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A Variant of Lehmer's Conjecture, II: The CM-case

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Abstract. Let f be a normalized Hecke eigenform with rational integer Fourier coefficients. It is an interesting question to know how often an integer n has a factor common with the n-th Fourier coefficient of f. It has been shown in previous papers that this happens very often. In this paper, we give an asymptotic formula for the number of integers n for which $(n, a(n)) = 1$, where $a(n)$ is the n-th Fourier coefficient of a normalized Hecke eigenform f of weight 2 with rational integer Fourier coefficients and having complex multiplication.

1 Introduction

The arithmetic of the Fourier coefficients of modular forms is intriguing and mysterious. For instance, consider the cusp form of Ramanujan:

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}.$$

The coefficients $\tau(n)$ have received extensive arithmetic scrutiny following the ground-breaking investigations of Ramanujan himself [11]. Here, we have one of the oft-quoted conjectures in number theory attributed to Lehmer [3, 4], which asserts that $\tau(p) \neq 0$, where p is a prime. Equivalently, for any $n \geq 1$, $\tau(n) \neq 0$. In general, proving such non-vanishing of all Fourier coefficients of a modular form is delicate and difficult. A more accessible problem is to study the arithmetic density of the non-zero coefficients. We refer to [7, 16] for results of this type.

In a recent work [10], a variant of Lehmer's conjecture was considered. More precisely, let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be the Fourier expansion of a normalized eigenform and suppose that the $a(n)$'s are rational integers for all n. Then it is natural to ask whether

$$\#\{ p \leq x \mid a(p) \equiv 0 \pmod{p} \} = o(\pi(x)).$$

Heuristically, if the weight is > 2, the number of such primes up to x may grow like $\log\log x$ though we do not even know if these are of density zero. In general, denoting (a, b) to be the greatest common divisor of a and b, one can ask whether

$$\#\{ n \leq x \mid (n, a(n)) \neq 1 \} = o(x),$$

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an assertion that turns out to be false. As mentioned in [10], the correct question in this context is the opposite assertion, namely whether it is true that

$$\#\{\, n \le x \mid (n, a(n)) = 1 \,\} = o(x).$$

This variant of Lehmer's conjecture appears to be amenable to study. In contrast to the prime case, $a(n)$ almost always has a factor in common with n. In particular, the following result was proved in [10].

Let us set $L_2(x) = \log\log x$ and for each $i \ge 3$, define $L_i(x) = \log L_{i-1}(x)$. In any occurence of an $L_i(x)$, we always assume that x is sufficiently large so that $L_i(x)$ is defined and positive.

Theorem 1.1 ([10]) *For a normalized eigenform f as above with rational integer Fourier coefficients $\{a(n)\}$, one has*

$$\#\{\, n \le x \mid (n, a(n)) = 1 \,\} \ll \frac{x}{L_3(x)}.$$

In the same paper, it was anticipated that if f has complex multiplication (CM), a stronger result should hold. The ethos of our present work is to vindicate this anticipation, at least in the case that f has weight 2. A modular form f is said to have CM if there is an imaginary quadratic field K and a Hecke character Ψ of K with conductor $\mathfrak{m}$ so that

$$f(z) = \sum_{\substack{\mathfrak{a} \\ (\mathfrak{a},\mathfrak{m})=1}} \Psi(\mathfrak{a}) e^{2\pi i \mathbb{N}(\mathfrak{a}) z}.$$

Here, the sum is over integral ideals $\mathfrak{a}$ of the ring of integers of K that are coprime to $\mathfrak{m}$, and $\mathbb{N}(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$. Thus

$$a(n) = \sum_{\substack{\mathbb{N}(\mathfrak{a})=n, \\ (\mathfrak{a},\mathfrak{m})=1}} \Psi(\mathfrak{a}).$$

In particular for a prime p, $a(p) = 0$ if p does not split in K and $a(n) = 0$ if $p || n$ (*i.e.*, $p \mid n$ but $p^2 \nmid n$) for some prime p for which $a(p) = 0$. It is well known that if we are given a set S of primes of positive density, the set of integers n with the property that $p||n$ for some $p \in S$ has density one. Thus $a(n) = 0$ for a set of n of density one. More precisely, let us set

$$M_{f,1}(x) = \#\{\, n \le x \mid a(n) \ne 0 \,\}.$$

Then we show that there is a constant u_f so that

$$M_{f,1}(x) = (1 + o(1)) \frac{u_f x}{\sqrt{\pi} (\log x)^{\frac{1}{2}}}.$$

We also show that there is a constant $\omega_f > 0$ so that

$$\prod_{\substack{p<x \\ a(p)\ne 0}} \Bigl(1 - \frac{1}{p}\Bigr) \sim \frac{\omega_f}{(\log x)^{\frac{1}{2}}},$$

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where $\omega_f = \mu_f \mu_2 \mu_3$,

$$\mu_2 = \begin{cases} \frac{1}{2} & \text{if } a(2) \neq 0, \\ 1 & \text{otherwise} \end{cases} \qquad \mu_3 = \begin{cases} \frac{2}{3} & \text{if } a(3) \neq 0, \\ 1 & \text{otherwise,} \end{cases}$$

and μ_f is given in Proposition 3.3. Finally, the main result of our paper is the following theorem.

Theorem 1.2 *Let f be a normalized eigenform of weight 2 with rational integer Fourier coefficients $\{a(n)\}$. If f is of CM-type, then there is a constant $U_f > 0$ so that*

$$\#\{n \le x \mid (n, a(n)) = 1\} = (1 + o(1)) \frac{U_f x}{\sqrt{\pi}(L_3(x) \log x)^{\frac{1}{2}}}.$$

The constant is given explicitly in terms of f during the course of the proof.

Our methods are based on the techniques of Erdös [1], Serre [14, 15] and those of Ram Murty and the second author [5, 6, 8–10]. Throughout this article, p and q will denote primes.

2 Divisibility of Fourier Coefficients

Let f be a normalized Hecke eigenform of weight 2 for $\Gamma_0(N)$ with CM and let K be the imaginary quadratic field associated with f. The Fourier expansion of f at infinity is given by

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z},$$

where we are assuming that the $a(n)$'s are rational integers.

