a triangle. The equivalence of using S or S' is clear. Passing between S' and S'' is described in Section 4.2.

Remark 1 We will see in Proposition 4.5 that the value of kos determines the congruence class of a vertex. Hence, the fact that the parameters in S'' determine the congruence class of a complex triangle is an analog of the fact that two side lengths and the shape of the included angle determine the congruence class of a Euclidean triangle.

Some aspects of the previous theorem extend to larger sets and spaces. The next result is an amalgam of the fact that up to rescaling the $H \in \mathcal{CPP}$ are exactly the spaces $DA(X)$ for X a finite set in $\mathbb{CH}^n$ [AM], the fact mentioned earlier that automorphisms of the ball induce rescalings of spaces $DA(X)$, and the description of congruence classes of finite sets in $\mathbb{CH}^n$ given in [BE, G, HS, Ro].

Theorem 3.2 *An n-dimensional $H \in \mathcal{RK}$ satisfies $H \in \mathcal{CPP}$ if and only if there is an $X = \{x_i\}_1^n \subset \mathbb{CH}^{n-1}$ with $H \sim DA(X)$.*

Given another n-dimensional $H' \in \mathcal{CPP}$ with $H' \sim DA(X')$ for $X' = \{x_i'\}_1^n \subset \mathbb{CH}^{n-1}$, the following are equivalent:

(1) *$H \sim H'$.*
(2) *$X \sim X'$.*
(3) *$DA(X) \sim DA(X')$.*
(4) *All the triangles of X are congruent to the triangles of X'; i.e., for each triple i, j, k, $1 \leq i, j, k \leq n$, there is a ball automorphisms taking $\{x_i, x_j, x_k\}$ to $\{x_i', x_j', x_k'\}$.*
(5) *The triangles of X which have one vertex at x_1 are congruent to the corresponding triangles of X'; for each pair $i, j, 1 < i < j \leq n$, the triangles $\{x_1, x_i, x_j\}$ and $\{x_1', x_i', x_j'\}$ are congruent.*
(6) *The regular three-dimensional subspaces of H are rescalings of the corresponding regular three-dimensional subspaces of H'.*
(7) *The regular three-dimensional subspaces of H which contain k_{x_1} are rescalings of the corresponding three-dimensional subspaces of H'.*

Corollary 3.3 *The relationship $H \sim DA(X)$ establishes a bijection between the class of n-dimensional $H \in \mathcal{CPP}$ modulo rescaling and the class of n point sets X in $\mathbb{CH}^k$, $k \geq n - 1$, modulo congruence. The correspondence respects passage to regular subspaces and to subsets.*

If we use S, S', or S'' to characterize the triangles in (4) of the theorem, we obtain $O(n^3)$ real numbers which describe X up to congruence. However, that list has repetitions and redundancies. Restricting to the triangles listed in (5) and noting that some side lengths are listed twice leads to a list of $(n-1)^2$ real parameters which determine X up to congruence. That number is optimal; triangles are determined by four parameters, tetrahedra by nine. There are also constraints on the parameters; the inequality (3.6) for triangles is the simplest example. We give analogous constraints for tetrahedra in Theorem 6.2. Also, noting the previous corollary, the same parameters (or, perhaps, similarly named parameters) describe spaces $DA(X)$ and spaces in $\mathcal{CPP}$.

3.1 Assembly and coherence

We might know that each of several spaces $\{J_i\}$ is equivalent under rescaling to a subspace H_i of some H; $J_i \leadsto H_i \subset H$. In that case, there are *coherence conditions* connecting the $\{J_i\}$; if $H_i \cap H_j = H_{ij}$, then the corresponding subspaces, $J_{i(j)} \subset J_i$ which corresponds to H_{ij} and $J_{j(i)} \subset J_j$ which also corresponds to H_{ij}, must be rescalings of each other. This condition is nontrivial if $\dim H_{ij} > 1$. If these conditions are met, then the $\{J_i\}$ are said to be a *coherent set* of spaces and we will write $\{J_i\} \Rightarrow H$.

Here is how this works in the context of Question 2. For a four-dimensional $H \in \mathcal{CPP}$ with $RK(H) = \{h_i\}_{i=1}^4$, denote the four regular three-dimensional subspaces $\{H_i\}_{i=1}^4$ by

$$RK(H_i) = \{h_r : 1 \leq r \leq 4, \ r \neq i\}.$$

Question 2 asks for necessary and sufficient conditions on four three-dimensional spaces $\{J_i\}_{i=1}^4 \subset \mathcal{RK}$ to insure that there is an $H \in \mathcal{CPP}$ with for $1 \leq i \leq 4$, $H_i \sim J_i$.

If there are such rescalings, then we suppose the indices on the reproducing kernels j_{ir} of J_i have been chosen so that for each i, r, the rescaling of H_i and J_i matches the kernel function h_r of H_i with the kernel function j_{ir} of J_i.

If $\{J_i\}$ is a coherent set of spaces, then the space J_1, with kernel functions $\{j_{12}, j_{13}, j_{14}\}$, is a rescaling of H_1 which has kernel functions $\{h_2, h_3, h_4\}$ with j_{1s} pairing with h_s; similarly for J_3 with kernels $\{j_{31}, j_{32}, j_{34}\}$ and H_3 with kernels $\{h_1, h_2, h_4\}$. Thus, subspace $J_{1,24}$ of J_1 with kernel functions $\{j_{12}, j_{14}\}$ and the subspace $J_{3,24}$ of J_3 with kernel functions $\{j_{32}, j_{34}\}$ are both rescalings of the subspace of H_{24} with kernel functions $\{h_2, h_4\}$, and hence $J_{1,24} \sim J_{3,24}$. The other coherence conditions are all of this form; given $p < q, r < s$ distinct elements of $\{1, 2, 3, 4\}$, we must have $J_{p,rs} \sim J_{q,rs}$. These conditions are necessary for there to be rescalings of $\{J_i\}$ into H. These conditions are the Hilbert space analogs of the matching side length conditions necessary to assemble triangles into a tetrahedron.

Notice that the conditions we obtained just now, $J_{p,rs} \sim J_{q,rs}$, did not involve the space H and in fact we can have a coherent set of spaces $\{J_i\}$ without having an H. It suffices to have a consistent set of coherence conditions, statements that certain subspaces of the various $\{J_i\}$ are rescalings of each other. The consistency requirement is that the set of specified rescalings must be compatible with the fact that rescaling is an equivalence relation which interacts coherently with passage to subspaces. That is automatic if $\{J_i\} \Rightarrow H$ but is also possible otherwise. In that case, we still call the $\{J_i\}$ a coherent set of spaces and we will write $\{J_i\} \Rightarrow \,??$.

Thus, the statement $\{J_i\} \Rightarrow \,??$ is the statement that there is a description (perhaps only implicit) of a type of target space H and an assembly scheme for constructing such an H. The statement $\{J_i\} \Rightarrow H$ is that there is such an H.

Given $\{J_i\} \Rightarrow H$, we would like to obtain information about H from the spaces $\{J_i\}$ together with the coherence data $\{J_i\} \Rightarrow \,??$. We know from Theorem 3.2 that the values of δ_{ij} and $\mathrm{kos}_i(j, k)$ for H describe H up to rescaling and would like to compute them from the $\{J_i\}$. Given $\{J_i\} \Rightarrow \,??$, we can construct the imputed value of $\delta_H(a, b)$ by selecting any J_s whose image under the rescaling $J_s \rightsquigarrow H$ contains the kernel functions h_a and h_b of $RK(H)$. If j_{ra} and j_{rb} are the elements of $RK(J_s)$ which correspond to h_a and h_b under that rescaling map, then $\delta_{J_r}(j_{ra}, j_{rb})$ is our candidate value for $\delta_H(a, b)$. Note that this is defined even if there is no H; however, if there is an H, then the rescaling $J_s \rightsquigarrow H$ insures that this value is $\delta_H(a, b)$. Also, if there is another possible choice for J_s, then the coherence conditions insure that choice will produce the same value of δ. If there is no such J_s, then we have no candidate for the value $\delta_H(a, b)$ and that value would be a free variable in our analysis; the free variable K_{42} in Corollary 6.3 is an example. The procedure for constructing our candidate values for $\mathrm{kos}_a(b, c)$ is the same except that we need to select a J_s whose image would contain the three elements of $RK(H)$ with indices a, b, and c.

The general pattern is that given $\{J_i\} \Rightarrow \,??$, we can use the coherence data and compute (some of) the entries $\mathrm{kos}_a(b, c)$ that the matrix $\mathrm{KOS}(H, a)$ will have if there is an H with $\{J_i\} \Rightarrow H$. (We are supposing that it has been specified which kernel functions (if any) of the spaces J_i are to be associated with the distinguished kernel k_a in H.) We denote the partially defined matrix computed this way by $\mathrm{KOS}(\{J_i\}, a)$. If $\{J_i\} \Rightarrow H$, then $\mathrm{KOS}(\{J_i\}, a) = \mathrm{KOS}(H, a)$. In Section 5.1, we give necessary and sufficient conditions on a matrix for it to be $\mathrm{KOS}(H, a)$ for an $H \in \mathcal{CPP}$. Those

conditions applied to $\mathrm{KOS}(\{J_i\}, a)$ give necessary and sufficient conditions for the assembly question $\{J_i\} \Rightarrow ??$ to have a positive solution.

By Corollary 3.3, statements about spaces, subspaces, rescalings, values of invariants, and coherence are equivalent to statements about sets in $\mathbb{CH}^m$, subsets, congruences, values of invariants, and an appropriate notion of coherence. We will use the same language and notation in both contexts. For instance, given sets $\{Y_i\}$ in $\mathbb{CH}^n$, we will write $\{Y_i\} \Rightarrow X$ if there are specified congruences of each Y_i into some X describing which points from the $\{Y_i\}$ are to be mapped to which points of X. Those specifications force congruences of various subsets of the various Y_i that are analogs to the rescaling statements for the subspaces $J_{p,rs}$ in the previous paragraphs. We write $\{Y_i\} \Rightarrow ??$ if we know the $\{Y_i\}$ satisfy those congruences even if we do not know that there is an X.

We will pass freely between these ideas for RKHS and for sets in $\mathbb{CH}^n$.

3.2 Equivalence of the two questions

Using Theorems 3.1 and 3.2, we can see that the two questions in the introduction are equivalent. Those theorems give us the following facts:

(1) Given a set of triangles $\{Y_i\}_{i=1}^4$ in $\mathbb{CH}^n$, there are three-dimensional $\{J_i\}_{i=1}^4 \subset \mathcal{CPP}$ such that

$$\text{(3.7)} \qquad J_i \sim DA(Y_i), \ i = 1, \ldots, 4.$$

In the other direction, given three-dimensional $\{J_i\}_{i=1}^4 \subset \mathcal{CPP}$, there are $\{Y_i\}_{i=1}^4$ such that (3.7) holds. In either case, (3.7) continues to hold if the $\{J_i\}$ are replaced by rescalings $\{J_i'\}$ or if the $\{Y_i\}$ are replaced by sets congruent to the $\{Y_i'\}$.

(2) Given a tetrahedron X in $\mathbb{CH}^n$, there is a four-dimensional $H \in \mathcal{CPP}$ such that

$$\text{(3.8)} \qquad H \sim DA(X).$$

In the other direction, given $H \in \mathcal{CPP}$, there is an X such that (3.8) holds. In either case, (3.8) continues to hold if H is replaced by a rescaling H' or X is replaced by a congruent X'.

Hence, we have the following proposition.

Proposition 3.4 *Given triangles $\{Y_i\}_{i=1}^4$ in $\mathbb{CH}^n$ and Hilbert spaces $\{J_i\}_{i=1}^4 \subset \mathcal{CPP}$ which are related as in (3.7), there is an X with $\{Y_i\} \Rightarrow X$ if and only if there is an $H \in \mathcal{CPP}$ with $\{J_i\} \Rightarrow H$. In that case, H and X satisfy (3.8).*

Proof This follows from statements (1) and (2) above together with the observation that for a finite $X \subset \mathbb{CH}^n$, the regular subspaces of $DA(X)$ are exactly the spaces $DA(Y)$ for Y a subset of X. ∎

In the proposition, the assumption that (3.7) holds insures the $J_i \in \mathcal{CPP}$. Even with that condition, having $H \in \mathcal{RK}$ and $\{J_i\} \Rightarrow H$ is not enough to insure that $H \in \mathcal{CPP}$. This is shown by, for instance, Quiggin's example which is discussed in Section 6.3.

4 Geometry and *Kos*

We will use the functional kos extensively below and so we pause now to develop the geometry associated with this functional.

4.1 Evaluating *Kos*

Because kos is an automorphism invariant, we can study it for a general triple by first using an automorphism to place our triple in the configuration Γ described in (3.1): $\{x_1, x_2, x_3\} = \{(0,0), (a,0), (x,b)\}$. In that case, computing using (2.14) gives

$$(4.1) \qquad \text{kos}_1(2,3) = \left\langle\!\!\left\langle \frac{(a,0)}{\|(a,0)\|}, \frac{(x,b)}{\|(x,b)\|} \right\rangle\!\!\right\rangle = \frac{a\bar{x}}{\|x_2\|\,\|x_3\|} = \frac{\bar{x}}{\|x_3\|} = \frac{|\bar{x}|}{\|x_3\|}\frac{\bar{x}}{|\bar{x}|}.$$

To evaluate $\text{kos}_1(2,3)$ for a general triple $\{x_1, x_2, x_3\}$, we want a description that is invariant under automorphism of hyperbolic space and which gives the value in (4.1) for the particular triple $\{(0,0), (a,0), (x,b)\}$. Given the general triple, let $G(1,2)$ be the complex geodesic which contains x_1 and x_2. Recall that $P_{G(1,2)}$ is the metric projection onto $G(1,2)$.

