Suppose that $A$ is a finite direct product of commutative rings. We show from first principles that a Gröbner basis for an ideal of $A[x_1, \ldots, x_n]$ can be easily obtained by 'joining' Gröbner bases of the projected ideals with coefficients in the factors of $A$ (which can themselves be obtained in parallel). Similarly for strong Gröbner bases. This gives an elementary method of constructing a (strong) Gröbner basis when the Chinese Remainder Theorem applies to the coefficient ring and we know how to compute (strong) Gröbner bases in each factor.

1. INTRODUCTION

Let $A$ be a commutative ring with $1 \neq 0$. We are interested in obtaining a (strong) Gröbner basis of a non-zero ideal $I$ of $A[x_1, \ldots, x_n]$ when $A = A_1 \times \cdots \times A_m$ is a direct product of rings and we know how to obtain (strong) Gröbner bases of the projected ideals $\pi_i(I)$ for $i = 1, \ldots, m$. We show that this can be done by 'joining' (strong) Gröbner bases for the $\pi_i(I)$ of $A_i[x_1, \ldots, x_n]$. Thus we can compute a (strong) Gröbner basis for $I$ when we know algorithms for computing a (strong) Gröbner basis for $\pi_i(I)$. As an application, we compute a (strong) Gröbner basis for $I$ when the Chinese Remainder Theorem applies to $A$ and we can compute (strong) Gröbner bases in each factor. Recall that if $A$ is a principal ideal ring, any non-zero ideal of $A[x]$ has a strong Gröbner basis [3, Algorithm 6.4]. We give another proof of this fact.

The preliminary Section 2 recalls the necessary background on (strong) Gröbner bases from [1, 3]. Section 3 discusses the join of Gröbner bases while Section 4 describes the strong join of strong Gröbner bases. In the final section, we assume that $A$ is a principal ideal ring.
2. Preliminaries

We have $A = A_1 \times \cdots \times A_m$ and we write $A[x]$ for $A[x_1, \ldots, x_n]$. The monoid of terms in $x_1, \ldots, x_n$ is denoted by $T$. Let $<$ be a fixed but arbitrary admissible order on $T$. Throughout the paper, we use the same term order $<$ on each $A_i[x]$ as on $A[x]$.

If $f = \sum_{t \in T} f_t A[x] \setminus \{0\}$ and $v = \max\{t \in T : f_t \neq 0\}$ then $v$ is the leading term, $f_v$ the leading coefficient and $f_v v$ the leading monomial of $f$, denoted $\text{lt}(f)$, $\text{lc}(f)$ and $\text{lm}(f)$ respectively. We also write $\text{lm}(S)$ for $\{\text{lm}(f) : f \in S\}$ where $S \subset A[x] \setminus \{0\}$.

Let $G \subset A[x] \setminus \{0\}$ be finite. Then $f \in A[x]$ has a standard representation with respect to $G$ if $f = \sum_{j = 1}^{k} c_j(t_j) g_j^*$ for some $c_j \in A \setminus \{0\}$, $t_j \in T$, $g_j^* \in G$ such that $t_j \text{lt}(g_j^*) \leq \text{lt}(f)$, [2, p. 218]. We write $\text{Std}(G)$ for the polynomials which have a standard representation with respect to $G$.

Also, if $G \subset A[x] \setminus \{0\}$ is finite, then $G$ is a Gröbner basis for a non-zero ideal $I \subset A[x]$ if and only if $I = \text{Std}(G)$, [1, Theorem 4.1.12]. If $A$ is Noetherian, every non-zero ideal of $A[x]$ has a Gröbner basis [1, Corollary 4.1.17].

Recall that if $G \subset A[x] \setminus \{0\}$ is finite, then $G$ is a strong Gröbner basis for $I = \langle G \rangle$ if and only if for any $f \in I$ there is a $g \in G$ such that $\text{lm}(g) \mid \text{lm}(f)$, [1, Definition 4.5.6]. If $A$ is a principal ideal ring, Algorithm 6.4 of [3] constructs a strong Gröbner basis for any non-zero ideal of $A[x]$. Also, a strong Gröbner basis $G$ is called minimal if no proper subset of $G$ is a strong Gröbner basis for $\langle G \rangle$.