For any prime p, let $\mathbb{Z}_p$ denote the ring of p-adic integers. By Eichler–Shimura–Deligne and since the Fourier coefficients of f are in $\mathbb{Z}$, there is a continuous representation

$$\rho_{p,f} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}_p).$$

This representation is unramified outside the primes dividing Np. This means that for any prime q that does not divide Np and for any prime $\mathfrak{q}$ of $\bar{\mathbb{Q}}$ over q, $\rho_{p,f}(\mathrm{Frob}_{\mathfrak{q}})$ makes sense. We note that while $\rho_{p,f}(\mathrm{Frob}_{\mathfrak{q}})$ does depend on the choice of $\mathfrak{q}$ over q, its characteristic polynomial depends only on the conjugacy class of $\rho_{p,f}(\mathrm{Frob}_{\mathfrak{q}})$(hence only on q) and is given by

$$T^2 - a(q)T + q. \tag{2.1}$$

We consider the reduction of the above representation modulo p

$$\bar{\rho}_{p,f} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_p).$$

The fixed field of the kernel of this representation determines a number field L that is a Galois extension of $\mathbb{Q}$ with group the image of $\bar{\rho}_{p,f}$.

We need to enumerate primes q as above for which $a(q) \equiv 0 \pmod{p}$. For this purpose, the following version of a theorem of Schaal [13] is useful.

Theorem 2.1 *Let $\mathfrak{f}$ be an integral ideal of a number field K of degree $n = r_1 + 2r_2$, where r_1, r_2 denote the number of real and complex embeddings, respectively. Also let $\beta \in K$ denote an integer with $(\beta, \mathfrak{f}) = 1$. Let $M_1, \dots, M_{r_1}$ be nonnegative and $P_1, \dots, P_n$ be positive real numbers with $P_l = P_{l+r_2}$, $l = r_1 + 1, \dots, r_1 + r_2$ and $P = P_1 \dots, P_n$. Consider the number B of integers $\omega \in K$ subject to the conditions*

$$\omega \equiv \beta \,(\mathrm{mod}\ \mathfrak{f}), \quad (\omega) \text{ a prime ideal}$$

$$M_l \le \omega^{(l)} \le M_l + P_l, \quad l = 1, \dots, r_1$$

for real conjugates of ω and for complex conjugates

$$|\omega^{(l)}| \le P_l, \quad l = r_1 + 1, \dots, n.$$

If $P \ge 2$ and the norm $\mathbb{N}\mathfrak{f}$ satisfies

$$\mathbb{N}\mathfrak{f} \le \frac{P}{(\log P)^{(2r_1+2r_2-2+2/n)}},$$

then one has

$$B \ll \frac{P}{\phi(\mathfrak{f}) \log \frac{P}{\mathbb{N}\mathfrak{f}}} \left\{ 1 + O\left(\log \frac{P}{\mathbb{N}\mathfrak{f}} \right)^{-1/n} \right\},$$

where the implied constants depend only on K and not on $\mathfrak{f}$.

Define

$$\pi^*(x, p) := \#\big\{ q \le x \mid a(q) \equiv 0 \,(\mathrm{mod}\ p),\ a(q) \ne 0 \big\}.$$

Now suppose that q is a prime that splits in K, say $q\mathcal{O}_K = \mathfrak{q}_1\mathfrak{q}_2$ and that $\pi_q, \bar{\pi}_q$ are roots of the characteristic polynomial (2.1). Then

$$a(q) = \pi_q + \bar{\pi}_q \quad \text{and} \quad q = \pi_q \bar{\pi}_q.$$

Also if $a(q) \ne 0$, then $\pi_q \in \mathcal{O}_K$ and $|\pi_q| = q^{1/2}$. If $a(q) \equiv 0 \,(\mathrm{mod}\ p)$, then $\pi_q^2 \equiv -q \,(\mathrm{mod}\ p)$. Thus, if in addition $q \equiv a \,(\mathrm{mod}\ p)$, then $\pi_q \,(\mathrm{mod}\ p)$ has a bounded number of possibilities (at most 4 in fact). Also, the ideal (π_q) is prime as $(\pi_q)(\bar{\pi}_q) = (q)$. Thus,

$$\sum_{\substack{q \le x \\ \pi_q \equiv \alpha \,(\mathrm{mod}\ p) \\ q \equiv a \,(\mathrm{mod}\ p) \\ q\mathcal{O}_K = \mathfrak{q}_1\mathfrak{q}_2}} 1 \le \sum_{\substack{\omega \in \mathcal{O}_K \\ (\omega) \text{ is prime} \\ |\omega| \le \sqrt{x} \\ \omega \equiv \alpha \,(\mathrm{mod}\ p)}} 1.$$

Applying Theorem 2.1 with $\mathfrak{f} = (p)$, the right-hand side is seen to be

$$\ll \frac{x}{p^2 \log \frac{x}{p^2}}$$

for $p^2 \le x / \log x$.

Now, summing over all $a \,(\mathrm{mod}\ p)$ yields the following proposition.

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Proposition 2.2 *Let f be a modular form as above. Then for $p^2 \leq x/\log x$, we have*

$$\pi^*(x,p) \ll \frac{x}{p\log\frac{x}{p^2}}.$$

Now using Proposition 2.2 and partial summation, we see that for primes $p \leq \sqrt{x/\log x}$,

$$\sideset{}{^*}\sum_{\substack{p^2\log p\leq q\leq x\\ a(q)\equiv 0 \pmod p}} \frac{1}{q} \ll \frac{1}{p}\int_{p^2\log p}^{x} \frac{dt}{t\log\frac{t}{p^2}} \ll \frac{1}{p}\log\log\frac{x}{p^2},$$

where $\sum^*_{y\leq q\leq x}$ means that the summation is over all primes $y \leq q \leq x$ for which $a(q) \neq 0$. Thus, we have the following result.

Proposition 2.3 *Let f be a modular form as above and also let $p^2 \leq x/\log x$ be a fixed prime. Then one has*

$$\sideset{}{^*}\sum_{\substack{p^2\log p\leq q\leq x\\ a(q)\equiv 0 \pmod p}} \frac{1}{q} \ll \frac{1}{p}L_2\left(\frac{x}{p}\right),$$

where $\sum^*_{y\leq q\leq x}$ means that the summation is over all primes $y \leq q \leq x$ for which $a(q) \neq 0$.

Remark 2.4 We note that the contribution from the remaining primes $q \leq p^2\log p$ is

$$\sideset{}{^*}\sum_{\substack{q\leq p^2\log p\\ a(q)\equiv 0 \pmod p}} \frac{1}{q} \ll \frac{L_2(p)}{\log p}.$$

However, we shall not make use of this estimate.