We will use the notion of angle between two geodesic segments in $G(1,2)$. Two geodesic segments in $\mathbb{CH}^n$ form a bivalent vertex; however, there is not a natural notion of "vertex angle" which determines the shape of the vertex, rather the geometry of the vertex is determined by the complex number kos, this is discussed in Section 4.3. If, however, the two geodesics lie in the same complex geodesic, then there is a conformal map of that complex geodesic to the Poincare disk. In that case, we take the angle between the two segments to be the angle between their images in the Poincare geometry of the disk (which is in fact the same as the Euclidean angle).

Here is the invariant statement we want.

Proposition 4.1 *Set $P_{G(1,2)}x_3 = y$. Writing $\text{kos}_{x_1}(x_2, x_3) = re^{i\theta}, r > 0, -\pi \le \theta < \pi$ and* angle *for the hyperbolic angle, we have*

$$(4.2) \qquad\qquad r = \frac{\delta(x_1, y)}{\delta(x_1, x_3)},$$

$$(4.3) \qquad\qquad \theta = \text{angle}(x_1 x_2, x_1 y).$$

Proof We need to show that the formula is correct for Γ and that it is invariant. For Γ, the complex geodesic $G(1,2)$ is the unit disk in the first coordinate line, and in that case, it is elementary to show that the metric projection of x_3 onto $G(1,2)$ is $(x,0)$. Also, for angles in the unit disk, the Euclidean angle and the hyperbolic angle are the same. These facts show that both (4.2) and (4.3) are correct for Γ.

To see that (4.2) is invariant, note that the statement $P_{G(1,2)}x_3 = y$ is invariant as is δ. The equality (4.3) is more subtle because there is no natural definition of the angle of intersection for two geodesics in $\mathbb{CH}^n$. However, the geodesic segments $x_1 P_{G(1,2)}x_3$ and $x_1 x_2$ are both in $G(1,2)$. Any complex geodesic, in particular $G(1,2)$, is conformally equivalent to the classical Poincaré disk which does carry an invariant notion of angle between intersecting curves. Using that notion of angle, we see that (4.3) is also invariant. ∎

Given points $\{x_1, x_2, x_3\}$, we are using kos to describe the geometry of the intersection of the geodesics $x_1 x_2$ and $x_1 x_3$. Other parameters can be used for the same purpose, for instance, in [Go, p. 88], Goldman uses angles, ϕ, θ which satisfy $\cos\phi = |\text{kos}_1(2,3)|$ and $\cos\theta = \text{Re kos}_1(2,3)$.

If the triangle is not in general position, we can say more about the relationship between the values of kos at the vertices and the shape of the triangle. Given a triangle $T = \{x_1, x_2, x_3\} \subset \mathbb{CH}^2$, we use an automorphism to suppose that x_1 is at the origin. Regard $\mathbb{CH}^2$ as $\mathbb{B}_2$ inside $\mathbb{C}^2$, denote by V the real linear span of the points of T, and set $W = V \cap \mathbb{B}_2$. It may be that W is a totally geodesic submanifold of $\mathbb{CH}^2$. In that case, with the origin in W, there are three possibilities. First, W may be a classical geodesic in which case it will be a line segment through the origin. Second, W may be a complex geodesic which contains the origin. In that case, we can suppose it is the unit disk in the plane of the first coordinate and hence it is the classical Poincaré disk with constant negative curvature -1. The final possibility is that W is a totally real totally geodesic disk and hence, after an automorphism, we can suppose it is $RB_2 = \{(r, s) \in \mathbb{B}_2 : r, s \in \mathbb{R}\}$, the Beltrami-Klein disk of constant curvature $-1/4$.

In the first case, noting (4.2) and (4.3), we see that $\text{kos}_1(2,3) = \pm 1$. The value -1 occurs when x_1 separates the other two points, the value $+1$ when it does not. In the second case, x_3 is in $G(1, 2)$ and so $P_{G(1,2)}x_3 = x_3$. From (4.2), we see that $|\text{kos}_{x_1}(x_2, x_3)| = 1$ and then from (4.3) that $\text{kos}_1(2,3) = e^{-i\gamma}$ where γ is the Euclidean angle between the segments $x_1 x_2$ and $x_1 x_3$. Thus, γ is the Euclidean and also the hyperbolic angle at vertex x_1. Similarly, the values of kos at the other two vertices give the other angles. The congruence class of a triangle in a plane of constant negative curvature is determined by its angles and hence in this case also by the three values of kos. Finally, in the third case, the triangle is in a totally real vector space and we are in the situation discussed in Proposition 6.7. By automorphism invariance, we may suppose x_1 is at the origin. From (2.14), we see that $\text{kos}_{x_1}(x_2, x_3) = \cos\theta$ where θ is the Euclidean angle of the vertex at the origin of the triangle with vertices $\{x_i\}_{i=1}^3$. However, the intersection of the real plane with those vertices with $\mathbb{CH}^2$ is the Beltrami–Klein model of $\mathbb{RH}^2$ which, at the origin, is conformal with the Euclidean geometry. Hence, θ is also the angle at x_1 of the hyperbolic triangle with vertices $\{x_i\}_{i=1}^3$ sitting in a copy of $\mathbb{RH}^2$. Similarly for the other two vertices. Hence, again, we know the three angles of a triangle in a plane of constant negative curvature and hence know its congruence class. Finally, looking backward, we see that the first case is the second and third cases holding simultaneously.

In each of these cases, the argument can be reversed. If $\text{kos}_1(2,3) = \pm 1$, then W is a line through the origin, if $|\text{kos}_{x_1}(x_2, x_3)| = 1$, then $P_{G(1,2)}x_3 = x_3$, and hence the triangle lies in a complex geodesic, and finally, again noting Proposition 6.6, if $\text{kos}_{x_1}(x_2, x_3)$ is real, then after automorphism, the triangle is in the position described.

4.2 *Kos* **and other invariants**

In Theorem 3.1, we recalled the result of Brehm [Br] that equality of the data sets S' is a congruence criterion for triangles in $\mathbb{CH}^n$. That criterion and variations are often used in describing the geometry of finite sets in $\mathbb{CH}^n$ [BE, CG, HS, G, Ro]. Here, we mainly use the congruence criterion given by the dataset S'' which is, as we noted in

Remark 1, an analog of the classical side-angle-side congruence criterion for Euclidean triangles.

It is straightforward to pass between S' and S''. The parameters are invariant, so we can assume that we are in the model case and the triangle is $\Gamma = \{x_1, x_2, x_3\} = \{(0,0), (a,0), (x,b)\}$. First, suppose we have the data S'', that is, $a = \delta(x_1, x_2)$, $\omega = \delta(x_1, x_3)$, and $\text{kos}_1(2,3)$ are known. Using those values and (2.14), we also know $a\bar{x}$. Hence, using (2.9), we can find the third side length using

$$(4.4) \qquad \delta^2(x_2, x_3) = 1 - \frac{(1 - \|x_2\|^2)(1 - \|x_3\|^2)}{|1 - \langle\langle x_2, x_3 \rangle\rangle|^2} = 1 - \frac{(1 - \alpha^2)(1 - \omega^2)}{|1 - \alpha\bar{x}|^2}.$$

To finish determining S', we need to find β, the angular invariant. For this triangle, $\beta = \arg(1 - a\bar{x})$, and, as we just mentioned, $a\bar{x}$ is known; hence, β is known.

In the other direction, to go from S' to S'', we need to find $\text{kos}_1(2,3)$. In this case, we know the three side lengths, and hence, noting (4.4), we know $|1 - \alpha\bar{x}|$. We also know the angular invariant $\beta = \arg(1 - a\bar{x})$. Combining these two, we know $a\bar{x}$. With that information, and using (4.1), we can find $\text{kos}_1(2,3)$.

4.3 *Kos* **and the geometry of vertices**

For a bivalent vertex in $\mathbb{R}^n$, the cosine of the vertex angle carries all the intrinsic geometric data about the vertex. The geometry of a trivalent vertex is determined by the geometry of the three component bivalent vertices. The values of kos provide similar information for vertices in $\mathbb{CH}^n$.

To set the stage, we first consider sets in $\mathbb{R}^n$. By a vertex **V** in $\mathbb{R}^n$, we mean a collection of two or more line segments, rays, with a common starting point the vertex point, or vertex, V. We call **V** bivalent if it has two rays, trivalent if there are three, multivalent in general. We suppose the rays of a bivalent vertex are ordered. (That will be important in the complex case.) The two rays of a bivalent vertex span an affine plane and form an angle in that plane, the vertex angle. We say two vertices are (Euclidean) congruent if there is a Euclidean isometry placing the second vertex point at the same position as the first and so that the initial segments of the rays of the second vertex coincide with the initial segments of the rays of the first.

Proposition 4.2 *Two bivalent vertices in $\mathbb{R}^n$ are congruent if and only if the cosines of their vertex angles are equal.*

If three Euclidean triangles have pairs of matching side lengths, then they can be put together as faces of a tetrahedron if and only if the three dihedral angles that would be joined at a vertex can, in fact, be joined to form a trivalent vertex. (The simple classical condition for this is in Corollary 6.9.) In particular, there is no constraint on the side lengths beyond the matching condition. Here is a precise statement.

Suppose we are given $\mathfrak{S}$, a set of three triangles $\{T_i\}_{i=1}^3$ each contained in some $\mathbb{R}^n$. Suppose for $i = 1, 2, 3$ that T_i has vertices $\{x_{i1}, x_{ia}, x_{ib}\}$, that the Euclidean length of the side $x_{i1}x_{ia}$ is l_{ia} and similarly for l_{ib}, and that $\mathbf{W}_i$ is the bivalent vertex in T_i with vertex point x_{i1} and rays ordered similarly to the sides of the triangle; $x_{i1}x_{ia}$ first, $x_{i1}x_{ib}$ second. It may or may not be possible to join these three triangles as faces of a tetrahedron in some $\mathbb{R}^n$. That is there may be or may not be a tetrahedron $\{y_1, y_2, y_3, y_4\}$ in

some $\mathbb{R}^n$ with the triangular face $\{y_1, y_2, y_3\}$ congruent to T_1, $\{y_1, y_3, y_4\}$ congruent to T_2, and $\{y_1, y_2, y_4\}$ congruent to the triangle $\widetilde{T}_3 = \{x_{31}, x_{3b}, x_{3a}\}$. (The triangle $\widetilde{T}_3$ has the same vertices as T_3 but in a different order.) Certainly, a necessary condition for building a tetrahedron is that certain side lengths match.

Definition 4.3 The *set of three triangles* $\mathfrak{S}$ is said to be a matched set if there are numbers $\{L_i\}_{i=1}^3$ such that

(4.5) $l_{1b} = l_{2a}, = L_1, \; l_{2b} = l_{3a} = L_2, \; l_{3b} = l_{1a} = L_3.$

Proposition 4.4 *Given a matched set of three Euclidean triangles $\mathfrak{S}$, the following are equivalent.*

(1) *The triangles of $\mathfrak{S}$ are congruent to the three faces of a tetrahedron.*
(2) *If $\mathfrak{S}'$ is another matched set of three triangles (with associated data denoted by primes) and for $i = 1, 2, 3$ the bivalent vertex $\mathbf{W}'_i$ is congruent to the bivalent vertex $\mathbf{W}_i$, then the triangles of $\mathfrak{S}'$ are congruent to the faces of a tetrahedron. That is, the previous conclusion holds for any choice of the lengths $\{L'_i\}$.*
(3) *There is a trivalent vertex $\mathbf{V}$ whose three component bivalent vertices are congruent to the three bivalent vertices $\{\mathbf{W}_i\}_{i=1}^3$.*
   In each case, the tetrahedron or the trivalent vertex is unique up to Euclidean congruence.

Proof That (2) implies (1) implies (3) is trivial. To finish, we show (3) implies (1). If (3) holds, we have a trivalent vertex $\mathbf{V}$. Extend the three segments forming $\mathbf{V}$ to half lines starting at the vertex point. Select points on those three half lines so that their distances from the vertex point are the values L'_i from (2). Those three points together with the vertex point give the four vertices of the desired tetrahedron. By construction, the side lengths meeting at the vertex point are correct. The congruence of the triangular faces meeting at that point to the target triangles is a consequence in each case of congruence of triangles with a pair of matching side lengths and equality of the included angel. The side lengths match by construction, and the angles are equal by the hypotheses in (3) on the bivalent vertices which form the given trivalent vertex.   ∎

We now prove analogs of the two previous propositions for $\mathbb{CH}^n$. By a vertex $\mathbf{V}$ in $\mathbb{CH}^n$, we mean a collection of two or more geodesic segments, *rays*, R_i, with a common starting point the vertex point, V. Again, a vertex may be bivalent, trivalent, or multivalent. We say two vertices are congruent if there is an automorphism placing the second vertex point on the first and so that initial segments of the two sets of rays overlap.

The geometry of a bivalent vertex in $\mathbb{CH}^n$ can be described by two real numbers or one complex number. We will use the complex quantity $\mathrm{kos}\,(\mathbf{V})$ which we define to be $\mathrm{kos}_V(x_1, x_2)$ where V is the vertex point and x_i is chosen on the ray R_i, $i = 1, 2$. The formula (2.14) shows that this value does not depend on those choices. Thus, if $\{x_1, x_2, x_3\}$ is a triangle and $\mathbf{V}$ the bivalent vertex with vertex point x_1, then $\mathrm{kos}\,(\mathbf{V}) = \mathrm{kos}_1(2, 3)$.