3. The Join

The projections $\pi_i : A \to A_i$ induce maps $\pi_i : A[x] \to A_i[x]$. It is straightforward to check that the induced map $\pi : A[x] \to A_1[x] \times \cdots \times A_m[x]$ given by $\pi(f) = (\pi_1(f), \ldots, \pi_m(f))$ and the map $\kappa : A_1[x] \times \cdots \times A_m[x] \to A[x]$, which collects coefficients of like terms, are mutually inverse ring homomorphisms. We relate Gröbner bases of $I \subset A[x]$ to Gröbner bases of $\pi_i(I) \subset A_i[x]$, where $1 \leq j \leq m$.

**Proposition 3.1.** If $G$ is a Gröbner basis for a non-zero ideal $I \subset A[x]$, then $\pi_i(G) \setminus \{0\}$ is a Gröbner basis for $\pi_i(I)$ in $A_i[x]$ for $i = 1, \ldots, m$.

**Proof:** We can assume that $i = 1$. Let $f_1 \in \pi_1(I) \setminus \{0\} \subset A_1[x]$ and put $G_1 = \pi_1(G) \setminus \{0\}$. We show that $f_1 \in \text{Std}(G_1)$. For let $f = \kappa(f_1, 0, \ldots, 0) \in I \setminus \{0\}$. We have $\text{lm}(f) = (\text{lc}(f_1), 0, \ldots, 0) \text{lt}(f_1)$, so that $\text{lt}(f) = \text{lt}(f_1)$. Since $G$ is a Gröbner basis for $I$, $f = \sum_{j = 1}^{k} c_j(t_j) g_j^*$ for some $c_j \in A \setminus \{0\}$, $t_j \in T$, $g_j^* \in G$ with $t_j \text{lt}(g_j^*) \leq \text{lt}(f) = \text{lt}(f_1)$. Then $f_1 = \sum_{j = 1}^{s} \pi_1(c_j(t_j)) \pi_1(g_j^*)$ for some $j_1, 1 \leq j_1 < \cdots < j_s \leq k$ with all $\pi_1(c_j(t_j))$ and $\pi_1(g_j^*)$ non-zero. We have $t_j \text{lt}(\pi_1(g_j^*)) \leq \text{lt}(\pi_1(g_j^*)) \leq \text{lt}(f_1) = \text{lt}(f_1)$, that is, $f_1 \in \text{Std}(G_1)$ and $G_1$ is a Gröbner basis for $\pi_1(I)$. \[\square\]
DEFINITION 3.2: Let $G_i \subset A_i[x] \setminus \{0\}$ for $i = 1, 2$. Then, $G_1 \cup G_2$, the join of $G_1$ and $G_2$ is the subset $G_1 \times \{0\} \cup \{0\} \times G_2$ of $A_1[x] \times A_2[x]$.

PROPOSITION 3.3. Let $I$ be a non-zero ideal of $A[x]$ and $G_i \subset A_i[x] \setminus \{0\}$ for $i = 1, \ldots, m$. Then $\kappa(G_1 \cup \cdots \cup G_m)$ is a Gröbner basis for $I$ if and only if $G_i$ is a Gröbner basis for $\pi_i(I)$ for $i = 1, \ldots, m$.

PROOF: Note first that $0 \not\in H = \kappa(G_1 \cup \cdots \cup G_m)$. We show $I \subset \text{Std}(H)$ if each $G_i$ is a Gröbner basis. Let $f \in I \setminus \{0\}$. Since $\pi_i(f) \in \pi_i(I) = \text{Std}(G_i)$, we can write $\pi_i(f) = \sum_{j=1}^{k_i} c_i^{(j)} t_i^{(j)} g_i^{(j)}$ for some $k_i \geq 1$, $c_i^{(j)} \in A_i \setminus \{0\}$, $t_i^{(j)} \in T$, $g_i^{(j)} \in G_i$ with $t_i^{(j)} \text{lt}(g_i^{(j)}) \leq \text{lt}(\pi_i(f)) \leq \text{lt}(f)$. Then

$$f = \kappa(\pi_1(f), \ldots, \pi_m(f))$$

$$= \kappa\left(\sum_{j=1}^{k_1} c_1^{(j)} t_1^{(j)} g_1^{(j)}, 0, \ldots, 0\right) + \cdots + \kappa\left(0, \ldots, 0, \sum_{j=1}^{k_m} c_m^{(j)} t_m^{(j)} g_m^{(j)}\right)$$

$$= \sum_{j=1}^{k_1} c_1^{(j)} t_1^{(j)} \kappa(g_1^{(j)}, 0, \ldots, 0) + \cdots + \sum_{j=1}^{k_m} c_m^{(j)} t_m^{(j)} \kappa(0, \ldots, 0, g_m^{(j)}).$$