3 Vanishing of $a(p)$

Let E be the elliptic curve defined over $\mathbb{Q}$ corresponding to the modular form f of level $N = N_E$. As f is of CM-type corresponding to the imaginary quadratic field K, we know that E has CM by an order in K. A prime p is supersingular for E if E has good reduction at p and its reduction E_p has multiplication by an order in a quaternion division algebra. It is well known that a prime p of good reduction is supersingular if and only if

$$|E(\mathbb{F}_p)| \equiv 1 \pmod p. \tag{3.1}$$

In particular, the set of primes supersingular for E only depends on the isogeny class of E. For $p \geq 5$, (3.1) is equivalent to the condition $a(p) = 0$.

Let $\pi_E(x)$ denote the number of primes $p \leq x$ such that p is a supersingular prime for E. We know that $\pi_E(x) \geq \pi_K^-(x)$, where $\pi_K^-(x)$ denotes the number of primes $p \leq x$ that remain prime in K. In fact, the following more precise result is due to Deuring (see [2, Ch. 13, Thm. 12]).

Proposition 3.1 *Let E be an elliptic curve defined over $\mathbb{Q}$ with multiplication by an order in an imaginary quadratic field K. Let p be a prime of good reduction for E. Then p is supersingular for E if and only if p ramifies or remains prime in K.*

In particular, this implies the following result.

Proposition 3.2 *Suppose that $p \geq 5$. With E as in the previous proposition, we have $a(p) = 0$ if and only if p is a prime of bad reduction or p does not split in K.*

As E has complex multiplication, it has additive reduction at primes of bad reduction and thus $a(p) = 0$. The rest follows from Deuring's result.

Finally, we record a result that will be useful in establishing the main result.

Proposition 3.3 *There is a constant $\mu_f > 0$ so that*

$$\prod_{\substack{5\leq p<z \\ a(p)\neq 0}} \left(1 - \frac{1}{p}\right) = \frac{\mu_f}{(\log z)^{\frac{1}{2}}} + O_f\left(\frac{1}{(\log z)^{3/2}}\right).$$

Proof Using Rosen [12, Thm. 2], we have

$$\prod_{\mathbb{N}\mathfrak{p}\leq z} \left(1 - \frac{1}{\mathbb{N}\mathfrak{p}}\right)^{-1} = e^{\gamma}\alpha_K \log z + O_K(1).$$

Here, the product is over primes $\mathfrak{p}$ of K and α_K is the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$. Note that $\alpha_K = L(1, \chi_K)$, where χ_K is the quadratic character defining K and $L(s, \chi_K)$ is the associated L-function. It follows that

$$\prod_{\mathbb{N}\mathfrak{p}\leq z} \left(1 - \frac{1}{\mathbb{N}\mathfrak{p}}\right) = \frac{e^{-\gamma}L(1,\chi_K)^{-1}}{\log z} + O_K\left(\frac{1}{(\log z)^2}\right).$$

Thus,

$$\prod_{\substack{p\leq z \\ p \text{ splits in } K}} \left(1 - \frac{1}{p}\right) = \frac{\beta_K}{(\log z)^{\frac{1}{2}}} + O_K\left(\frac{1}{(\log z)^{3/2}}\right),$$

where

$$\beta_K = e^{-\gamma/2}L(1,\chi_K)^{-1/2} \prod_{p \text{ inert}} \left(1 - \frac{1}{p^2}\right)^{-\frac{1}{2}} \prod_{p\mid d_K} \left(1 - \frac{1}{p}\right)^{-\frac{1}{2}}.$$

By Proposition 3.2, for $p \geq 5$, we have $a(p) \neq 0$ if and only if p is a prime of good reduction and splits in K. This proves the result with

$$\mu_f = \beta_K \prod_{\substack{p \text{ splits} \\ p\mid 6N}} \left(1 - \frac{1}{p}\right)^{-1}.$$

■

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4 The Number of Non-Zero Fourier Coefficients

We begin by considering a slightly more general setting as in Serre [15, §6], which parts of this section follow closely. Let $n \mapsto a(n)$ be a multiplicative function and define the multiplicative function

$$a^0(n) = \begin{cases} 1 & \text{if } a(n) \neq 0, \\ 0 & \text{if } a(n) = 0. \end{cases}$$

We want the asymptotic behaviour of

$$M_{a,d}(x) := \#\{n \leq x \mid a(n) \neq 0, \ \ d|n\} = \sum_{dn \leq x} a^o(dn),$$

for any positive integer d.

4.1 The Case $d = 1$

Consider the Dirichlet series

$$\phi(s) = \sum_n \frac{a^0(n)}{n^s} = \prod_p \phi_p(s),$$

where

$$\phi_p(s) = \sum_{m=0}^{\infty} a^0(p^m) p^{-ms}.$$

Let $P_a(x) = \#\{p \leq x \mid a(p) = 0\}$. Suppose we know that

$$P_a(x) = \lambda \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^{1+\delta}}\Big) \tag{4.1}$$

for some $\delta > 0$ and $\lambda < 1$. Then

$$\sum_{p \leq x} a^0(p) = (1 - \lambda) \frac{x}{\log x} + O\Big(\frac{x}{(\log x)^{1+\delta}}\Big)$$

and

$$\sum_p \frac{a^0(p)}{p^s} = (1 - \lambda) \log\left(\frac{1}{s-1}\right) \ + \ \epsilon_1(s),$$

where $\epsilon_1(s)$ is analytic in a neighbourhood of $s = 1$. Moreover,

$$\log(\phi(s)) = \sum_p \log(\phi_p(s)) = \sum_p \frac{a^0(p)}{p^s} + \epsilon_2(s),$$

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where $\epsilon_2(s)$ is also analytic in a neighbourhood of $s = 1$. Thus,

$$\log(\phi(s)) = (1-\lambda)\log\Big(\frac{1}{s-1}\Big) + \epsilon_3(s)$$

and

$$\phi(s) = \frac{e^{\epsilon_3(s)}}{(s-1)^{1-\lambda}}.$$

A set of primes P is called *frobenien* (in the sense of Serre [14, Thm. 3.4]) if there is a finite Galois extension $K/\mathbb{Q}$ and a conjugacy-stable subset $H \subseteq G = \mathrm{Gal}(K/\mathbb{Q})$ such that for p sufficiently large, $p \in P$ if and only if $\sigma_p(K/\mathbb{Q}) \subseteq H$. Here $\sigma_p(K/\mathbb{Q})$ denotes the conjugacy class of Frobenius automorphisms associated to p. If the set of primes enumerated by P_a is frobenien, we have

$$M_{a,1}(x) = \frac{u_a\, x}{\Gamma(1-\lambda)(\log x)^{\lambda}} + O\Big(\frac{x}{(\log x)^{\lambda+1}}\Big), \tag{4.2}$$

where $u_a = e^{\epsilon_3(1)}$. Moreover, in the case that $\lambda = 0$, if one has the additional hypothesis that

$$\sum_{a(p)=0} \frac{1}{p} < \infty, \tag{4.3}$$

then [15, p. 167] states that

$$u_a = \prod_{a(p)=0}\Big(1 - \frac{1}{p}\Big). \tag{4.4}$$

Remark 4.1 If we do not assume that P_a enumerates a frobenien set of primes, we can still invoke a Tauberian theorem to get an asymptotic formula

$$M_{a,1}(x) \sim \frac{u_a x}{\Gamma(1-\lambda)(\log x)^{\lambda}}.$$

In the next two subsections, we consider those arithmetic functions for which P_a is frobenien.