Proposition 4.5 *Two bivalent vertices,$\mathbf{V}$, $\mathbf{W}$, in $\mathbb{CH}^n$ are congruent if and only if $\mathrm{kos}\,(\mathbf{V}) = \mathrm{kos}\,(\mathbf{W})$.*

Proof If the vertices are congruent, then the conclusion is clear. In the other direction, let v be the vertex point of $\mathbf{V}$, pick ε small, and pick x and y on the two rays of $\mathbf{V}$ at distance ε from v. Let w be the vertex point $\mathbf{W}$ and pick points r, s on the rays forming $\mathbf{W}$ at distance ε from w. The triangles $\{v, x, y\}$ and $\{w, r, s\}$ give the same values for (3.3), the dataset S'' of Theorem 3.1, and hence, by that theorem, they are congruent. That congruence of triangles also gives the required congruence of the vertices.

There is a contrast between the real and complex cases. For a bivalent vertex $\mathbf{V}$, let $\mathbf{V}_{\mathrm{rev}}$ be the vertex obtained from $\mathbf{V}$ by reversing the order of the two rays. For vertices in $\mathbb{RH}^n$, $\mathbf{V}_{\mathrm{rev}}$ is congruent to $\mathbf{V}$. However, if $\mathbf{V} \subset \mathbb{CH}^n$, then $\mathbf{V}_{\mathrm{rev}}$ is generically not congruent to $\mathbf{V}$, and it is congruent to $\mathbf{V}^*$, the vertex obtained by conjugating the coordinates of $\mathbf{V}$. This can be seen by computing that $\mathrm{kos}(\mathbf{V}_{\mathrm{rev}}) = \overline{\mathrm{kos}(\mathbf{V})} = \mathrm{kos}(\mathbf{V}^*)$. ∎

Now, suppose we are given $\mathfrak{S}$, a set of three triangles each in $\mathbb{CH}^n$. We continue the earlier notation and terminology, but now we use the pseudohyperbolic side lengths $\delta(\cdot, \cdot)$. With that change, the analog of Proposition 4.4 holds.

Proposition 4.6 *Given a matched set of triangles $\mathfrak{S}$ in $\mathbb{CH}^n$, the statements (1)–(3) of Proposition 4.4 are equivalent. If the conditions hold, then the congruence class of the tetrahedra and of the trivalent vertex are uniquely determined. In particular, the congruence class of three faces of a tetrahedron together with the congruence class of the vertex at which they meet determines the congruence class of the fourth face.*

Proof It is immediate that $(2) \implies (1) \implies (3)$. To see that $(1) \implies (2)$, suppose from (1) that we have $\mathfrak{S}$ and the associated tetrahedron Λ and that we are given a new set of lengths $\{L_i'\}$ from (2). Use an automorphism to replace Λ with the tetrahedron Δ with coordinates given by (3.2). For $i = 2, 3, 4$, select $\gamma_i > 0$ so that the tetrahedron $\Delta' = \{x_1, \gamma_2 x_2, \gamma_3 x_3, \gamma_4 x_4\}$ has $\|\gamma_i x_i\| = L_i'$. The bivalent vertices of Δ' at the origin are obtained from those of Δ by changing the lengths of the rays which does not change the congruence class of the vertices. Hence, those vertices have the desired congruence classes. Also, the lengths of the edges of Δ' which meet at the origin match the $\{L_i'\}$. Thus, by the results on S'' in Theorem 3.1, the triangles of $\mathfrak{S}'$ are congruent to the faces of Δ', establishing (2). To show that $(3) \implies (1)$, first use an automorphism to place the trivalent vertex $\mathbf{V}$ at the origin with its rays in the directions of the rays of Δ of (3.2). Next, select a point on each ray whose distance from the origin is the appropriate L_i. The triangles with vertex at the origin are, again by Theorem 3.1, congruent to the triangles of $\mathfrak{S}$ and hence the origin together with those three new points are the vertices of the tetrahedron required to show that (1) holds.

For uniqueness, first consider two trivalent vertices $\mathbf{W}$ and $\mathbf{W}'$ which satisfy condition (3). Pick a small ε and pick a point on each ray at distance ε from the vertex. Let Σ be the tetrahedron determined by those four points, and let Σ' be the similar tetrahedron constructed using $\mathbf{W}'$. We will be done if we show that Σ and Σ' are congruent. The argument which shows they are congruent also gives the uniqueness in statements (1) and (2). First, note that the results on S'' in Theorem 3.1 insures that the triangular faces of Σ meeting at the vertex point $\mathbf{W}$ are congruent to those in Σ'

meeting at the vertex point of $\mathbf{W}'$. By condition (5) of Theorem 3.2, this is enough to show that the tetrahedra are congruent. ∎

In particular, note that in both the real and complex cases, the congruence class is completely described by the shapes of three faces and the geometry of the vertex where they meet. There need not be any mention of the geometry of the fourth face.

The analogous congruence result for multivalent vertices in $\mathbb{CH}^n$ is given in Corollary 5.3.

4.4 *Kos* **and area**

There is not a natural notion of area for a general triangle T in $\mathbb{CH}^m$; however, there is a related invariant, the symplectic area of T, obtained by integrating the symplectic form of $\mathbb{CH}^n$ over a real two manifold bounded by the sides of T.

However, if T sits inside a complex geodesic $A \subset \mathbb{CH}^m$, then we can define and compute its area as follows. After an automorphism, we can suppose T is inside a copy of $\mathbb{CH}^1$ inside $\mathbb{CH}^n$. That copy of $\mathbb{CH}^1$ is isometric to the classical Poincare disk of curvature -1. We define the area of T, Area(T), to br the area of that copy of T in the Poincare disk. This definition is an automorphism invariant, and it can be shown that Area(T) equals both the symplectic area of T and is also twice the angular invariant, α, of T (defined in Section 2.3) [Go, HM].

We can also evaluate Area(T) using kos .

Proposition 4.7   *If $T = \{x_1, x_2, x_3\}$ is a triangle in $\mathbb{CH}^n$ which sits inside a complex geodesic, then*

$$(4.6) \qquad \pi - \arg\left(\mathrm{kos}_1(2,3)\,\mathrm{kos}_2(3,1)\,\mathrm{kos}_3(1,2)\right) = 2\alpha(x_1, x_2, x_3) = \mathrm{Area}(T).$$

Proof   After an automorphism of $\mathbb{CH}^n$, we suppose that T is in the unit disk of $\mathbb{C}$ which we identify with $\mathbb{CH}^1$. From (2.13), we see that $\mathrm{kos}_1(2,3)$ is a positive multiple of

$$\kappa_{123} = \langle\langle \phi_{x_1}(x_2), \phi_{x_1}(x_3) \rangle\rangle .$$

Hence, in computing the left-hand side, *LHS*, of (4.6), we can replace kos$(2,3)$ with κ_{123} and similarly for other indices. The conformal involutions of the disk are given by Blaschke factors. Hence, noting that for a, b in the disk $\langle\langle a, b \rangle\rangle = a\bar{b}$, we find that $\kappa_{123} = \frac{x_1 - x_2}{1 - x_1\overline{x_2}} \frac{\overline{x_1 - x_3}}{1 - x_1\overline{x_3}}$. Thus,

$$LHS = \pi - \arg \frac{-\Pi\,|x_i - x_j|^2}{\Pi\left(1 - x_i\overline{x_j}\right)^2}$$

with both products over the index pairs $\{(1,2),(2,3),(3,1)\}$. The positive factor $\Pi\,|x_i - x_j|^2$ does not affect the value of arg, and hence we continue with

$$LHS = \pi - \left(\pi + 2\arg \Pi k(x_i, x_j)\right) = 2\alpha,$$

the last equality by (2.10).

To finish, we need to know that $2\alpha = \mathrm{Area}(T)$. That is in [Go]. Alternatively, going back to (4.6), let γ_i be the angle at vertex x_i. Because $T \subset \mathbb{CH}^1$, we see from Proposition 4.1 that $\mathrm{kos}_1(2,3) = e^{i\gamma_1}$ where γ_1 is the angle at x_1 of the triangle T.

Similarly for the other indices. Using that, we see that *LHS* in (4.6) equals $\pi - (\gamma_1 + \gamma_2 + \gamma_3)$, which equals Area($T$) by the classical result based on the Gauss–Bonnet theorem.

In contrast to T, consider now a triangle R that sits inside a totally real submanifold M. (If M exists, it can be taken to have real dimension 2.) The previous discussion does not apply, and, in fact, the symplectic area of R is 0. However, we have the following observation. We can suppose M is two-dimensional, and using a conformal automorphism, we can move M to $RB_2 = \{(x, y, 0, ..., 0) \in \mathbb{B}_n : x, y \in \mathbb{R}\}$ inside $\mathbb{CH}^n = \mathbb{B}_n$. That subspace of complex hyperbolic space is, isometrically, a copy of the Beltrami–Klein model of $\mathbb{RH}^2$ which has constant curvature $-1/4$. As such, it carries a natural area measure, and we let $Area(R)$ denote the area of R as a triangle in that space. This quantity is also invariant under conformal automorphisms of $\mathbb{CH}^n$. ∎

Proposition 4.8 *Suppose $R = \{x_1, x_2, x_3\}$ sits in a totally real totally geodesic submanifold of $\mathbb{CH}^n$. Then*

$$4(\pi - (\cos^{-1}\mathrm{kos}_1(2,3) + \cos^{-1}\mathrm{kos}_3(1,2) + \cos^{-1}\mathrm{kos}_2(3,1))) = Area(R).$$

Proof Without loss of generality, $M = RB_2$. The equality is a consequence of two facts. First, taking note of (2.14), $\cos^{-1}\mathrm{kos}(2,3)$ is the angle between the geodesics x_1x_2 and x_1x_3, and similarly for the other indices. Second, by the Gauss–Bonnet theorem, the area of a triangle with angles α, β, γ in a plane of constant curvature $-1/4$ is $4(\pi - (\alpha + \beta + \gamma))$. ∎

One proposition gives a result involving a sum of values of $\arg\mathrm{kos}$ and the other a sum of values of $\cos^{-1}\mathrm{kos}$. It would be interesting to have a general result which unifies the two.

5 Finite sets in $\mathbb{CH}^n$

5.1 Describing sets by their triangles

In Theorem 3.2, we saw that the congruence class of a finite $X \subset \mathbb{CH}^n$ is determined by the congruence classes of those of its subtriangles which share a specified designated vertex. From Theorem 3.1, we know that the congruence class of each of those triangles can be described using side lengths and the angular invariant. Those parameters have been used in describing the congruence class of finite $X \subset \mathbb{CH}^n$ [BE, CG, G, HS, Ro] and have also been used for more general geometric questions, [C, CG]. When those parameters are restricted to sets in real hyperbolic space, the angular invariant trivializes leading to descriptions of polyhedra in $\mathbb{RH}^n$ in terms of edge lengths.

Here, instead of side lengths and the angular invariant, we use side lengths and kos to describe the constituent triangles of X. This alternative description emphasizes a different type of geometric data. For instance, when restricted to sets in real hyperbolic space, it produces a description of polyhedra using side lengths and vertex angles. The parameters only have natural geometric constraints, and their values interact well with passage to subsets. That lets us give explicit answers to the questions in the introduction.

Suppose we are given $X = \{x_i\}_{i=1}^{n+1} \subset \mathbb{CH}^n$. If we connect x_1 to each of the other x_i by a geodesic γ_{1i}, then the point x_1 will be the vertex point of an n-valent vertex $\mathbf{V}$ which is composed of the bivalent vertices $\mathbf{V}_{ij}$, $2 \le i, j \le n+1$ having γ_{1i} as a first ray and γ_{1j} as a second. We can describe X using the distances between x_1 and the other x_j and the numbers $K_{ij} = \text{kos}(\mathbf{V}_{ij})$. Specifically, recalling the definition (2.15), we define the n-vector $\rho(X)$ and the $n \times n$ matrix $\mathcal{M}(X)$ by

$$(5.1) \qquad \rho(X) = (\delta(x_1, x_2), ..., \delta(x_1, x_{n+1})), \text{ and}$$

$$(5.2) \qquad \mathcal{M}(X) = \left(K_{ij}\right)_{i,j=2}^{n+1} = \left(\text{kos}\left(\mathbf{V}_{ij}\right)\right)_{i,j=2}^{n+1} = \text{KOS}(DA(X), 1).$$

The functionals that give the entries of the vector ρ and the matrix $\mathcal{M}$ are defined for any $H \in \mathcal{RK}$, and hence those same definitions can be used to produce $\rho(H)$ and $\mathcal{M}(H)$.

The following result parametrizes congruence classes of finite sets in $\mathbb{CH}^n$ as well as rescaling equivalence classes of $\mathcal{RK}$ with the CPP.