Now $\kappa(0, \ldots, 0, g_i^{(j)}, 0, \ldots, 0) \in H$, $t_i^{(j)} \text{lt}(\kappa(0, \ldots, 0, g_i^{(j)}, 0, \ldots, 0)) = t_i^{(j)} \text{lt}(g_i^{(j)}) \leq \text{lt}(f)$ for $j = 1, \ldots, k_i$ and $i = 1, \ldots, m$, so that $f \in \text{Std}(H)$. The converse follows immediately from Proposition 3.1.

EXAMPLE 3.4. Let $f = 2x^2 + 3x + 1 \in \mathbb{Z}_6[x]$. We obtain a Gröbner basis for $\langle f \rangle$ as follows. The usual isomorphism $\chi : \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$ induces an isomorphism $\chi : \mathbb{Z}_6[x] \to (\mathbb{Z}_2 \times \mathbb{Z}_3)[x]$ and $\chi(f) = (0, 2)x^2 + (1, 0)x + (1, 1)$. We have $\pi\chi(f) = (x + 1, 2x^2 + 1) \in \mathbb{Z}_2[x] \times \mathbb{Z}_3[x]$ and clearly $(x + 1)$ and $\{x^2 + 2\}$ are Gröbner bases in $\mathbb{Z}_2[x]$ and $\mathbb{Z}_3[x]$ respectively. By Proposition 3.3, $\kappa(\{x+1\} \cup \{x^2+2\}) = \{(1,0)x + (1,0), (0,1)x^2 + (0,2)\}$ is a Gröbner basis for $\langle \chi(f) \rangle$ and we deduce that $\chi^{-1}\kappa(\{x+1\} \cup \{x^2+2\}) = \{3(x+1), 4x^2 + 2\}$ is a Gröbner basis for $\langle f \rangle$.

4. THE STRONG JOIN

First note that $G = \{3(x+1), 4x^2 + 2\}$ is not a strong Gröbner basis for $\langle G \rangle$ in Example 3.4: $x^2 - 3x + 2 = 4x^2 + 2 - 3x(x + 1) \in \langle G \rangle$, but 3 and 4 are not units in $\mathbb{Z}_6$, so there is no $g \in G$ such that $\text{lm}(g) | \text{lm}(x^2 - 3x + 2)$. We shall now show how to obtain a strong Gröbner basis in $A[x]$ from strong Gröbner bases in the $A_i[x]$.

PROPOSITION 4.1. If $G$ is a strong Gröbner basis for a non-zero ideal $I \subset A[x]$ then $\pi_i(G) \setminus \{0\}$ is a strong Gröbner basis for $\pi_i(I)$ in $A_i[x]$ for $i = 1, \ldots, m$.

PROOF: We take $i = 1$. Let $G$ be a strong Gröbner basis and let $f_1 \in \pi_i(I) \setminus \{0\} \subset A_1[x]$. Put $f = \kappa(f_1, 0, \ldots, 0)$ as in Proposition 3.1. There is a $g \in G$ such that
\[\text{lm}(g) \mid \text{lm}(f), \text{so } \pi_1(\text{lm}(g)) \neq 0, \text{so } \pi_1(g) \neq 0\] and \[\pi_1(\text{lm}(g)) = \text{lm}(\pi_1(g)).\] Since \[\text{lm}(\pi_1(g)) \mid \text{lm}(f_1)\] and \[\pi_1(g) \in \pi_1(G) \setminus \{0\}, \pi_1(G) \setminus \{0\}\] is a strong Gröbner basis for \(\langle \pi_1(G) \rangle = \pi_1(I)\).

**Definition 4.2.** Let \(G_i \subset A_i[x] \setminus \{0\}\) for \(i = 1, 2\). Then \(G_1 \uplus G_2\), the strong join of \(G_1, G_2\) is the subset \(G_1 \cup G_2 \cup \{t_1g_1, t_2g_2 : g_1 \in G_1, t_1 = \text{lcm}(\text{lt}(g_1), \text{lt}(g_2)) / \text{lt}(g_1)\}\) of \(A_1[x] \times A_2[x]\).