4.2 Convolution with a Secondary Function

Now consider another function $n \mapsto b(n)$ with the following properties:

(i) There is an integer d so that $b(n) \neq 0$ implies that all prime divisors of n are prime divisors of d.

(ii) We have $|b(n)| \leq 4^{\nu(n)}$, where $\nu(n)$ is the number of distinct prime divisors of n.

Let us set

$$\xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

We see that

$$\sum_{m\le x} |b(m)| \le \sum_{p|m \Rightarrow p|d} 4^{\nu(m)}(x/m)^{1/4} = x^{1/4} \prod_{p|d}\Big(1 + \frac{4}{p^{1/4}-1}\Big).$$

We observe that

$$\prod_{p|d}\Big(1 + \frac{4}{p^{1/4}-1}\Big) \ll 2^{\nu(d)},$$

and so

$$\sum_{m\le x} |b(m)| \ll x^{1/4} 2^{\nu(d)}. \tag{4.5}$$

Moreover, using (4.5), we have

$$\sum_{z<m<2z} \frac{|b(m)|}{m} \ll z^{-3/4} 2^{\nu(d)}. \tag{4.6}$$

Let $c = a^0 * b$ be the Dirichlet convolution and consider the function

$$\psi(s) = \sum_n \frac{c(n)}{n^s} = \phi(s)\xi_d(s).$$

Then, we have

$$\sum_{n\le x} c(n) = \sum_{m\le x} b(m) \sum_{r\le x/m} a^0(r).$$

The contribution from terms with $\sqrt{x} \le m \le x$ is

$$\le x \sum_{\sqrt{x}\le m\le x} \frac{|b(m)|}{m}.$$

Decomposing the sum into dyadic intervals $U < m \le 2U$ and using (4.6) show that the summation is $O(x^{-3/8}2^{\nu(d)})$ and hence the whole expression is $O(x^{5/8}2^{\nu(d)})$. Assuming that (4.2) holds (that is, that P_a enumerates a frobenien set of primes), we have

$$\sum_{n\le x} c(n) = \tag{4.7}$$

$$\sum_{m\le\sqrt{x}} b(m)\Big\{\Big(\frac{u_a}{\Gamma(1-\lambda)} + O\Big(\frac{1}{\log x}\Big)\Big)\frac{x}{m(\log x/m)^{\lambda}}\Big\} + O\big(x^{5/8}2^{\nu(d)}\big).$$

Note that

$$\Big(\log\frac{x}{m}\Big)^{-\lambda} = (\log x)^{-\lambda} + O((\log m)(\log x)^{-\lambda-1}).$$

Using this and (4.6), the right-hand side of (4.7) is equal to

$$\Big(\frac{u_a}{\Gamma(1-\lambda)}+O\Big(\frac{1}{\log x}\Big)\Big)\frac{x}{(\log x)^{\lambda}}\Big(\xi_d(1)+O\big(x^{-3/8}(\log x)^{-1}2^{\nu(d)}\big)\Big)+O\big(x^{5/8}2^{\nu(d)}\big).$$

Summarizing this discussion, we have proved the following.

Proposition 4.2 *We have*

$$\sum_{n\leq x} c(n) = \frac{u_a\xi_d(1)}{\Gamma(1-\lambda)}\frac{x}{(\log x)^{\lambda}} + O\Big(\frac{x2^{\nu(d)}}{(\log x)^{\lambda+1}}\Big)$$

uniformly in d.

4.3 The Case of General d

Consider the Dirichlet series

$$\psi_d(s) = \sum_n \frac{a^0(dn)}{n^s}.$$

We may write it as

$$\Big(\sum_{\substack{n_1=1\\ p\mid n_1\Rightarrow p\mid d}}^{\infty}\frac{a^0(dn_1)}{n_1^s}\Big)\Big(\sum_{\substack{n_2=1\\ (n_2,d)=1}}^{\infty}\frac{a^0(n_2)}{n_2^s}\Big).$$

Thus, we see that $\psi_d(s) = \phi(s)\xi_d(s)$, where

$$\phi(s) = \sum_{n_3=1}^{\infty}\frac{a^0(n_3)}{n_3^s}$$

as in Section 4.1 and

$$\xi_d(s) = \Big(\sum_{\substack{n_1=1\\ p\mid n_1\Rightarrow p\mid d}}^{\infty}\frac{a^0(dn_1)}{n_1^s}\Big)\Big(\sum_{\substack{n_2=1\\ p\mid n_2\Rightarrow p\mid d}}^{\infty}\frac{a^0(n_2)}{n_2^s}\Big)^{-1}.$$

We have a factorization

$$\xi_d(s) = \prod_{p\mid d}\xi_{p,d}(s),$$

where

$$\xi_{p,d}(s) = \Big(\sum_{m=0}^{\infty}a^0(p^{m+\mathrm{ord}_p d})p^{-ms}\Big)\Big(\sum_{m=0}^{\infty}a^0(p^m)p^{-ms}\Big)^{-1}.$$

We record the following estimate for later use.

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Lemma 4.3 $\xi_{p,d}(1) = a^0(p^{\mathrm{ord}_p d}) + O\big(\frac{1}{p}\big)$.

We write

$$\xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

and suppose that $\xi_d(s)$ (that is, the coefficients $\{b(n)\}$) satisfies the conditions of Section 4.2. Recall that

$$M_{a,d}(x) := \#\{n \le x \mid a(n) \ne 0, d|n\}.$$

We have

$$M_{a,d}(x) = \sum_{dn \le x} a^0(dn)$$

and by Proposition 4.2, we deduce the following.