Theorem 5.1 (1) *Given $X = \{x_i\}_{i=1}^{n+1} \subset \mathbb{CH}^n$, each entry of $\rho(X)$ is between 0 and 1, and $\mathcal{M}(X)$ is a positive-semidefinite matrix with 1's on the diagonal.*
(2) *Conversely, given such a ρ and $\mathcal{M}$, there is an X so that $\rho = \rho(X)$ and $\mathcal{M} = \mathcal{M}(X)$.*
(3) *Given $Y \subset \mathbb{CH}^n$, $X \sim Y$ if and only if $\rho(X) = \rho(Y)$ and the matrices $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ are unitarily equivalent.*
(4) *If $H \in \mathcal{RK}$, then H has the CPP if and only if $\rho(H)$ and $\mathcal{M}(H)$ satisfy the conditions in (1). If that happens, then $H \sim DA(X)$ with X given by condition (2).*

Proof In (1), the claim for $\rho(X)$ is clear. The matrix $\mathcal{M}(X)$ is invariant under automorphisms of $\mathbb{CH}^n$, and hence we can suppose x_1 is at the origin. Let $\widehat{X}$ be set of radial projections of the remaining points onto the unit sphere, $\widehat{X} = \{\widehat{x_2}, ..., \widehat{x_{n+1}}\} \subset \partial \mathbb{B}_n$. We then see from (2.14) that $K_{ij} = \langle \langle \widehat{x_i}, \widehat{x_j} \rangle \rangle$ for $2 \le i, j \le n+1$. Thus, $\mathcal{M}(X)$ is the Gram matrix of the set of vectors $\widehat{X}$ and hence is positive semidefinite. For (2), a matrix with the properties of $\mathcal{M}$ must be the Gram matrix of a set $W = \{w_i\}_{i=2}^{n+1} \subset \mathbb{C}^n$, unique up to unitary equivalence. The 1's on the diagonal of $\mathcal{M}$ insure that $W \subset \partial \mathbb{B}_n$. We now form X by designating the origin as x_1 and for $2 \le i \le n+1$ picking x_i on the line segment $[0, w_i]$ with $|x_i| = \delta(x_1.x_i)$. It is straightforward that X has the required properties.

For (3), first suppose Y is congruent to X; that is, Y is the image of X under an automorphism of the ball. The entries of ρ and $\mathcal{M}$ are automorphism invariants, and this gives the desired equalities. In the other direction, suppose the data associated with X equal the data associated with Y. Without loss of generality, we can suppose x_1 is at the origin in which case $\mathcal{M}(X)$ is the Gram matrix of the set $\widehat{X} \subset \partial \mathbb{B}_n$. Similarly for Y and $\widehat{Y}$. Thus, $\widehat{X}$ and $\widehat{Y}$ have the same Gram matrix, and hence there is a unitary map $\mathbb{C}^n$ which takes $\widehat{X}$ to $\widehat{Y}$. That unitary is also an automorphism $\mathbb{CH}^n$ and so takes each segment connecting the origin to a point of $\widehat{X}$ to a segment connecting the origin to a point of $\widehat{Y}$. Given the further assumption that $\rho(X) = \rho(Y)$, it must take X to Y, as required.

The final statement follows from the first three together with the fact that $\rho(H)$ and $\mathcal{M}(H)$ determine H up to rescaling. ∎

Thus the congruence class of a set X of k ($= n + 1$) points in $\mathbb{CH}^m$, $m \geq k - 1$, is determined by the $k - 1$ real numbers in $\rho(X)$ together with the $(k-1)(k-2)/2$ complex numbers which specify, $\mathcal{M}(X)$ a positive semidefinite matrix with ones on the diagonal. Together these give $(k-1)^2$ real parameters, as expected. We should emphasize that $\rho(X)$ and $\mathcal{M}(X)$ only depend on the congruence class of *ordered* set X. The choice of which element is placed at the origin by an automorphism substantially affects all the values in $\rho(X)$ and $\mathcal{M}(X)$. Once that point is specified, the remaining ordering only affects the ordering of the entries in $\rho(X)$ and $\mathcal{M}(X)$.

Details of $\mathcal{M}(X)$ contain information about the geometry of X. From the previous theorem and proof, we have the following corollary.

Corollary 5.2 *If the rank of $\mathcal{M}(X)$ is m, then X is congruent to a set in $\mathbb{CH}^m$ but is not congruent to a set in $\mathbb{CH}^j$ for any $j < m$. In particular, if the rank is 1, then X is contained in a complex geodesic.*

Here are the details of $\rho(X)$ and $\mathcal{M}(X)$ in some simple cases. Suppose $n + 1 = 3$. From Theorem 3.1, we know that the congruence class of the triangle $X = \{x_1, x_2, x_3\}$ is described by the set $S'' = (\delta_{12}, \delta_{13}, \mathrm{kos}_1(2, 3))$. In the notation of the previous theorem,

$$\rho(X) = (\delta_{12}, \delta_{13}), \quad \mathcal{M}(X) = \begin{pmatrix} 1 & \mathrm{kos}_1(2,3) \\ \mathrm{kos}_1(3,2) & 1 \end{pmatrix}.$$

In this case, the condition $\mathcal{M}(X) \succcurlyeq 0$ is equivalent to $|\mathrm{kos}_1(2,3)| \leq 1$, which is the condition (3.5) in Theorem 3.1.

The case of $n + 1 = 4$ points is discussed in Section 6.

For $X \subset \mathbb{CH}^1$, we use the complex coordinate of $\mathbb{C}^1$ and write $X = \left\{ r_s e^{i\theta_s} \right\}_{s=1}^{n+1}$ with $r_1 = 0$. Using the computations in Section 4.1, we see that

$$\rho(X) = (r_2, ..., r_{n+1}), \quad \mathcal{M}(X) = \left(\exp i \left(\theta_s - \theta_t \right) \right)_{s,t=2}^{n+1}.$$

The automorphisms of $\mathbb{CH}^1$ which fix the base point are rotations. They do not change the entries in $\mathcal{M}(X)$ or the congruence class of X. On the other hand, complex conjugation, which is not in $\mathrm{Aut}\,\mathbb{B}_1$, can change the matrix entries and the congruence class.

For $Y \subset RB_2$, the Beltrami–Klein model of $\mathbb{RH}^2$, we use the polar coordinates of the containing $\mathbb{R}^2$. We have $Y = \{(r_s, \theta_s)\}_{s=1}^{n+1}$ with $r_1 = 0$. Now, using Section 4.1 gives

$$\rho(Y) = (r_2, ..., r_{n+1}), \quad \mathcal{M}(Y) = \left(\cos \left(\theta_s - \theta_t \right) \right)_{s,t=2}^{n+1}.$$

In this case, the map $(r, \theta) \to (r, -\theta)$, which looks like complex conjugation, is the restriction of an element of $\mathrm{Aut}\,\mathbb{B}_2$ to RB_2, namely, the map $(z, w) \to (z, -w)$. That map changes the sign of the $\theta's$, but that does not change $\mathcal{M}(Y)$ or the congruence class of Y.

The previous theorem also gives the extension to n-valent vertices of the results in Section 4.3 for bivalent and trivalent vertices. Suppose $\mathbf{V}$ is an n-valent vertex in $\mathbb{CH}^n$ with vertex point x_1 and rays $\{\gamma_{1i}\}_{i=2}^{n+1}$. We are only interested in congruence classes, and hence we suppose x_1 is at the origin. For $i = 2, ..., n + 1$, select a point x_i

on γ_{1i} and set $X = \{x_i\}_{i=1}^{n+1}$. From the previous theorem, we know the congruence class of X is determined by $\mathcal{M}(X)$ and $\rho(X)$. Looking at that proof, we see that knowing $\mathcal{M}(X)$ is equivalent to knowing the congruence class of the projected set $\widehat{X}$. From the definitions, we see that knowing $\widehat{X}$ is equivalent to knowing $\mathbf{V}$. Hence, defining $\mathcal{M}(\mathbf{V})$ to be $\mathcal{M}(X)$, we have the following corollary of the previous theorem.

Corollary 5.3 (1) *Given an n-valent vertex $\mathbf{V}$ in $\mathbb{CH}^n$, $\mathcal{M}(\mathbf{V})$ has 1's on the diagonal and $\mathcal{M}(\mathbf{V}) \geqslant 0$.*

(2) *Given $\mathcal{N}$ which satisfies those conditions, there is a $\mathbf{V}$ in $\mathbb{CH}^n$ with $\mathcal{M}(\mathbf{V}) = \mathcal{N}$.*

(3) *Two such vertices are congruent if and only if they give the same matrix $\mathcal{M}$.*

(4) *Given $\{z_i\}_{i=2}^{n+1} \subset \partial \mathbb{B}_n$ there is a $\mathbf{V}$ in $\mathbb{CH}^n$ with $\mathcal{M}(\mathbf{V})$ equal to the Gram matrix of $\{z_i\}$.*

5.2 Comparison with the McCullough–Quiggin theorem

Theorem 5.1 has some similarity to the McCullough–Quiggin theorem. Here is brief informal discussion of that relation; more details are in Chapters 7 and 8 of [AM] and the Historical Notes to those chapters.

Recall that we are only considering finite-dimensional spaces.

The first observation is that our "definition of convenience," that H has the CPP if and only if it is a rescaling of a space $DA(X)$, is actually inconvenient in this context. We need to distinguish between that definition and the actual complete Pick property (ACPP) defined in terms of extension properties of certain matrix multipliers on H. That property is defined and discussed in Chapter 5 of [AM].

Given H and $1 \leq s \leq \dim H$, we can form the matrices $KOS(H, s)$ as described in (2.15).

Consider now the following four conditions:

(1) H has the ACPP.

(2) H has the CPP.

(3) For each s, $1 \leq s \leq \dim H$, $KOS(H, s) \geqslant 0$.

(4) For some s, $1 \leq s \leq \dim H$, $KOS(H, s) \geqslant 0$.

We saw that if $H = DA(X)$, then $KOS(H, s)$ is a Gram matrix; hence, statement 2 implies statements 3 and 4.

The result of McCullough and Quiggin, Theorem 7.6 in [AM], is that statements 1 and 3 are equivalent. (Their result actually uses an equivalent formulation based on matrices $MQ(H, s)$ defined at (2.16). Their result also involves matrices of all sizes, but in this finite-dimensional case, it suffices to only consider the matrices of maximum size.) It is a result of Agler and McCarthy, Theorem 8.2 in [AM], that conditions 1 and 2 are equivalent. Combining these two results, we see that 3 implies 2.

On the other hand, by Theorem 5.1, if condition 4 holds, then 2 holds; H is a rescaling of some $DA(X)$. Considering the details of the proof of Theorem 5.1, this is a simpler path to that conclusion than the path through condition 1. Also, it only requires information about a single matrix, condition 4 rather than 3. However, this path says nothing about the relationship between conditions 1 and 2, which is one of the centerpieces of the theory of spaces with the ACPP.

5.3 Assembly questions

We now return to the assembly and coherence question we discussed in Section 3.1. We will consider questions of congruence involving subsets of sets in $\mathbb{CH}^n$; however, recall, as we commented earlier, that these are equivalent to questions about rescaling of regular subspaces of spaces in $\mathcal{CPP}$.

The generalized version of Question 1 from the Introduction is: given a finite collection of finite sets $\{Y_i\}$ in some $\mathbb{CH}^n$, is there an X in some $\mathbb{CH}^m$ which contains congruent copies of the various Y_i? We may also impose requirements on the overlap of the copies of the $\{Y_i\}$ inside of X. In Section 3.1, we saw that meeting such overlap conditions may require congruence of certain subsets of the various Y_i. If all those congruence constraints are satisfied, we write $\{Y_i\} \Rightarrow ??$ and if there is such an X, we write $\{Y_i\} \Rightarrow X$.

For $X \subset \mathbb{CH}^n$ with distinguished base point x_1, recall that we write $\mathrm{KOS}(X,1)$ as a shorthand for $\mathrm{KOS}(DA(X),1)$. For each i, let 1_i be that point in Y_i which is mapped to the base point x_1 of X under the assumed congruences, with 1_i an arbitrary point of Y_1 if that definition is unfilled.

If $\{Y_i\} \Rightarrow X$, then, as we noted in Section 3.1, we can use data from the matrices $\mathrm{KOS}(Y_i,1_i)$ to compute some of the entries $\mathrm{KOS}(X,1)$ and those computations can be done even without knowing if there is an X. The computations produce a (possibly only partially filled) matrix, which we denote $\mathrm{KOS}(\{Y_i\},1)$,

We now study the possibility that there is an X by comparing $\mathrm{KOS}(\{Y_i\},1)$ with the properties which we know from Theorem 5.1 that $\mathrm{KOS}(X,1)$ must have. More specifically, from our previous analysis culminating in Theorem 5.1, we know that the congruence class of X is described by a vector of side lengths, $\rho(X)$, and the matrix $\mathcal{M}(X) = \mathrm{KOS}(X,1)$. We extract information from $\mathrm{KOS}(X,1)$ by working with the principal submatrices. From the definitions, we have the following lemma.

Lemma 5.4 *If $H \in \mathcal{RK}$, then the principal submatrices, $\mathcal{PS}(\mathrm{KOS}(H,1))$, are the matrices $\mathrm{KOS}(J,1)$ for J which are regular subspaces of H which contain the kernel function k_1. In particular, if $H = DA(X)$, then they are the matrices $\mathrm{KOS}(Y,1)$ for each Y which is a subset of X which contains the distinguished point.*

The following classical result will let us relate the fact that $\mathcal{M}(X) \succcurlyeq 0$ to properties of its principal submatrices $\mathcal{PS}(\mathcal{M}(X))$.

Lemma 5.5 (Sylvester's criterion) *An $n \times n$ matrix A satisfies $A \succcurlyeq 0$ if and only if $\det B \geq 0$ for all $B \in \mathcal{PS}(A)$, that is, if and only if the principal minors of A are nonnegative. $A > 0$ if and only if the leading principal minors are positive.*

Now, we look at three special cases, the case where the $\{Y_i\}$ are three point sets whose congruent images fill an $n + 1$ point set, the question of specifying the geometry of the n point subsets of a set of $n + 1$ points, and after those general questions, we consider specific ad hoc variation chosen to show how these ideas work in a more complicated situation.

In the next section, we use these ideas for a systematic study of the question in the introduction of assembling four triangles into a tetrahedron.