**Proposition 4.3.** \(\kappa(\langle G_1 \cup G_2 \rangle \cup G_3) = \kappa(G_1 \cup \kappa(G_2 \cup G_3))\).

**Proof:** Use the fact that in \(\kappa(G_1 \cup G_2), \text{lt}(\kappa(t_1g_1, t_2g_2)) = \text{lcm}(\text{lt}(g_1), \text{lt}(g_2))\) and that the lcm of leading terms is associative.

For \(m \geq 3\) we define \(\kappa(G_1 \cup \cdots \cup G_m)\) inductively to be \(\kappa(\langle G_1 \cup \cdots \cup G_{m-1} \rangle \cup G_m)\).

**Theorem 4.4.** Let \(I\) be a non-zero ideal in \(A[x]\) and \(G_i \subseteq \pi_i(I) \setminus \{0\}\) for \(i = 1, \ldots, m\). Then \(\kappa(G_1 \cup \cdots \cup G_m)\) is a strong Gröbner basis for \(I\) if and only if \(G_i\) is a strong Gröbner basis for \(\pi_i(I)\) for \(i = 1, \ldots, m\).

**Proof:** It suffices to prove the result for \(m = 2\), as the general case follows inductively. Assume that \(G_i\) is a strong Gröbner basis for \(\pi_i(I)\) for \(i = 1, 2\). We shall prove that for any \(f \in I \setminus \{0\}\) there is a \(g \in \kappa(G_1 \cup G_2)\) such that \(\text{lm}(g) \mid \text{lm}(f)\). For \(i = 1, 2\), put \(\pi_i(f) = f_i\). We consider several cases.

(i) \(f_1 \neq 0\) and \(f_2 = 0\). Then \(\text{lm}(f) = (\text{lcm}(f_1), 0) \text{lt}(f_1)\). Since \(G_1\) is a strong Gröbner basis for \(\pi_1(I)\), there is a \(g_1 \in G_1\) such that \(\text{lm}(g_1) \mid \text{lm}(f_1)\). Putting \(g = \kappa(g_1, 0) \in \kappa(G_1 \cup G_2)\), we have \(\text{lm}(g) = (\text{lcm}(g_1), 0) \text{lt}(g_1)\) and so \(\text{lm}(g) \mid \text{lm}(f)\).

(ii) \(f_1 \neq 0, f_2 \neq 0\) and \(\text{lt}(f_1) > \text{lt}(f_2)\): this is similar to case (i) since \(\text{lm}(f) = (\text{lcm}(f_1), 0) \text{lt}(f_1)\).

(iii) \(f_1 = 0\) and \(f_2 \neq 0\): this is analogous to case (i).

(iv) \(f_1 \neq 0, f_2 \neq 0\) and \(\text{lt}(f_1) < \text{lt}(f_2)\): see case (iii).

(v) \(f_1 \neq 0, f_2 \neq 0\) and \(\text{lt}(f_1) = \text{lt}(f_2)\). Then \(\text{lm}(f) = (\text{lcm}(f_1), \text{lcm}(f_2)) \text{lt}(f_1)\).

For the converse, assume that \(\kappa(G_1 \cup G_2)\) is a strong Gröbner basis for \(I\) and fix \(i \in \{0, 1\}\). Let \(H_i = \pi_i(G_1 \cup G_2) \setminus \{0\}\), which is a strong Gröbner basis for \(\pi_i(I)\) by Proposition 4.1.

From the definition of \(G_1 \cup G_2, G_i \subseteq H_i\) and any \(h_i \in H_i \setminus G_i\) is of the form \(h_i = t_ig_i\) for some \(t_i \in T\). Thus \(\langle G_i \rangle = \langle H_i \rangle\) and for any \(f \in \pi_i(I) = \langle G_i \rangle\), there is an \(h_i \in H_i\) and a \(g_i \in G_i\) such that \(\text{lm}(g_i) \mid \text{lm}(h_i)\mid \text{lm}(f)\). Hence \(G_i\) is a strong Gröbner basis for \(\pi_i(I)\).

**Theorem 4.4** thus gives an iterative algorithm for computing a strong Gröbner basis in \(A[x]\), provided we have an algorithm (SGB_i say) that computes a strong Gröbner
basis in each $A_i[x]$ for $1 \leq i \leq m$. The SGB$_i$ can be done in parallel and the complexity of computing $\kappa(G_1 \cup \cdots \cup G_m)$ from $G_1, \ldots, G_m$ is $O\left(\prod_{i=1}^{m} |G_i|\right)$. The latter can be improved by first minimising each $G_i$. We note that $\kappa(G_1 \cup \cdots \cup G_m)$ may not be minimal, so in general, a further minimisation step will be necessary. We formalise this as follows.