Proposition 4.4 *If ξ_d satisfies the hypotheses of Section 4.2, then we have*

$$M_{a,d}(x) = \frac{u_a \xi_d(1)}{\Gamma(1-\lambda)} \frac{x/d}{(\log x/d)^{\lambda}} + O\Big(\frac{x 2^{\nu(d)}}{d(\log x/d)^{\lambda+1}}\Big)$$

uniformly in d.

4.4 Application to Modular Forms

Now let f be a normalized Hecke eigenform of weight $k \ge 2$ and let $a(n) = a_f(n)$ denote the n-th Fourier coefficient of f. In this case, let us denote the constant u_a of the previous paragraph by u_f, and the function $M_{a,d}$ by $M_{f,d}$.

In some cases, u_f can be made explicit. If f does not have CM and $d = 1$, then condition (4.3) holds (see [8]) and so u_f is given by (4.4). We shall discuss the case that f has CM.

In this case the assumption (4.1) made on $P_a(x)$ is true with $\lambda = \frac{1}{2}$ and so

$$M_{f,1}(x) \sim \frac{u_f x}{\sqrt{\pi}(\log x)^{\frac{1}{2}}}.$$

(Here, we have used the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.) If we assume that f is of weight 2 and has integer Fourier coefficients, then by Proposition 3.2, the "frobenien" condition is satisfied apart from a finite set of primes. If we can show that the conditions of Section 4.2 are satisfied, then specializing Proposition 4.4 to this case, we can deduce the following.

Proposition 4.5 *We have*

$$M_{f,d}(x) = \#\{n \le x \mid a_f(n) \ne 0, d|n\} = \frac{u_f x \xi_d(1)}{\sqrt{\pi} d\, (\log x/d)^{\frac{1}{2}}} + O\Big(\frac{x 2^{\nu(d)}}{d(\log x/d)^{3/2}}\Big)$$

where u_f is a constant depending on f.

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We begin with some preliminary results. Let us set $i_f(p)$ to be the least integer $i \geq 1$ for which $a_f(p^i) = 0$. If for a given p, there is no such i, then let us set $i_f(p) = 0$. In particular, if p divides the level N of f, then $i_f(p) = 1$.

Lemma 4.6 *For $p \nmid N$, we have*

(i) $i_f(p) \in \{0, 1, 2, 3, 5\}$.
(ii) *If $i_f(p) > 0$, then $a_f(p^i) = 0$ for every $i > 0$ with*

$$i + 1 \equiv 0 \,(\mathrm{mod}\ i_f(p) + 1).$$

(iii) *If $a_f(p^i) = 0$ for some $i > 0$, then $i + 1 \equiv 0 \pmod{i_f(p) + 1}$.*
(iv) *For p sufficiently large (depending on f), we have $i_f(p) \in \{0, 1\}$.*

Proof Let us suppose that $i_f(p) > 0$. Thus, $a_f(p^i) = 0$ for some $i \geq 1$. Let us write α_p and β_p for the roots of $X^2 - a_f(p)X + p$. Then, we have

$$a_f(p^i) = \frac{\alpha_p^{i+1} - \beta_p^{i+1}}{\alpha_p - \beta_p}. \tag{4.8}$$

Thus, $\alpha_p = \zeta\beta_p$ where $\zeta^{i+1} = 1$. Since $\zeta \in \mathbb{Q}(\alpha_p, \beta_p) = \mathbb{Q}(\alpha_p)$ and $[\mathbb{Q}(\alpha_p) : \mathbb{Q}] = 2$, we must have $\zeta^2 = 1$ or $\zeta^4 = 1$ or $\zeta^6 = 1$. This means that one of $\{\zeta + 1, \zeta^2 + 1, \zeta^2 + \zeta + 1, \zeta^2 - \zeta + 1\}$ is zero. This in turn means that one of $\{a_f(p), a_f(p^3), a_f(p^2), a_f(p^5)\}$ is zero. This proves the first assertion. The second follows from (4.8). For the third assertion, we note that $\alpha_p = \zeta\beta_p$ where $\zeta^{i+1} = 1$. We also have $\zeta^{i_f(p)+1} = 1$. Hence, $\zeta^j = 1$ where $i + 1 \equiv j \pmod{i_f(p) + 1}$. If $j > 0$, then $a_f(p^{j-1}) = 0$. But $0 \leq j - 1 < i_f(p)$, a contradiction unless $j = 1$. But then $a_f(1) = 0$ which is also a contradiction. Hence, we must have $j = 0$, proving the third assertion. The fourth assertion follows from [6, Lemma 2.5]. ∎

As before, let us set

$$\phi_p(s) = \sum_{m=0}^{\infty} a^0(p^m)p^{-ms}.$$

From the above lemma, we deduce the following.

Lemma 4.7 *We have for $p \nmid N$,*

$$\phi_p(s) = \begin{cases} \left(1 - \frac{1}{p^s}\right)^{-1} & \text{if } i_f(p) = 0, \\ p^s\left(\frac{1}{p^s - 1} - \frac{1}{p^{(i_f(p)+1)s} - 1}\right) & \text{if } i_f(p) > 0. \end{cases}$$

Note $\phi_p(s) = 1$ for $p \mid N$.

Next, we evaluate $\xi_d(1)$. We have the following.

Proposition 4.8 *Writing*

$$\xi_d(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

we have that

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(i) *$b(n) = 0$ if n is divisible by a prime that does not divide d, and*
(ii) *if $p|d$, we have $|b(p^m)| \le 4$ for all m.*

In particular, the function $n \mapsto b(n)$ satisfies the conditions of Section 4.2. Moreover, we have for $p \nmid N$,

$$\xi_{p,d}(1) = \begin{cases} 1 & \text{if } i_f(p) = 0, \\ 1 + p^{-1} - p^{\nu - 2k_0 + 1} & \text{if } i_f(p) = 1, \\ \frac{1 + p + \cdots + p^{i_f(p)} - p^{\nu - (k_0 - 1)(i_f(p)+1)}}{p + \cdots + p^{i_f(p)}} & \text{if } i_f(p) > 1. \end{cases}$$

Here $\nu = \operatorname{ord}_p d$ and k_0 is the smallest integer $\ge \frac{\nu+1}{i_f(p)+1}$.