5.3.1 Variation 1

We know from Theorem 3.2 that the congruence class of a set X is determined by the congruence classes of the triangles in X which contain a specified base point. We now ask if a given set of triangles can be congruent to those faces of some X. We want to know if we can map triangles $\{Y_i\}_{i=1}^n$ in $\mathbb{CH}^n$,

$$Y_j = \left\{ y_{j1}, y_{j,j+1}, y_{j,j+2} \right\}, \quad j < n,$$
$$Y_n = \left\{ y_{n1}, y_{n,n+1}, y_{n2} \right\},$$

into a set $X = \{x_1, ..., x_{n+1}\}$ with each y_{jk} is mapped to x_k. That is, each image has its first vertex at x_1 and the images fill X with each segment $x_1 x_t$ in X covered twice. The coherence conditions are that two triangle sides that cover the same $x_1 x_t$ must be the same length.

Recall that the matrix $\mathrm{KOS}(\{Y_i\}, 1)$ is defined in (2.15).

Theorem 5.6   *Given the coherence data $\{Y_i\} \Rightarrow ??$ just described, the following are equivalent:*

(1) $\exists X, \{Y_i\} \Rightarrow X$.
(2) $\mathrm{KOS}(\{Y_i\}, 1) \succcurlyeq 0$.
(3) $\forall A \in \mathcal{PS}(\mathrm{KOS}(\{Y_i\}, 1)), \det A \geq 0$.
(4) $\forall S \subset \{1, ..., n\}, 1 \in S, \det \mathrm{KOS}(\{Y_i\}_{y \in S}, 1) \geq 0$.

Proof   This is a direct consequence of Theorem 5.1 and Lemmas 5.4 and 5.5.   ∎

Taking note of Corollary 3.3, we also have the same result for spaces $J_i \in \mathcal{CPP}$ and the question of moving from $\{J_i\} \Rightarrow ??$ to $\{J_i\} \Rightarrow H$ with $H \in \mathcal{CPP}$.

For both $\{Y_i\}$ and $\{J_i\}$, the condition $\det A \geq 0$ is automatic for the A that are 1×1 principal submatrices and the 2×2 case is insured by Theorem 3.1. For A that are 3×3, the situation is more complicated. For instance, if we suppose $\{J_i\} \subset \mathcal{RK}$ but not necessarily in $\mathcal{CPP}$, then $\det A \geq 0$ is not automatic; see the comment after Theorem 6.4.

The case $n = 3$ of this result is the tetrahedron assembly question of the introduction. We discuss it in more detail in the next section.

5.3.2 Variation 2

Suppose we are given $\{Y_i\}_{i=1}^n$, sets of size n in $\mathbb{CH}^n$. Write $Y_i = \left\{ y_{ij} : 1 \leq j \leq n+1, j \neq i+1 \right\}$. We impose the coherence conditions $\{Y_i\} \Rightarrow ??$ that would hold if there were a set

$$X = \{x_1, ..., x_{n+1}\}$$

and congruences $Y_i \rightsquigarrow X$ which mapped the points y_{is} to x_s, for all $s \neq i$. Informally, we are trying to specify the congruence type of all the subsets of size n inside a set of size $n + 1$. This coherence condition $\{Y_i\} \Rightarrow ??$ requires strong interrelations between the Y_i. Given Y_r and Y_s, there are $Y_{rs} \subset Y_r$ and $Y_{sr} \subset Y_s$ both of size $n - 1$ with $Y_{rs} \sim Y_{sr}$. Also, note that every $A \in \mathcal{PS}(\mathrm{KOS}(\{Y_i\}, 1))$ which is not maximal, $A \neq \mathrm{KOS}(\{Y_i\}, 1)$, satisfies $A \in \mathcal{PS}(\mathrm{KOS}(Y_r, 1))$ for some individual

Y_r. We know $\mathrm{KOS}(Y_r, 1) \succcurlyeq 0$, and hence, by Sylvester's criterion, $\det A \geq 0$. In sum, the only $A \in \mathcal{PS}(\mathrm{KOS}(\{Y_i\}, 1))$ for which we do not know $\det A \geq 0$ is the matrix $\mathrm{KOS}(\{Y_i\}, 1)$ itself. This discussion, together with Theorem 5.1, and the previous two lemmas, complete the proof of the following.

Theorem 5.7 *Given $\{Y_i\} \Rightarrow ??$, there is an X so that $\{Y_i\} \Rightarrow X$ if and only if* $\det \mathrm{KOS}(\{Y_i\}, 1) \geq 0$.

5.3.3 Variation 3

We just looked at cases where the coherence requirements on the $\{Y_i\}$ were minimal and maximal. We now look at an intermediate case which is rich enough to display some structure and simple enough for explicit computations.

Suppose we are given two four-point sets in $\mathbb{CH}^n$, $Y_A = \{a_1, a_2, a_3, a_4\}$ and $Y_B = \{b_1, b_3, b_4, b_5\}$. The coherence requirements, $\{Y_A, Y_B\} \Rightarrow ??$, we impose are the congruences that would hold if we had maps $Y_A, Y_B \rightsquigarrow X = \{x_1, ..., x_5\}$ which respect the subscripts of the points. If that holds, then the triangles $A = \{a_1, a_3, a_4\}$ and $B = \{b_1, b_3, b_4\}$ are congruent, and that congruence is the only coherence requirement.

We should not expect to fill the matrix $\mathrm{KOS}(\{Y_A, Y_B\}, 1)$. The set X has five points and so is determined by $(5-1)^2 = 16$ real parameters. On the other hand, each of Y's provides nine parameters, but four of those are pinned by the fact that triangles A and B are congruent, leaving 14. This suggests that our description is two real or one complex parameter short of being able to fully describe X. In fact, we cannot construct the entry $\mathrm{kos}_1(2, 5)$ in the matrix $\mathrm{KOS}(\{Y_A, Y_B\}, 1)$ because neither the image of Y_A nor of Y_B contain $\{x_1, x_2, x_5\}$. To move forward, we introduce a parameter z and fill the matrix $\mathrm{KOS}(\{Y_A, Y_B\}, 1)$ to a matrix $\mathcal{Y} = \mathrm{KOS}(\{Y_A, Y_B, z\}, 1)$ obtained from $\mathrm{KOS}(\{Y_A, Y_B\}, 1)$ by putting z in the place where the $\mathrm{kos}_1(2, 5)$ entry would be, and $\bar{z}$ where $\mathrm{kos}_1(5, 2)$ would be. The values of z for which $\mathcal{Y} = \mathrm{KOS}(\{Y_A, Y_B, z\}, 1) \succcurlyeq 0$, if any, will parameterize inequivalent possible constructions of the desired X.

We need to study the determinants of the matrices in $\mathcal{PS}(\mathcal{Y})$. $\mathcal{Y}$ is a 4×4 matrix with rows and columns indexed by the set $\{2, 3, 4, 5\}$. The matrices in $\mathcal{PS}(\mathcal{Y})$ are determined by the 15 nonempty subsets of that index set. We denote those matrices by $\mathcal{Y}$ with subscripts denoting the rows, and hence also columns, of $\mathcal{Y}$ that are retained. There are four single element subsets to consider, for each of them the resulting matrix has the single entry 1 and hence a positive determinant. There are six possibilities with two subscripts. The matrix $\mathcal{Y}_{34}$ will be a submatrix of both $\mathrm{KOS}(Y_A, 1)$ and $\mathrm{KOS}(Y_B, 1)$ and hence, by Sylvester's criterion, will have a positive determinant. The matrix $\mathcal{Y}_{23}$ is not a submatrix of $\mathrm{KOS}(Y_B, 1)$, but it is a submatrix of $\mathrm{KOS}(Y_A, 1)$ and that is enough to insure it has a positive determinant. The same holds for $\mathcal{Y}_{24}$, and a similar argument applies $\mathcal{Y}_{35}$ and $\mathcal{Y}_{45}$ but with the roles of A and B reversed. The remaining matrix of that size is $\mathcal{Y}_{25}$; it cannot be studied using either Y_A or Y_B. It is a 2×2 matrix with 1's on the diagonal and z and $\bar{z}$ as off diagonal elements, and it must be dealt with separately. If we do not know about $\det \mathcal{Y}_{25}$, then we also cannot know the positive semidefinite nature of the matrices of which it as a submatrix, and hence those matrices must also be studied separately. They are $\mathcal{Y}_{253}$, $\mathcal{Y}_{254}$, and $\mathcal{Y}_{2534} = \mathcal{Y}$. The

two remaining submatrices are $\mathcal{Y}_{234} = \mathrm{KOS}(Y_A, 1)$ and $\mathcal{Y}_{345} = \mathrm{KOS}(Y_B, 1)$, which we know are positive semidefinite.

Theorem 5.8   *There is an X so that $\{Y_A, Y_B\} \Rightarrow X$ if and only if there is a z so that $\mathcal{Y} = \mathrm{KOS}(\{Y_A, Y_B, z\}, 1) \succcurlyeq 0$, equivalently if and only if $\mathcal{Y}$ and the submatrices $\mathcal{Y}_{25}$, $\mathcal{Y}_{253}$, and $\mathcal{Y}_{254}$ have positive determinants. The condition on $\mathcal{Y}_{25}$ holds exactly if $|z| \leq 1$; the other two determinants can be studied using* (6.1).

6   Tetrahedra

We now make the earlier conditions more explicit for four point sets and answer Questions 1 and 2 of the introduction. We also use the results to analyze a family of four-dimensional $\mathcal{RK}$ introduced by Quiggin.

6.1   Question 1

We are given $\{T_i\}_1^4$ a set of four triangles in $\mathbb{CH}^n$ which satisfy the coherence conditions for assembly into a tetrahedron, $\{T_i\}_{i=2}^4 \Rightarrow ??$, and we want to know if they can, in fact, be assembled into a tetrahedron $X \subset \mathbb{CH}^n$, $\{T_i\}_{i=2}^4 \Rightarrow X$. That can be done if and only if the parameter values imputed to X from the details of the $\{T_i\}$ and the coherence conditions describe a possible tetrahedron. We begin by reviewing those parameters.

Any tetrahedron in $\mathbb{CH}^n$ is congruent to one of the forms $X = \{0, x_2, x_3, x_4\} \subset \mathbb{CH}^2$. Here, we identify $\mathbb{CH}^2$ with the unit ball in $\mathbb{C}^2$ and also regard the $\{x_i\}$ as points in that space. Let $\mathbf{V}_{ij}$ be the bivalent vertex $x_i 0 x_j$. From Theorem 5.1, we see that we must have each $|x_i| < 1$ and that the matrix

$$\mathcal{M} = \mathcal{M}(X) = \left(\mathrm{kos}\left(\mathbf{V}_{ij}\right)\right)_{i,j=2}^4 = \left(\langle\langle \widehat{x_i}, \widehat{x_j}\rangle\rangle\right)_{i,j=2}^4$$

must be positive semidefinite. We will make that last condition more explicit.

Lemma 6.1   *Given $\{a_i\}_{i=1}^3 \subset \mathbb{C}$ with each $|a_i| \leq 1$, set*

$$\mathcal{N} = \begin{pmatrix} 1 & a_1 & a_2 \\ \overline{a_1} & 1 & a_3 \\ \overline{a_2} & \overline{a_3} & 1 \end{pmatrix}.$$

The following are equivalent:

(1) $0 \preccurlyeq \mathcal{N}$.
(2) $0 \leq \det \mathcal{N}$.
(3) $0 \leq 1 + 2 \operatorname{Re} a_1 \overline{a_2} a_3 - |a_1|^2 - |a_2|^2 - |a_3|^2$.
(4) $|a_1 \overline{a_2} - a_3|^2 \leq (1 - |a_1|^2)(1 - |a_2|^2)$.

Proof   By Sylvester's criterion, the first condition implies the second. The second and third conditions are equivalent by definition. That the third and fourth are equivalent can be seen by expanding both sides of the fourth statement giving

$$|a_1 \overline{a_2}|^2 - 2 \operatorname{Re} a_1 \overline{a_2} a_3 + |a_3|^2 \leq 1 - |a_1|^2 - |a_2|^2 + |a_1 a_2|^2.$$

Cancellation and rearrangement shows that is equivalent to the third statement.

To go in the other direction, we again use Sylvester's criterion and show the second statement implies the first. That states that the first statement is a consequence of the nonnegativity of the seven principal minors. Three are the determinants of the 1×1 matrices given by the diagonal entries and they are positive. The determinants of the three 2×2 submatrices are accounted for by the assumption on the size of the a_i. Finally, the positivity of $\det \mathcal{N}$ is the second statement. ∎

We want to know if a set of four triangles in $\mathbb{CH}^n$, $\{T_i\}_1^4$ which satisfy $\{T_i\}_{i=1}^4 \rightrightarrows$??, might satisfy $\{T_i\}_{i=1}^4 \rightrightarrows X$ for some tetrahedron $X = \{0, x_2, x_3, x_4\}$. By congruence invariance, we may suppose that for $i = 2, 3, 4$, the triangle T_i has vertices $\{0, x_{i1}, x_{i2}\}$ for various x_{ij} of length at most one. The coherence requirements on the set of triangles insure that $|x_{22}| = |x_{31}|$, $|x_{32}| = |x_{41}|$, and $|x_{42}| = |x_{21}|$. If the assembly of the triangles into an X is possible, these three quantities will be the side lengths $|x_2|$, $|x_3|$, and $|x_4|$, of X, and those lengths, the values of the ρ of Theorem 5.1, can be any numbers between zero and one. Hence, it only remains to check if the values of $\mathrm{kos}(\mathbf{V}_{ij})$ imputed from the triangles lead to a matrix $\mathcal{M}$ with the required properties. If assemble is possible, then the vertex $\mathbf{V}_{23}$ of X will be (congruent to) the vertex of T_2 at the origin, similarly for $\mathbf{V}_{34}$ and T_3, $\mathbf{V}_{42}$ and T_4. Thus, $\mathcal{M}(\{T_i\})$, the imputed value of $\mathcal{M}(X)$, is given by

$$(6.1) \qquad \mathcal{M}(\{T_i\}) = \begin{pmatrix} 1 & \mathrm{kos}(T_2) & \overline{\mathrm{kos}(T_4)} \\ \overline{\mathrm{kos}(T_2)} & 1 & \mathrm{kos}(T_3) \\ \mathrm{kos}(T_4) & \overline{\mathrm{kos}(T_3)} & 1 \end{pmatrix} = \begin{pmatrix} 1 & K_{23} & K_{24} \\ K_{32} & 1 & K_{34} \\ K_{42} & K_{43} & 1 \end{pmatrix},$$

where the last equality defines the matrix (K_{ij}).