**Algorithm 4.5.**

Input: $F \subseteq A[x] \setminus \{0\}$, $F$ finite, $A = \prod_{i=1}^{m} A_i$ and we have an algorithm strong SGB$_i$ which computes a strong Grobner basis in $A_i[x]$ for $1 \leq i \leq m$.

Output: $G$, a minimal strong Grobner basis for $(F)$.

**begin**

for $i \leftarrow 1$ to $m$ do

$G_i \leftarrow$ SGB$_i(\pi_i(F))$

minimise $G_i$

end for

$G \leftarrow G_1$

for $i \leftarrow 2$ to $m$ do

$G \leftarrow \kappa(G \cup G_i)$

end for

minimise $G$

return $(G)$

**end**

Finally, we note that in computing $G = \kappa(G_1 \cup \cdots \cup G_m)$ we can first compute $\text{Im}(G)$ to preselect the polynomials of $G$ belonging to a minimal strong Grobner basis. Only these polynomials need then be computed in full. See Example 5.3.

5. **The Principal Ideal Ring Case**

In this final section, we restrict $A$ to be a principal ideal ring. We give an alternative proof that any non-zero ideal of $A[x]$ has a strong Grobner basis and conclude with some examples.

**Corollary 5.1.** (Compare [3, Algorithm 6.4].) If $A$ is a principal ideal ring then any non-zero ideal of $A[x]$ has a strong Grobner basis.

**Proof:** We have $A \cong \prod_{i=1}^{m} A_i$, where each $A_i$ is a principal ideal domain or a finite-chain ring by [4, Theorem 33, Section 15, Chapter 4]. We can obtain a strong Gröbner basis over a principal ideal domain using for example, [2, Algorithm D-Grobner, p. 461]). Over a finite-chain ring any Gröbner basis is a strong Gröbner basis by [3, Proposition...
3.9, so it suffices to compute a Gröbner basis, using for example [3, Algorithm 6.1] which computes a Gröbner basis over any principal ideal ring. Hence by Theorem 4.4 we can compute a strong Gröbner basis for any non-zero ideal of \( A[x] \).

An improved strong Gröbner basis algorithm for finite-chain rings is described in the Appendix.

**Example 5.2.** (Compare [3, Example 7.3].) Let \( F = \{2x^2 + 3x + 1\} \subset \mathbb{Z}_6[x] \) as in Example 3.4. We obtain a strong Gröbner basis for \( \langle F \rangle \) by applying Algorithm 4.5 to \( \chi(F) \). Firstly, \( \pi_\chi(F) = (x + 1, 2x^2 + 1) \) and trivially \( \{x + 1\} \) and \( \{x^2 + 2\} \) are minimal strong Gröbner bases in \( \mathbb{Z}_2[x] \) and \( \mathbb{Z}_3[x] \) respectively. We have \( \{x + 1\} \cup \{x^2 + 2\} = \{(x + 1, 0), (0, x^2 + 2), (x^2 + x, x^2 + 2)\} \) and \( G = \kappa(\{x + 1\} \cup \{x^2 + 2\}) = \{(1, 0)x + (1, 0), (0, 1)x^2 + (0, 2), (1, 1)x^2 + (1, 0)x + (0, 2)\} \) is a strong Gröbner basis for \( \langle \chi(F) \rangle \). We minimise \( G \) to obtain \( H = \{(1, 0)x + (1, 0), (1, 1)x^2 + (1, 0)x + (0, 2)\} \). Finally \( \chi^{-1}(H) = \{x^2 + 3x + 2, 3(x + 1)\} \) is a minimal strong Gröbner basis for \( \langle F \rangle \).

In the next example, we use Algorithm SGB-FCR of the Appendix.