Proof By a calculation similar to that of Lemma 4.7, we see that

$$\sum_{m=0}^{\infty} a^0(p^{m+\nu}) p^{-ms} = \begin{cases} \left(1 - \frac{1}{p^s}\right)^{-1} & \text{if } i_f(p) = 0, \\ p^s\left(\frac{1}{p^s - 1} - \frac{p^{\{\nu - (k_0-1)(i_f(p)+1)\}s}}{p^{(i_f(p)+1)s} - 1}\right) & \text{if } i_f(p) > 0. \end{cases}$$

Hence, writing $i = i_f(p)$, we have

$$\xi_{p,d}(s) = \frac{p^{(i+1)s} - 1 - p^{\{\nu+1-(k_0-1)(i+1)\}s} + p^{\{\nu-(k_0-1)(i+1)\}s}}{p^{(i+1)s} - p^s}$$

which is equal to

$$\left(1 - \frac{1}{p^{\{k_0(i+1)-\nu-1\}s}} + \frac{1}{p^{\{k_0(i+1)-\nu\}s}} - \frac{1}{p^{(i+1)s}}\right)\left(1 - \frac{1}{p^{is}}\right)^{-1}$$

from which it follows that $|b(p^m)| \le 4$. Moreover, as

$$\xi_d(s) = \prod_{p|d} \xi_{p,d}(s),$$

it follows also that $b(n) = 0$ unless every prime divisor of n also divides d. The last assertion of the lemma follows from the above formulas.

Remark 4.9 Note that the dependence of $\xi_{p,d}$ on d is only through $\operatorname{ord}_p d$. Thus, where the meaning is clear, for $p|d$ and d squarefree, we shall write ξ_p.

In the remainder of this section, we will elaborate on the constant u_f and, in particular, relate it to L-function values. From Lemma 4.7, we have

$$\log \phi(s) = -\sum_{i_f(p)=0} \log\left(1 - \frac{1}{p^s}\right) - \sum_{i_f(p)=1} \log\left(1 - \frac{1}{p^{2s}}\right) + \sum_{i_f(p)>1} \log \phi_p(s).$$

Note that by Lemma 4.6(iv) the third sum on the right-hand side ranges over a finite set of primes p.

Denote by χ_K the quadratic Dirichlet character that defines K and $L(s,\chi_K)$ the associated Dirichlet series. Let us denote by S, I, R the set of primes that split, stay inert, or ramify in K (respectively). Then we have

$$-\sum_{p\in S}\log\Big(1-\frac{1}{p^s}\Big) = \frac{1}{2}\log\zeta(s)+\frac{1}{2}\log L(s,\chi_K)+\frac{1}{2}\sum_{p\in I}\log\Big(1-\frac{1}{p^{2s}}\Big) + \frac{1}{2}\sum_{p\in R}\log\Big(1-\frac{1}{p^s}\Big)$$

Moreover, if $i_f(p)=0$, then $a(p)\neq 0$ and for $p\nmid 6N$, this means that p is a prime of good reduction and splits in K. Therefore,

$$-\sum_{\substack{i_f(p)=0\\ p\nmid 6N}}\log\Big(1-\frac{1}{p^s}\Big) = -\sum_{\substack{p\in S\\ p\nmid 6N}}\log\Big(1-\frac{1}{p^s}\Big)+\sum_{\substack{i_f(p)>1\\ p\nmid 6N}}\log\Big(1-\frac{1}{p^s}\Big).$$

Since $i_f(p)=1 \Leftrightarrow a(p)=0$, we can write

$$-\sum_{\substack{i_f(p)=1\\ p\nmid 6N}}\log\Big(1-\frac{1}{p^{2s}}\Big) = -\sum_{\substack{a(p)=0\\ p\nmid 6N}}\log\Big(1-\frac{1}{p^{2s}}\Big).$$

After a straightforward (but tedious) computation, one sees that

$$\log\phi(s) = \frac{1}{2}\log\frac{1}{s-1}+\frac{1}{2}\log\big(\zeta(s)(s-1)\big)+\frac{1}{2}\log L(s,\chi_K) + \frac{1}{2}\sum_{p\in I}\log\Big(1-\frac{1}{p^{2s}}\Big)+\log C(s),$$

where

$$C(s)=\prod_{\substack{a(p)=0\\ p\nmid 6N}}\Big(1-\frac{1}{p^{2s}}\Big)^{-1}\prod_{p\in R}\Big(1-\frac{1}{p^s}\Big)^{\frac{1}{2}}\prod_{\substack{p\in S\\ p\mid 6N}}\Big(1-\frac{1}{p^s}\Big) \prod_{\substack{i_f(p)>1\\ p\nmid 6N}}\Big\{\Big(1-\frac{1}{p^s}\Big)\phi_p(s)\Big\}\prod_{p\mid 6N}\phi_p(s).$$

Putting the above discussion together, we see that

$$\phi(s)=\frac{\epsilon(s)}{(s-1)^{1/2}},$$

where

$$u_f=\epsilon(1)=L(1,\chi_K)^{1/2}\prod_{p\in I}\Big(1-\frac{1}{p^2}\Big)^{1/2}C(1).$$

5 A Sieve Lemma

We record a simple consequence of Proposition 4.5 that will be used in Section 8.

Lemma 5.1 *Let f be as in the previous section, that is, a normalized Hecke eigenform of weight ≥ 2 with complex multiplication. Let $y_1 = L_2(x)^{1+\epsilon}$ and set*

$$N_{y_1}(x) = \{n \leq x\colon q|n \Rightarrow q \geq y_1, a_f(n) \neq 0\}. \tag{5.1}$$

Then

$$N_{y_1}(x) = \frac{U_f x}{\sqrt{\pi}(L_3(x)\log x)^{\frac{1}{2}}} + O\Big(\frac{xL_3(x)^2}{(\log x)^{3/2}}\Big),$$

where

$$U_f = \frac{u_f \mu_f c_f}{\sqrt{\pi}} \prod_{\substack{p<y_1 \\ i_f(p)>1}} \Big(1 - \frac{\xi_{p,d}(1)}{p}\Big) \prod_{\substack{p\in\{2,3\} \\ i_f(p)=0}} \Big(1 - \frac{1}{p}\Big).$$

Note that the last two products are over a finite number of primes and

$$c_f = \prod_{\substack{5\leq p<y_1 \\ i_f(p)\geq 2}} \Big(1 - \frac{1}{p}\Big)^{-1} \prod_{\substack{p<y_1 \\ i_f(p)=1}} \Big(1 - \frac{1}{p^2}\Big).$$

Proof Set $P_{y_1} = \prod_{p<y_1} p$. By the principle of inclusion-exclusion, we have

$$N_{y_1}(x) = \sum_{d|P_{y_1}} \mu(d) M_{f,d}(x).$$

Since $P_{y_1} \ll e^{y_1}$, we see that for any $d|P_{y_1}$, we have $\log x \ll \log x/d \ll \log x$. Now using Proposition 4.5, the right hand side is

$$= \frac{u_f x}{\sqrt{\pi}(\log x)^{\frac{1}{2}}} \sum_{d|P_{y_1}} \frac{\mu(d)}{d}\Big(\xi_d(1) + O\Big(\frac{2^{\nu(d)}}{(\log x)}\Big)\Big).$$