We have collected all of the pieces to answer Question 1.

Theorem 6.2 *With the numbering and naming scheme just described:*

(1) *There is an X such that $\{T_i\}_{i=1}^4 \rightrightarrows X$ if and only if there is an X such that $\{T_i\}_{i=2}^4 \rightrightarrows X$.*
(2) *If $\{T_i\} \rightrightarrows X$, then $\mathcal{M}(\{T_i\}) = \mathcal{M}(X) \succcurlyeq 0$.*
(3) *If $\mathcal{M}(\{T_i\}) \succcurlyeq 0$, then there is an X with $\{T_i\} \rightrightarrows X$.*
(4) *$\mathcal{M}(\{T_i\}) \succcurlyeq 0$ if and only if $\det \mathcal{M}(X) \geq 0$, equivalently if and only if*

$$(6.2) \qquad \left| K_{34} - \overline{K_{23}} K_{24} \right|^2 \leq (1 - |K_{23}|^2)(1 - |K_{24}|^2).$$

Proof The first statement is a consequence of the observation that our computation of the imputed values of $\rho(X)$ and $\mathcal{M}(X)$ never used data from T_1. The next two statements follow from the discussion before the theorem together with Theorem 5.1. The final statement follows from the previous lemma together with statement (6) of Theorem 3.1, which insures that each $|K_{ij}| \leq 1$. ∎

The condition (6.2) was obtained by specializing general results. However, in this low-dimensional case, we could have worked directly with the coordinates of $X = \Delta$ as described in (3.2). In that case, the K_{ij} can be computed using the points $\{\widehat{x_i}\}_{i=2}^4$ on the unit sphere with coordinates

$$\widehat{x}_2 = (1, 0, 0), \ \widehat{x}_3 = (\xi, \beta, 0), \ \widehat{x}_4 = (\eta, \zeta, \gamma);$$
$$\beta, \gamma \geq 0; \ \xi, \eta, \zeta \in \mathbb{C};$$

(6.3)
$$|\xi|^2 + \beta^2 = |\eta|^2 + |\zeta|^2 + \gamma^2 = 1.$$

For $2 \leq i, j \leq 4$, we have $K_{ii} = 1$, $K_{ij} = \overline{K_{ji}}$. The rest of the story is given by

$$K_{23} = \bar{\xi}, \ \ K_{24} = \bar{\eta}, \ \ K_{34} = \xi \bar{\eta} + \beta \bar{\zeta}.$$

We also have

$$|\beta|^2 = 1 - |K_{23}|^2; \ \ |\bar{\zeta}|^2 = 1 - |K_{24}|^2 - \gamma^2.$$

Hence, noting that for all i, j, $|K_{ij}| \leq 1$, we must have

(6.4)
$$\left| K_{34} - \overline{K_{23}} K_{24} \right|^2 \leq (1 - |K_{23}|^2)(1 - |K_{24}|^2),$$

which is (6.2). In the other direction, it is not hard to start from $\{K_{ij}\}$, which satisfy these conditions and find coordinates of points on the sphere which generate these data.

Our path to answering Question 1 had many digressions, and it is perhaps worth-while to summarize the essential steps. First, if there is a tetrahedron, then the interpretation of kos in (2.14) together with the definition $\mathcal{M} = \mathcal{M}(\{T_i\})$ insures that $\mathcal{M}$ is a Gram matrix and hence is positive definite. In the other direction, given that $\mathcal{M}$ is positive definite, it must be the Gram matrix of a three-point set on the sphere. Reversing the construction of that three-point set determines three points in the ball which, together with the origin, are the vertices of our candidate tetrahedron. To finish, we need to show that this candidate tetrahedron has faces in the correct congruence classes. For the faces that meet at the origin, this holds by the complex analog of the side-angle-side criterion for congruence of triangles, Remark 1. Proposition 4.6 then shows that that data determine the congruence class of the fourth face and the proof of that same proposition shows that that same congruence class is forced by the data of the first three triangles and the assumed coherence conditions.

Here is a variation on these ideas. Given two triangles, T_2 and T_3, with one pair of matching side lengths, is it possible to find a third triangle T_4 so that $\{T_i\}_2^4 \Rightarrow X$ for some tetrahedron X? Again the question is whether the imputed parameters for X are an allowable set. If T_2 and T_3 are given and have a pair of matching side lengths, then all of the data for a putative $\rho(X)$ are specified and, as required, each value is between zero and one. The value of kos at the distinguished vertex of T_2 will give the value of K_{23} for the potential matrix $\mathcal{M}$, similarly for T_3 and K_{24}. However, the data from T_2 and T_3 are not enough to compute K_{42}, which would be the value of kos at its distinguished vertex of T_4. Hence, the congruence class of T_4 is indeterminate. To go forward, we set $K_{42} = z$, and using z, we complete the matrix $\mathcal{M}(\{T_2, T_3\})$ to a matrix $\mathcal{M}$ which has no missing values. We then apply the previous theorem to that matrix.

Corollary 6.3 *If $\{T_2, T_3\} \Rightarrow ??$, then there are a third triangle T_4 and a tetrahedron X with $\{T_2, T_3, T_4\} \Rightarrow X$ if and only if the Euclidean ball in $\mathbb{C}^1$*

$$B = B \left(\overline{K_{23}} K_{24}, (1 - |K_{23}|^2)^{1/2} (1 - |K_{24}|^2)^{1/2} \right)$$

is nonempty. In that case, the pairing of z with the value K_{34} establishes a one-to-one correspondence between $z \in B$ and the congruence class of the possible third triangle T_4. If B is empty, then there is no such T_4.

Proof Putting z into (6.2) gives $\left| z - \overline{K_{23}} K_{24} \right|^2 \le \left(1 - \left| K_{23} \right|^2 \right) \left(1 - \left| K_{24} \right|^2 \right)$. ∎

6.2 Question 2

We saw in Section 3.2 that Question 2 is equivalent to Question 1. Having answered Question 1, we now reformulate that answer in the context of Question 2. We will be informal.

We start with four three-dimensional $\{J_i\}_{i=1}^4 \subset \mathcal{CPP}$ and want to know if there is an $H \in \mathcal{CPP}$ whose four regular subspaces are rescalings of the $\{J_i\}$. Let $\{h_i\}_{i=1}^4 = RK(H)$, and for $r = 1, \ldots, 4$, let $\{j_{rs}\}_{s=1}^3 = RK(J_r)$. The chart below gives the numbering scheme for the supposed rescalings and notation for certain values of kos. For instance, the first row indicates that rescaling takes the three kernel functions of J_1, in the indicated order, to the indicated kernel functions of H. The last entry of the line introduces notation. The other lines are similar. We will not need to consider the values of kos associated with the last row.

$$
\begin{array}{lllll}
J_1 & j_{11}, j_{12}, j_{13} & \rightarrow & h_1, h_2, h_3, & L_{23} = \operatorname{kos}_{j_{11}}(j_{12}, j_{13}), \\
J_2 & j_{21}, j_{22}, j_{23} & \rightarrow & h_1, h_3, h_4, & L_{34} = \operatorname{kos}_{j_{21}}(j_{22}, j_{23}), \\
J_3 & j_{31}, j_{32}, j_{33} & \rightarrow & h_1, h_4, h_2, & L_{42} = \operatorname{kos}_{j_{31}}(j_{32}, j_{33}), \\
J_4 & j_{41}, j_{42}, j_{43} & \rightarrow & h_2, h_3, h_4.
\end{array}
$$

The required coherence is that when rescaled images overlap, they must be compatible; so, for instance, we must have $\delta(j_{11}, j_{13}) = \delta(j_{21}, j_{22})$ because both pairs are mapped to (h_1, h_3).

Define $\mathcal{M}(\{J_i\})$ by

$$(6.5) \qquad \mathcal{M}(\{J_i\}) = \begin{pmatrix} 1 & L_{23} & L_{24} \\ L_{32} & 1 & L_{34} \\ L_{42} & L_{43} & 1 \end{pmatrix}.$$

The upper-right entries in this matrix are defined in the previous display. The lower-left entries are their complex conjugates. If there is an H, then this matrix will equal $KOS(H, 1)$.

Because $\{J_i\}_{i=2}^4 \subset \mathcal{CPP}$, we know from Theorem 3.1 that each $\left| L_{ij} \right| \le 1$. Hence, Lemma 6.1 can be applied and that gives several conditions equivalent to $\mathcal{M}(\{J_i\}) \succcurlyeq 0$ including $\det \mathcal{M}(\{J_i\}) \ge 0$ and the analog of (6.2). We can now translate Theorem 6.2 to this context.

Theorem 6.4 *Given $\{J_i\}_{i=2}^4 \subset \mathcal{CPP}$ and $\{J_i\}_{i=1}^4 \Rightarrow$??, there is a four-dimensional $H \in \mathcal{CPP}$ with $\{J_i\}_{i=2}^4 \Rightarrow H$ if and only if $\mathcal{M}(\{J_i\}) \succcurlyeq 0$, or, equivalently $\det \mathcal{M}(\{J_i\}) \ge 0$, or, equivalently, the numbers $\{L_{ij}\}_{i,j=2,3,4}$ satisfy the analog of (6.2).*

(If we only knew that $\{J_i\}_{i=2}^4 \subset \mathcal{RK}$, then deriving $\mathcal{M}(\{J_i\}) \succcurlyeq 0$ from $\det \mathcal{M}(\{J_i\}) \ge 0$ requires the additional assumption that the $\left| L_{ij} \right| \le 1$. However, this is not actually a different formulation. By Theorem 3.1, adding the assumptions

that $\left|L_{ij}\right| \le 1$ is equivalent to passing from the assumption that $\{J_i\}_{i=2}^4 \subset \mathcal{RK}$ to the assumption that $\{J_i\}_{i=2}^4 \subset \mathcal{CPP}$.)

6.3 Quiggin's example

There is a simple criterion for determining if a three-dimensional $H \in \mathcal{RK}$ is in $\mathcal{CPP}$. From Theorem 3.1, we see that $H \in \mathcal{CPP}$ if and only if $|\mathrm{kos}_1(2,3)| \le 1$. The analogous question for four-dimensional H is more complicated. It is clearly necessary that each regular three-dimensional subspace of H be in $\mathcal{CPP}$, but knowing if that condition is sufficient is essentially Question 2 of the introduction. The first example showing the condition is not sufficient is due to Quiggin [Q], [AM, p. 94]. He constructed a family H_x, $0 < x < 1$ of spaces H_x in $\mathcal{RK}$ each having the Pick property, a weaker statement than the CPP and which we will not detail here, and showed that $H_{1/4} \notin \mathcal{CPP}$. Here, we will use the results of the previous sections to read off directly the facts that for each $0 < x < 1$ the regular subspaces of H_x have the CPP but H_x does not.

Following Quiggin, we introduce a family $\{H_x : 0 < x < 1\} \subset \mathcal{RK}$ of four-dimensional spaces by specifying their Gram matrices, $\mathrm{Gr}(H_x)$. For $0 < x < 1$ and $s = (1-x)\sqrt{x}$, set

$$\mathrm{Gr}(H_x) = \begin{pmatrix} 1 & x & x & x+is \\ x & 1 & x-is & x \\ x & x+is & 1 & x \\ x-is & x & x & 1 \end{pmatrix}.$$

To show this is the Gram matrix of an $\mathcal{RK}$, we need to show that $\mathrm{Gr}(H_x) > 0$. By Lemma 5.5, we can do that by checking the signs of the leading principal minors. They are

$$(1+x)^2 (1-x)^4, (1+x)(1-x)^2, (1+x)(1-x), 1$$

and, by inspection, are all positive for $0 < x < 1$. (Those computations and the determinant computations below were done using computer algebra.)

Earlier we used the matrices $\mathrm{KOS}(\mathrm{Gr}(H_x),1)$ from (6.5). Here, for ease in computing, we use the matrices $\mathrm{MQ}(H_x,1) = \left(\delta_{1i}\delta_{1j}\,\mathrm{kos}_1(i,j)\right)_{i,j=2}^4$ mentioned in (2.16). The two have determinants of the same sign as do their square submatrices.