**Example 5.3.** As in [1, Example 4.2.12], let \( F = \{4xy + x, 3x^2 + y\} \subset \mathbb{Z}_{20}[x, y] \). Using lexicographic order with \( x > y \), they obtain a Gröbner basis \( G' = \{3x^2 + y, 4xy + x, 5x, Ay^2 + y, I5y\} \) via the method of syzygy modules. This is not a strong Gröbner basis since \( xy - x = 5xy - (4xy + x) \) is not strongly reducible with respect to \( G' \). Likewise for \( y^2 - y = 5y^2 - (4y^2 + y) \). (We note that [3, Corollary 5.12] shows that \( \{x^2 + 7y, xy - x, 5x, y^2 - y, 5y\} \) is a minimal strong Gröbner basis.) Instead, we compute a strong Gröbner basis for \( \langle F \rangle \) from scratch using the usual isomorphism \( \chi : \mathbb{Z}_{20} \rightarrow \mathbb{Z}_4 \times \mathbb{Z}_5 \) and Algorithm 4.5. We have \( \pi_\chi(F) = \{(x, 4xy + x), (3x^2 + y, 3x^2 + y)\} \subset \mathbb{Z}_4[x] \times \mathbb{Z}_5[x] \).

We obtain \( G_1 = \{x, y\} \) as a strong Gröbner basis for \( \{x, 3x^2 + y\} \) using Algorithm SGB-FCR; alternatively \( G_1 \) is a Gröbner basis by [3, Theorem 4.10] and it is a (minimal) strong Gröbner basis by [3, Proposition 3.9]. In \( \mathbb{Z}_5[x, y] \), we work with \( \{xy + 4x, x^2 + 2y\} \).

A minimal strong Gröbner basis is \( G_2 = \{xy + 4x, x^2 + 2y, y^2 + 4y\} \). First computing \( \text{Im}(\kappa(G_1 \cup G_2)) \) yields \( H = \{(1, 1)x^2 + (0, 2)y, (1, 1)xy + (0, 4)x, (1, 0)x, (1, 1)y^2 + (0, 4)y, (1, 0)y\} \) as a minimal strong Gröbner basis for \( \langle \chi(F) \rangle \). So \( \chi^{-1}(H) = \{x^2 + 12y, xy + 4x, 5x, y^2 + 4y, 5y\} \) is a minimal strong Gröbner basis for \( \langle F \rangle \).

### 6. Appendix

We derive an algorithm for computing a strong Gröbner basis over a finite-chain ring \( R \) from [3, Algorithm 6.1], using the definitions and notation of [3, Sections 3.1, 4.3]. In particular, for \( f, f_1, f_2 \in R[x] \setminus \{0\} \) and a finite set \( G \) of non-zero polynomials, \( \text{Spol}(f_1, f_2), \text{Apol}(f), \text{Rem}(f, G), \text{SRem}(f, G) \) denote the set of S-polynomials of \( f_1, f_2 \), the set of A-polynomials of \( f \), the remainder and the strong remainder of \( f \) with respect to \( G \), respectively.
Algorithm 6.1 of [3] computes a Gröbner basis over any principal ideal ring, so in particular over $R$. We know that any Gröbner basis over $R$ is a strong Gröbner basis by [3, Proposition 3.9]. We also know that $f$ is reducible with respect to $G$ if and only if $f$ is strongly reducible with respect to $G$ by [3, Proposition 3.2], so that $\text{SRem}(f, G) \subseteq \text{Rem}(f, G)$. So over $R$ we only need to use strong reduction, which is more efficient than reduction. The improved algorithm follows.

**Algorithm 6.1.**

$G \leftarrow \text{SGB-FCR}(F)$

**Input:** $F$ a finite subset of $R[x] \setminus \{0\}$, where $R$ is a computable finite-chain ring.

**Output:** $G$ a strong Gröbner basis for $\langle F \rangle$.

**Notes:**
- $B$ is the set of pairs of polynomials in $G$ whose S-polynomials still have to be computed.
- $C$ is the set of polynomials in $G$ whose A-polynomials still have to be computed.

**begin**

$G \leftarrow F$

$B \leftarrow \{(f_1, f_2) : f_1, f_2 \in G, f_1 \neq f_2\}$

$C \leftarrow F$

while $B \cup C \neq \emptyset$ do

if $C \neq \emptyset$ then

select $f$ from $C$

$C \leftarrow C \setminus \{f\}$

compute $h \in \text{Apol}(f)$

else

select $(f_1, f_2)$ from $B$

$B \leftarrow B \setminus \{(f_1, f_2)\}$

compute $h \in \text{Spol}(f_1, f_2)$

end if

compute $g \in \text{SRem}(h, G)$

if $g \neq 0$ do

$B \leftarrow B \cup \{(g, f) : f \in G\}$

$C \leftarrow C \cup \{g\}$

$G \leftarrow G \cup \{g\}$

end if

end while

return($G$)

**end**
REFERENCES