The main term is

$$\begin{aligned}
&= \frac{u_f x}{\sqrt{\pi}(\log x)^{\frac{1}{2}}} \prod_{p<y_1}\Big(1 - \frac{\xi_{p,d}(1)}{p}\Big) \\
&= \frac{u_f x}{\sqrt{\pi}(\log x)^{\frac{1}{2}}} \prod_{\substack{5\leq p<y_1 \\ i_f(p)=0}}\Big(1 - \frac{1}{p}\Big) \prod_{\substack{p<y_1 \\ i_f(p)\geq 1}}\Big(1 - \frac{\xi_{p,d}(1)}{p}\Big) \prod_{\substack{p\in\{2,3\} \\ i_f(p)=0}}\Big(1 - \frac{1}{p}\Big) \\
&= \frac{u_f x}{\sqrt{\pi}(\log x)^{\frac{1}{2}}} \prod_{\substack{5\leq p<y_1 \\ i_f(p)=0}}\Big(1 - \frac{1}{p}\Big) \prod_{\substack{p<y_1 \\ i_f(p)=1}}\Big(1 - \frac{1}{p^2}\Big) \\
&\qquad \prod_{\substack{p<y_1 \\ i_f(p)>1}}\Big(1 - \frac{\xi_{p,d}(1)}{p}\Big) \prod_{\substack{p\in\{2,3\} \\ i_f(p)=0}}\Big(1 - \frac{1}{p}\Big).
\end{aligned}$$

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Note that if $i_f(p) = 1$ and d is squarefree, we have $\xi_{p,d}(1) = \frac{1}{p}$ by Proposition 4.8. Also note that by Lemma 4.6, there are only finitely many primes p for which $i_f(p) > 1$, ensuring the convergence of

$$\prod_{i_f(p)>1} \left(1 - \frac{\xi_{p,d}(1)}{p}\right).$$

Now using Proposition 3.3, we see that the above sum is

$$\frac{U_f x}{\sqrt{\pi}(L_3(x)\log x)^{\frac{1}{2}}}.$$

The error term is

$$\ll \frac{x}{(\log x)^{3/2}} \sum_{d|P_{y_1}} \frac{|\mu(d)|}{d} 2^{\nu(d)}.$$

The sum over d is

$$\ll \prod_{\ell<y_1} \left(1 + \frac{2}{\ell}\right) \ll \prod_{\ell<y_1} \left(1 - \frac{1}{\ell}\right)^{-2} \ll L_3(x)^2.$$

This proves the result. ■

We record here a variant of the above result.

Lemma 5.2 *Suppose that $p \le y_1$. We have*

$$\#\{n \le x \mid p|n, a_f(n) \ne 0, q|n \Rightarrow q \ge p\} \ll \frac{x}{p(\log x)^{\frac{1}{2}}} \prod_{\substack{\ell \le p \\ \ell \text{ prime}}} \left(1 - \frac{1}{l}\right) + \frac{x}{(\log x)^{3/2}} e^{4\sqrt{p}} \frac{\log p}{p}.$$

6 Siegel Zeros

Let $L/\mathbb{Q}$ be a Galois extension of number fields with group G and n_L, d_L be the degree and the absolute value of the discriminant of $L/\mathbb{Q}$, respectively. Suppose that Artin's conjecture on the holomorphy of Artin L-functions is known for $L/\mathbb{Q}$. Set

$$\log \mathcal{M} = 2\left(\sum_{p|d_L} \log p + \log n_L\right).$$

Also, denote by d the maximum degree and by $\mathcal{A}$ the maximum Artin conductor of an irreducible character of G.

Let C be the set of elements in G that map to the Cartan subgroup and also have trace zero. Then C is stable under conjugation and thus C is a union of conjugacy classes. Denote by $\pi(x, C)$ the number of primes $p \le x$ with $\mathrm{Frob}_p \in C$. Then, [8, Thm. 4.1] asserts that for $\log x \gg d^4(\log \mathcal{M})$, there is an absolute and effective constant $c > 0$ so that

$$\pi(x, C) = \frac{|C|}{|G|}\operatorname{Li} x - \frac{|C|}{|G|}\operatorname{Li} x^{\beta} + O\Big(|C|^{\frac{1}{2}} x (\log x\mathcal{M})^2 \exp\Big\{ \frac{-c\log x}{d^{3/2}\sqrt{d^3(\log \mathcal{A})^2 + \log x}} \Big\} \Big).$$

The term involving β is present only if the Dedekind zeta function $\zeta_L(s)$ of L has a real zero β (the Siegel zero), in the interval

$$1 - \frac{1}{4\log d_L} \leq \Re(s) < 1.$$

Let L be the fixed field of the kernel of $\bar{\rho}_{p,f}$. (Recall that $\bar{\rho}_{p,f}$ was introduced in Section 2.) Now, let $G = \operatorname{Gal}(L/\mathbb{Q})$ (viewed as a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p)$) and let C be the subset of elements of G of trace zero. It is known that the subgroup $H = \operatorname{Gal}(L/K)$ is Abelian and maps to a Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p)$. The image of G maps to the normalizer of this subgroup. As G has an Abelian normal subgroup of index 2, it is well known that all irreducible characters of G are monomial, and so Artin's holomorphy conjecture holds for it.

Thus, we can appeal to the above version of the Chebotarev density theorem. The extension L/K is unramified outside of primes dividing pN, where N is the level of f. We have $d = 2$, and $\log \mathcal{M} \ll \log pN$ as well as $\log \mathcal{A} \ll \log pN$. For p sufficiently large, it is known that G maps *onto* the normalizer of a Cartan subgroup, and hence $p^2 \ll |G| \ll p^2$. Moreover, the size of $|C|$ satisfies $p \ll |C| \ll p$. Thus, if we set $\delta(p) = \frac{|C|}{|G|}$, we have $\frac{1}{p} \ll \delta(p) \ll \frac{1}{p}$ for p sufficiently large. Thus, we have the following result.

Theorem 6.1 *Let f be a CM form of level N as before. Then for $\log x \gg (\log pN)^2$, we have*

$$\pi^*(x, p) = \delta(p)\operatorname{Li} x - \delta(p)\operatorname{Li} x^{\beta} + O(xe^{-c\sqrt{\log x}}),$$

where $\frac{1}{p} \ll \delta(p) \ll \frac{1}{p}$ and the implied constant is absolute and effective.