From the definitions, we have

$$\mathrm{MQ}(H_x,1) = \begin{pmatrix} 1-x^2 & 1-\frac{x^2}{x-is} & 1-x-is \\ 1-\frac{x^2}{x+is} & 1-x^2 & 1-x-is \\ 1-x+is & 1-x+is & (1-x)(1+x^2) \end{pmatrix}.$$

Fix x. We want to know that $\{J_{xi}\}_1^4$, the regular three-dimensional subspaces of H_x, have the CPP. By (5) of Theorem 3.1, we know that $J_{x2} \in \mathcal{CPP}$ if the matrix $\mathcal{J}_2$ obtained by deleting the first row and first column of $\mathrm{MQ}(H_x,1)$ satisfies $\mathcal{J}_2 \succcurlyeq 0$. That will follow if we show $\det \mathcal{J}_2 \ge 0$. Similarly for J_{x3} and J_{x4}. For J_{x1}, we follow the same path but starting with $\mathrm{MQ}(H_x,2)$ rather than $\mathrm{MQ}(H_x,1)$. To show that $H_x \notin \mathcal{CPP}$, we will show that $\det \mathrm{MQ}(H_x,1) < 0$ and hence $\mathrm{MQ}(H_x,1) \succcurlyeq 0$ fails. All these things

can be seen in the explicit formulas for the determinants. Note that for $0 < x < 1$, we have $x^2 - x + 1 > 0$. We have

$$\det \mathcal{J}_1 = \det \mathcal{J}_2 = \det \mathcal{J}_3 = x^2 \left(x+1\right) \left(x-1\right)^2,$$

$$\det \mathcal{J}_4 = \frac{x^3 \left(x+1\right) \left(x-1\right)^2}{x^2 - x + 1},$$

$$\det \mathrm{MQ}(H_x, 1) = \frac{\left(\begin{array}{c} 2s^4 x^2 - s^2 x^7 + s^2 x^6 + s^2 x^5 + 3s^2 x^4 \\ -4s^2 x^3 - x^9 + x^8 + 2x^7 - 2x^6 - x^5 + x^4 \end{array} \right)}{s^2 + x^2}$$

$$= -\frac{x^3 \left(x+1\right)^2 \left(x-1\right)^4}{x^2 - x + 1}.$$

This shows that the matching distances property together with the cocycle property are not sufficient to insure that a set of four three-dimensional spaces with the CPP can be assembled into a four-dimensional space with the CPP. For $x = 1/4$, the following result is due to Quiggin [Q], [AM].

Proposition 6.5 *Fix x, $0 < x < 1$, and let $\{J_{xi}\}_{i=1}^4$ be the three-dimensional regular subspaces of H_x. Then:*

(1) *The $\{J_i\}_{i=1}^4 \subset \mathcal{CPP}$ and $\{J_i\}_{i=1}^4 \Rightarrow$??.*
(2) *There is an $H \in \mathcal{RK}$ with $\{J_i\}_{i=1}^4 \Rightarrow H$.*
(3) *There is no $H \in \mathcal{CPP}$ with $\{J_i\}_{i=1}^4 \Rightarrow H$.*

Proof We verified above that the $\{J_i\}$ all have the CPP. The coherence is automatic because the $\{J_i\}$ are the three-dimensional regular subspaces of a four-dimensional $\mathcal{RK}$. The second statement is automatic; $H = H_x$ will suffice. It is included to emphasize that there is no obstruction to assembling the $\{J_i\}$ into an $H \in \mathcal{RK}$, just not an $H \in \mathcal{CPP}$. ∎

6.4 Tetrahedra in $\mathbb{RH}^k$

6.4.1 Preliminaries

In this section, we specialize the previous results to real hyperbolic triangles and tetrahedra, those sitting in some $\mathbb{RH}^k$, or, equivalently, inside a copy of $\mathbb{RH}^k$ inside some $\mathbb{CH}^n$. The study of polyhedra in $\mathbb{RH}^k$ is an active research topic with a rich history; references include the books [An, F, Go], surveys [J, MP], and research papers [Di, HR, W]. Also, there is some study of the relation between RKHSs and sets in $\mathbb{RH}^k$ ([BIM, Ro, Sec. 7]). Some of the results below are in those references or can be developed efficiently using those techniques. Our goal here is to show how results for real hyperbolic tetrahedra can be seen as specialization of results for complex hyperbolic tetrahedra.

The following simple consequence of Lemma 2.1 of [BI] lets us tell when $X \subset \mathbb{CH}^n$ is actually in a copy of $\mathbb{RH}^n$.

Proposition 6.6 $X = \{x_i\}_{i=1}^s \subset \mathbb{CH}^n$ *is inside a copy of* $\mathbb{RH}^{s-1} \subset \mathbb{CH}^n$ *if and only if all the numbers* $\operatorname{kos}_i(p,q)$ *are real, or, equivalently, if and only if the entries of the Gram matrix of* $DA(X)$ *are real.*

Hence, the results in this section also apply to Hilbert spaces $H \in \mathcal{CPP}$ which are rescalings of spaces with real Gram matrices. Those spaces are the finite-dimensional regular subspaces of the diameter spaces discussed in [ARS].

Given $\{x_1, x_2, x_3\} \subset \mathbb{RH}^k$, we denote the vertex angle between geodesics $x_1 x_2$ and $x_1 x_3$. by va_{23}.

Corollary 6.7   *The triangle* $T = \{x_1, x_2, x_3\} \subset \mathbb{CH}^n$ *is in a copy of* $\mathbb{RH}^2$ *if and only if* $\operatorname{kos}_1(2,3)$ *is real. In that case,* $\operatorname{kos}_1(2,3) = \cos va_{23}$.

Proof   Using the model triangle Γ in (3.1), it is easy to check that if $\operatorname{kos}_1(2,3)$ is real, then the coordinates of the points of Γ are real and hence also so are the other values of kos. It then follows from the previous proposition that T is in a copy of $\mathbb{RH}^k$. Because Γ only has three points, we can take $k = 2$. Using (2.14), we see that $\operatorname{kos}_1(2,3) = \langle\langle \widehat{x_2}, \widehat{x_3} \rangle\rangle$. That inner product equals the cosine of the Euclidean angle at the origin of $\mathbb{R}^2$ between the segments $0\widehat{x_2}$ and $0\widehat{x_3}$. On T, the Euclidean metric on $\mathbb{R}^2$ is conformal with the hyperbolic metric, and hence the Euclidean angle whose cosine we found is also the hyperbolic angle.   ∎

If triangles can be assembled into a real hyperbolic tetrahedron, then it can be done in $\mathbb{RH}^3$. The model we will use for $\mathbb{RH}^3$ is the unit ball in $\mathbb{R}^3$ with the Poincare metric. We write $\mathbb{S}_2$ for the unit sphere in $\mathbb{R}^3$.

Because the Poincare ball model for $\mathbb{RH}^3$ is conformal with Euclidean space, some of our results here also apply to structures in Euclidean space. For instance, the condition of vertex angles on the bivalent vertices that are necessary to form a trivalent vertex is the same in both cases.

If we start with four real hyperbolic triangles which satisfy the matching side conditions necessary for assembly into a tetrahedron X, then Theorem 6.2 gives conditions for there to be an X in terms of a matrix $\mathcal{M}(X)$ whose entries are values of kos. Because we are in real hyperbolic space, those values are cosines of vertex angles. To emphasize this, we introduce name and notation for that type of $\mathcal{M}(X)$:

$$(6.6) \qquad \mathcal{M}_{CVA}(X) = \begin{pmatrix} 1 & \cos va_{23} & \cos va_{24} \\ \cos va_{32} & 1 & \cos va_{34} \\ \cos va_{42} & \cos va_{43} & 1 \end{pmatrix}$$

with the subscript CVA referring to the fact the entries are cosines of vertex angles. Theorem 6.2 specializes as the following.

Theorem 6.8   *Given triangles* $\{T_i\}_{i=1}^4$ *in* $\mathbb{RH}^n$ *with* $\{T_i\}_{i=2}^4 \Rightarrow$ *??, there is a tetrahedron* X *in* $\mathbb{RH}^n$ *such that* $\{T_i\}_{i=2}^4 \Rightarrow X$ *if and only if* $\det \mathcal{M}_{CVA}(\{T_i\}_{i=2}^4) \geq 0$.

6.4.2 The triangle inequality for angles

We saw conditions for three bivalent vertices in $\mathbb{CH}^n$ to be assembled into a trivalent vertex. Those conditions specialize to vertices in $\mathbb{RH}^n$ and, because the geometry of

our model of $\mathbb{RH}^n$ is conformal with Euclidean geometry, the same conditions apply to vertices in Euclidean space. The question of when three bivalent vertices in $\mathbb{R}^3$ can be assembled into a trivalent vertex is a straightforward question in Euclidean solid geometry and it is not surprising that it has a simple answer. We now look at that briefly.

Corollary 6.9 (The triangle inequality for angles)   *The numbers $0 \le \alpha, \beta, \gamma \le \pi$ are the (real hyperbolic or Euclidean) angles of a trivalent vertex in (real hyperbolic or Euclidean) space if and only if $\alpha \le \beta + \gamma$.*

Proof   There is such a vertex if and only if the vertex can be realized as part of a real hyperbolic tetrahedron X. By the previous theorem, that can happen if and only if the matrix $\mathcal{M}_{CVA}(X)$ of (6.6) has a positive determinant. We use Lemma 6.1 to rewrite that determinant condition, and then we compute

$$(6.7) \qquad (\cos\alpha - \cos\beta\cos\gamma)^2 \le (1 - \cos^2\beta)(1 - \cos^2\gamma),$$
$$|\cos\alpha - \cos\beta\cos\gamma| \le \sin\beta\sin\gamma,$$
$$-\sin\beta\sin\gamma + \cos\beta\cos\gamma \le \cos\alpha \le \sin\beta\sin\gamma + \cos\beta\cos\gamma,$$

$$(6.8) \qquad \cos(\beta + \gamma) \le \cos\alpha \le \cos(\beta - \gamma)$$

If $\gamma + \beta < \pi$, then the three angles in the previous line are in the range $(0, \pi)$ where the cosine is monotone decreasing. In that case, the first inequality gives $\alpha \le \beta + \gamma$. In the other case, we have $\alpha \le \pi \le \beta + \gamma$. In both cases, we have the desired inequality. The argument is reversible. ∎

In Section 6.4, we will present the triangle for which the previous corollary is a "triangle inequality."

There is an interesting identity that can be used to give an alternate proof of the corollary. Set $s = (\alpha + \beta + \gamma)/2$. By trigonometric analysis [Co, J] or, as Roeder notes in [Roe], by computation with complex exponentials, we have

$$\det\mathcal{M}_{CVA}(X) = 4\sin(s)\sin(s - \alpha)\sin(s - \beta)\sin(s - \gamma).$$

If we know $\det\mathcal{M}_{CVA}(X) \ge 0$, then an analysis of cases shows that all the factors on the right side are nonnegative. Knowing that gives the conclusion of the corollary.

Finally, we have taken a long route to what is a rather obvious fact of solid geometry. Consider the task of building a model of a trivalent vertex from three wedges of paper. Certainly the job is impossible if one wedge is wider than the other two combined.

6.4.3 Dihedral angles

In real hyperbolic space, any three points sit in a totally geodesically embedded hyperbolic plane and hence each edge is in the intersection of two such planes. We define the dihedral angle at that edge to be the angle of intersection of the two planes. Thus, at each trivalent vertex, we have three vertex angles and three dihedral angles.

In fact, the dihedral angles are used more commonly than vertex angles in describing real hyperbolic polyhedra (see, for instance, [FG, HR, Roe, W] and the references therein). In this section and the next, we look briefly at the relation between

the two types of angles and at how our earlier results translate to results involving dihedral angles. Some of the results we obtain are classical facts from solid geometry or spherical trigonometry. The work in this section is influenced by the work of Roeder in [Roe] and there are overlaps. We will be sketchy.

For the moment, we will use indices r, s, t to denote three different indices from the set $\{2, 3, 4\}$. Given the tetrahedron $X = \{x_1, x_2, x_3, x_3\} \subset \mathbb{RH}^3$ with x_1 at the origin, denote the triangular face with vertices $\{x_1, x_i, x_j\}$ by F_{ij}. The dihedral angle along edge s, $s = 2, 3, 4$, is the angle da_{rt} between the faces F_{rs} and F_{st}.

As we mentioned, the angles in our ball model of $\mathbb{RH}^3$ agree with the Euclidean angles, and hence it suffices to do the Euclidean computation of the da. To find da_{rt}, we first find the inward pointing unit normals, n_{rs} for the face F_{rs}, and similarly n_{st}, and then use the fact that $-\cos da_{rt} = \langle n_{rs}, n_{st} \rangle$. The requirement that n_{rs} be inward pointing is the requirement that $\langle n_{rs}, x_t \rangle \geq 0$. However, the formula for da_{rt} is unchanged if the normals are replaced by their negatives, and hence it is enough to construct the normals so that the inner products $\langle n_{rs}, x_t \rangle$ all have the same sign. Taking note of the fact that $\langle a, b \times c \rangle = \langle b, c \times a \rangle$ for vectors in $\mathbb{R}^3$, we see that the choices $\langle n_{rs}, x_t \rangle$

$$(6.9) \qquad n_{rs} = \frac{x_r \times x_s}{\|x_r \times x_s\|} = \frac{\widehat{x_r} \times \widehat{x_s}}{\|\widehat{x_r} \times \widehat{x_s}\|}$$

satisfy that requirement. Using these normals, we compute

$$(6.10) \qquad \cos da_s = -\langle\langle n_{rs}, n_{st} \rangle\rangle$$
$$= -\left\langle\left\langle \frac{\widehat{x_r} \times \widehat{x_s}}{\|\widehat{x_r} \times \widehat{x_s}\|}, \frac{\widehat{x_s} \times \widehat{x_t}}{\|\widehat{x_s} \times \widehat{x_t}\|} \right\rangle\right\rangle$$
$$= -\frac{\langle\langle \widehat{x_r}, \widehat{x_s} \rangle\rangle \langle\langle \widehat{x_s}, \widehat{x_t} \rangle\rangle - \langle\langle \widehat{x_r}, \widehat{x_t} \rangle\rangle \langle\langle \widehat{x_s}, \widehat{x_s} \rangle\rangle}{\|\widehat{x_r} \times \widehat{x_s}\| \|\widehat{x_s} \times \widehat{x_t}\|}$$
$$= \frac{\cos va_{rt} - \cos va_{rs} \cos va_{st}}{\sin va_{rs} \sin va_{st}}.$$

(Note that if the vertices are ordered, and hence the angles are signed, the cosine terms are independent of the ordering, but the sine factors are not. However, that product of sines is unchanged by reordering the vertices. The same comments apply to the expressions below.)

Definition 6.10 Suppose that for $2 \leq i, j \leq 4$, we are given angles $\{va_{ij}\}$ and $\{da_{ij}\}$. For all i, we suppose $va_{ii} = 0, da_{ii} = \pi$ and set $VA_{ii} = 0, DA_{ii} = \pi$. For $r \neq s$, we define

$$(6.11) \qquad DA_{rs} = \frac{\cos va_{rs} - \cos va_{tr} \cos va_{ts}}{\sin va_{tr} \sin va_{ts}},$$

$$(6.12) \qquad VA_{rs} = \frac{\cos da_{rs} + \cos da_{tr} \cos da_{ts}}{\sin da_{tr} \sin da_{ts}}.$$

Lemma 6.11 (Hyperbolic law of cosines)   *If at the vertex point x_1 the tetrahedron $X = \{x_1\}_{i=1}^{4} \subset \mathbb{RH}^n$ has vertex angles $\{va_{ij}\}_{i,j=2}^{4}$ and dihedral angles $\{da_{ij}\}_{i,j=2}^{4}$, then, with*

the notation (6.11) and (6.12),

$$\cos da_{ij} = DA_{ij}, \quad \cos va_{ij} = VA_{ij}, \quad 2 \le i, j \le 4.$$

We separated the definition from the lemma because we can use the definitions even if the va_{ij} are not known to be data from a tetrahedron. If we have a set of triangles $\{T_i\}_{i=2}^4$ in $\mathbb{RH}^k$ which satisfy the coherence conditions for assembly into a tetrahedron, $\{T_i\}_{i=2}^4 \Rightarrow$??, then we can construct the matrix $\mathcal{M}_{CVA}(\{T_i\}_{i=2}^4)$. Using that data in (6.11), we can compute imputed values of the DA_{ij}; the values $\cos da_{ij}$ would have if assembly were possible. If $\{T_i\}_{i=2}^4 \Rightarrow X$ for a tetrahedron X, then those values will be cosines of dihedral angles and satisfy $\left| D_{ij} \right| \le 1$.

Corollary 6.12 *If $\{T_i\}_{i=2}^4 \Rightarrow$?? and DA_{ij} is computed using the va_{ij} values from $\mathcal{M}_{CVA}(\{T_i\}_{i=2}^4)$ and (6.11), then there is a tetrahedron X with $\{T_i\}_{i=2}^4 \Rightarrow X$ if and only if for some i, j $\left| DA_{ij} \right| \le 1$.*

Proof From the formula (6.11), we see that $\left| DA_{ij} \right| \le 1$ holds if and only if (6.7) holds. That last condition is equivalent to knowing $\det \mathcal{M}_{CVA}(\{T_i\}_{i=2}^4) \ge 0$, which, by Theorem 6.8, is equivalent to their being an X. ∎

Informally, a value $\left| DA_{ij} \right| > 1$ is not possible for any dihedral angle; hence, in that case, there is no tetrahedron.

6.4.4 Spherical geometry

We can also use spherical geometry to relate the vertex angles and dihedral angles. Consider the tetrahedron $X = \{0, x_2, x_3. x_4\}$ in $\mathbb{RH}^3$ and the set $\widehat{X} = \{\hat{x}_2, \hat{x}_3, \hat{x}_4\}$ in the boundary sphere $\mathbb{S}_2$. We regard $\widehat{X}$ as the set of vertices of a spherical triangle, also called $\widehat{X}$.

A fundamental relation between X and $\widehat{X}$ is that the side lengths and angle measures of $\widehat{X}$ are the sizes of the vertex angles and dihedral angles, respectively, of the trivalent vertex of X at the origin. (The analog for the Euclidean tetrahedron is perhaps visually clear.) Hence, in particular, the "Triangle Inequality for Angles," Corollary 6.9, is literally the triangle inequality for the side lengths of the triangle $\widehat{X}$.

Associated with a spherical triangle with vertices $\widehat{X} = \{\widehat{x_2}, \widehat{x_3}, \widehat{x_4}\}$ is its polar dual $\widehat{X}^{\#}$. We forego the description of $\widehat{X}^{\#}$ using spherical geometry and just note that it is the triangle with vertices $\{n_{34}, n_{24}, n_{23}\}$ given by the formula (6.9). It is straightforward to see that this is an actual duality; $\widehat{X}^{\#\#} = \widehat{X}$.

The following is a fundamental relation between the geometries of a triangle and its polar dual.

Theorem 6.13 *Suppose $\widehat{X}$ has angles $\{a_i\}_{i=1,2,3}$ and side lengths $\{\ell_i\}_{i=1,2,3}$ and $\widehat{X}^{\#}$ has angles $\{a_i^{\#}\}_{i=1,2,3}$ and side lengths $\{\ell_i^{\#}\}_{i=1,2,3}$ with all the lengths and angles selected between 0 and π; then, for $i = 1, 2, 3$,*

$$\alpha_i^{\#} = \pi - \ell_i,$$
$$\ell_i^{\#} = \pi - \alpha_i.$$

We saw in Theorem 6.8 that angles $\{va_i\}_{i=1,2,3}$ are the vertex angles of a trivalent vertex if and only if the matrix of their cosines, $\mathcal{M}_{CVA}(X)$ given by (6.6), satisfies $\det \mathcal{M}_{CVA}(X) \geq 0$. Using Lemma 6.11, we can form the analogous result using negatives of cosines of dihedral angles. Set

$$(6.13) \qquad \mathcal{M}_{-CDA}(X) = \begin{pmatrix} 1 & -\cos da_{23} & -\cos da_{24} \\ -\cos da_{32} & 1 & -\cos da_{34} \\ -\cos da_{42} & -\cos da_{43} & 1 \end{pmatrix}.$$

If we are given angles $\{da_{ij}\}$ that are candidates for being the dihedral angles of a tetrahedron in $\mathbb{RH}^3$, then we write $\mathcal{M}_{-CDA}(?)$ for the matrix on the right-hand side of (6.13). Here is the analog of Theorem 6.2 for dihedral angles.

Theorem 6.14 *Given a tetrahedron $X \subset \mathbb{RH}^3$, we have $\det \mathcal{M}_{-CDA}(X) \geq 0$. Conversely, if angles $\{da_{ij}\}$ produce a matrix $\mathcal{M}_{-CDA}(?)$ with $\det \mathcal{M}_{-CDA}(?) \geq 0$, then there is a tetrahedron $X \subset \mathbb{RH}^3$ with those dihedral angles.*

Proof First, suppose we are given $X \subset \mathbb{RH}^3$. We then have the triangle with vertices $\widehat{X}$. Let $\widehat{X}^\#$ be the polar dual of $\widehat{X}$, and let $X^\#$ be any tetrahedron in $\mathbb{RH}^3$ with $\widehat{X^\#} = \widehat{X}^\#$. Consider the matrix $\mathcal{M}_{CVA}(X^\#)$. Its nondiagonal entries are of the form $\cos va_i^\#$ where $va_i^\#$ is a vertex angle of $X^\#$. That vertex angle is also the length $\ell_i^\#$ of the triangle $\widehat{X}^\#$. By Theorem 6.13, that length is given by $\ell_i^\# = \pi - a_i$ where a_i is an angle in the polar dual triangle $(\widehat{X}^\#)^\# = \widehat{X}^{\#\#} = \widehat{X}$. For any θ, we have $\cos\theta = -\cos(\pi - \theta)$. Collecting these facts and comparing matrix entries, we see that $\mathcal{M}_{CVA}(X^\#) = \mathcal{M}_{-CDA}(X)$, Hence, $\det \mathcal{M}_{-CDA}(X) \geq 0$ is equivalent to $\det \mathcal{M}_{CVA}(X^\#) \geq 0$. To see that that condition holds, recall that we are in real hyperbolic space and hence $\mathcal{M}_{CVA}(X^\#) = \mathcal{M}(X^\#)$. By Theorem 6.2, that last matrix is positive semidefinite and hence has a positive determinant.

The argument in the other direction is similar. Suppose we are given angles da_{ij} which produce $\mathcal{M}_{-CDA}(?)$ with $\det \mathcal{M}_{-CDA}(?) \geq 0$. Consider the supplementary angles $va_{ij}^\# = \pi - da_{ij}$. By Theorem 5.1, the cosines of those angles can be used to form a matrix $\mathcal{M}(X)$ associated with a tetrahedron X. We now pass successively to $\widehat{X}$, the spherical triangle associated with X, then to $\widehat{X}^\#$, its polar dual, and finally Y, a tetrahedron whose associated spherical triangle $\widehat{Y}$ equals $\widehat{X}^\#$. Tracking the changes, we see that we have passed to supplementary angles twice and hence the dihedral angles of Y are the da_{ij} and Y is the desired tetrahedron. ∎

7 Final comments

7.1 Cayley equations

If we replace the trigonometric variables in the expansion of $\det \mathcal{M}_{CVA}(\{T_i\}_{i=2}^4)$ with algebraic variables, we obtain

$$p(x, y, z) = 1 + 2xyz - x^2 - y^2 - z^2.$$

This polynomial was studied by Cayley in his classic study of cubic equations and sometimes carries his name [H]. For us, the region in $\mathbb{R}^3$ where the variables have absolute value at most one and $p(x, y, z) > 0$ parameterizes nondegenerate tetrahedra in $\mathbb{R}\mathbb{H}^3$. The boundary surface Ω, where $p(x, y, z) = 0$, corresponds to degenerate tetrahedra. The smooth points of Ω correspond to simple degenerations, degenerate tetrahedra that become nondegenerate when a single vertex is moved a small amount. The singular points of Ω correspond to more complicated, nongeneric, degeneracies. For instance, let T be a triangle $\{w, y, z\}$ in the ball model of $\mathbb{R}\mathbb{H}^3$ which is in the plane $\mathbb{R}\mathbb{H}^2$ specified by the vanishing of the third coordinate and which has the origin of that plane in its interior. Form tetrahedra X_ε by adjoining a fourth vertex, which will be the distinguished vertex x_1, with Euclidean coordinates $(0, 0, \varepsilon)$ for a small positive ε. The X_ε are proper tetrahedra, but the limiting X_0 whose vertices are the three starting points together with the origin as the distinguished vertex is degenerate. For X_0, the values of kos_1 are the cosines of the angles formed by connecting the origin to the other vertices. We are in a plane, so those angles sum to 2π. Thus, the corresponding (x, y, z) values are $\Lambda = (\cos\alpha, \cos\beta, \cos(2\pi - \alpha - \beta))$ for some angles α and β. Taking note of the fact that

$$\cos(2\pi - \alpha - \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta,$$

it is straightforward to check that Λ is a point in the surface Ω. It is a smooth point, and the degeneracy of X_0 can be removed by moving the vertex at the origin slightly to obtain an X_ε.

The singular points of Ω are the points $(\pm 1, \pm 1, \pm 1)$ with an even number of minus signs. The corresponding tetrahedra have four points on a single real geodesic. Those tetrahedra have nongeneric degeneracy, they remain degenerate if any one of the points is moved slightly.

Some related discussion is in [H].

7.2 Vertices at infinity

The study of tetrahedra in hyperbolic space, real or complex, is not restricted to classical bounded tetrahedra but also includes consideration of ideal tetrahedra, tetrahedra with one or more vertices in the ideal boundary (i.e., the "sphere at infinity," $\partial\mathbb{B}_n$). Although some of the previous discussion extends to those contexts, it is not clear if there are objects similar to the $DA(X)$ associated with these ideal tetrahedra. There is analysis of congruence of finite sets in the closure, $\overline{\mathbb{C}\mathbb{H}^n}$, in several places including [CG, G, Go, HS].

7.3 The physics literature

The question of characterizing triples of triangles in $\mathbb{R}\mathbb{H}^3$ which can be assembled as part of a tetrahedron is also studied in the physics literature, sometimes with the name "closure questions," for instance, [BDGL, CL, HHR]. In contrast to the work here, those papers make substantial use of the descriptive and analytical properties of the automorphism group of hyperbolic space.

7.4 Other Hilbert spaces and other geometries

We have worked with the Drury–Arveson kernel $k_z(w) = k(w, z) = (1 - \langle\langle w, z \rangle\rangle)^{-1}$. Similar questions can be considered for the Hilbert spaces generated by kernel functions $(1 - \langle\langle w, z \rangle\rangle)^{-t}$ for $t > 0$. Those functions are the reproducing kernels for various Besov–Sobolev spaces [ARSW2] and also arise in other contexts [M]. In [OS], it is shown that analogs of parts of Theorem 3.2 hold for those spaces if $t \le 2$ but not $t > 2$.

We have focused on the relationship between $DA(X)$ spaces and hyperbolic geometry. There are similar relationships between other classes of Hilbert spaces and other geometries, for instance, between the Segal–Bargmann–Fock spaces and the Hermitian geometry of $\mathbb{C}^n$ [ARSW, Ro], and between the Hilbert spaces of spin coherent states, and the geometry of complex spheres and projective spaces [BZ].

More general relations between geometry and spaces such as $DA(X)$ for X in $\mathbb{R}\mathbb{H}^n$ and $\mathbb{C}\mathbb{H}^n$ are suggested by the work in [M].

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