From the discussion above, we know that the stated bounds on $\delta(p)$ hold for p sufficiently large. To deduce that they hold for all p, it suffices to show that $\delta(p) > 0$ holds for all p. This inequality follows from the fact that the image of complex conjugation is an element of trace zero in the Galois group.

If the Dedekind zeta function $\zeta_L(s) = 0$ has a Siegel zero β with $1 - \frac{1}{4\log d_L} \leq \Re(s) < 1$, then by a result of Stark [17, p. 145] we know that there is a quadratic field M contained in L such that $\zeta_M(\beta) = 0$. Further [17, p. 147], for such M

$$\beta < 1 - \frac{1}{\sqrt{d_M}}.$$

Let $[L : M] = n$. Since $d_L \geq d_M^n$, we have

$$\beta < 1 - \frac{1}{d_L^{1/2n}}.$$

Now by an inequality of Hensel [15, p. 129], $\log d_L \leq 2n \log pn_L$ and so $\frac{1}{2n}\log d_L \leq \log pn_L$. Hence

$$\beta < 1 - \frac{1}{pn_L}. \tag{6.1}$$

7 Intermediate Results

As before

$$\pi^*(x,p) = \#\{q \leq x \mid a(q) \equiv 0 \,(\mathrm{mod}\ p), a(q) \neq 0\}.$$

Proving Theorem 1.2 requires the following lemmas. Let $0 < \epsilon < 1/2$ and set $y = L_2^{1-\epsilon}(x)$.

Lemma 7.1 *Let $p < y$ be a fixed prime. Then we have*

$$\sideset{}{^*}\sum_{\substack{q \leq x \\ a(q)\equiv 0 \,(\mathrm{mod}\ p)}} \frac{1}{q} = \delta(p)L_2(x) + O(L_3(x)),$$

where $\sum^*_{q\leq x}$ means that the summation is over all primes $q \leq x$ for which $a(q) \neq 0$.

Proof By partial summation, the sum is

$$\sideset{}{^*}\sum_{\substack{q \leq x \\ a(q)\equiv 0 \,(\mathrm{mod}\ p)}} \frac{1}{q} = \frac{\pi^*(x,p)}{x} + \int_2^x \frac{\pi^*(t,p)}{t^2}dt.$$

But $\int_2^x \frac{\pi^*(t,p)}{t^2}dt$ can be written as

$$\int_2^{(\log x)^\gamma} \frac{\pi^*(t,p)}{t^2}dt + \int_{(\log x)^\gamma}^x \frac{\pi^*(t,p)}{t^2}dt,$$

where γ is chosen in such a way that for $(\log x)^\gamma \leq t \leq x$, we have $\log t \gg (\log pN)^2$. The first integral is

$$\leq \int_2^{(\log x)^\gamma} \frac{\pi(t)}{t^2}dt \ll L_3(x), \quad \text{where } \pi(t) = \#\{p \leq t \mid p \text{ prime}\},$$

and the second integral is

$$\int_{(\log x)^\gamma}^x \frac{1}{t^2}\big(\delta(p)\mathrm{Li}(t) - \delta(p)\mathrm{Li}(t^\beta) + O(te^{-c\sqrt{\log t}})\big)\ dt, \quad \text{by Theorem 6.1.}$$

The first term is equal to

$$\delta(p)\int_{(\log x)^\gamma}^x \frac{dt}{t\log t} + O(L_3(x)) = \delta(p)L_2(x) + O(L_3(x)).$$

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Next, consider the term with the Siegel zero. Since by (6.1), $\beta < 1 - \frac{1}{pn_L}$, therefore the second term is

$$\delta(p)\int_{(\log x)^\gamma}^{x} \frac{1}{t^2}\mathrm{Li}(t^\beta)\,dt = \delta(p)\int_{(\log x)^\gamma}^{x} \frac{dt}{t^2}\int_2^{t^\beta}\frac{du}{\log u}$$
$$= \delta(p)\int_2^{x^\beta}\frac{du}{\log u}\int_{\max\left((\log x)^\gamma, u^{\frac{1}{\beta}}\right)}^{x}\frac{dt}{t^2}.$$

We split the range of integration of u into two integrals:

(I) $2 \le u \le (\log x)^{\gamma\beta}$,
(II) $(\log x)^{\gamma\beta} \le u \le x^\beta$.

The first range gives rise to the integral

$$\delta(p)\int_2^{(\log x)^{\gamma\beta}}\frac{du}{\log u}\left\{\frac{1}{(\log x)^\gamma} - \frac{1}{x}\right\} \ll \delta(p)(\log x)^{\gamma(\beta-1)} \ll 1.$$

The second range gives rise to the integral

$$\delta(p)\int_{(\log x)^{\gamma\beta}}^{x^\beta}\frac{du}{\log u}\left\{\frac{1}{u^{\frac{1}{\beta}}} - \frac{1}{x}\right\}.$$

Set $v = u^{\frac{1}{\beta}}$. Then $v^\beta = u$ and $\beta\log v = \log u$. Moreover, $du = \beta v^{\beta-1}dv$. Hence the integral is

$$\delta(p)\int_{(\log x)^\gamma}^{x}\frac{\beta v^{\beta-1}dv}{\beta\log v}\left(\frac{1}{v} - \frac{1}{x}\right) \ll \frac{\delta(p)}{(\log x)^{\gamma(1-\beta)}}\int_{(\log x)^\gamma}^{x}\frac{dv}{v\log v}$$
$$\ll \frac{\delta(p)L_2(x)}{(\log x)^{\frac{\gamma}{pn_L}}} \ll \frac{\delta(p)L_2(x)}{e^{\frac{\gamma}{n_L}L_2(x)^\epsilon}} \ll 1.$$

Finally, using the elementary estimate $e^{c\sqrt{u}} \gg u^2$, we deduce that the O-term is

$$\ll \int_{L_2(x)}^{\log x}\frac{du}{u^2} \ll 1.$$

The term $\pi^*(x,p)/x$ is of smaller order. This proves the lemma. ∎

Define $\nu(p,n) = \#\{q^m || n \mid a(q^m) \equiv 0 \pmod{p}\}$.

Lemma 7.2 *Assume that $p < y$. Then we have*

$$\sum_{n\le x}^{*}\nu(p,n) = (1+o(1))\frac{u_f\delta(p)xL_2(x)}{\sqrt{\pi}\log x} + O\left(\frac{xL_3(x)}{\sqrt{\log x}}\right),$$

where $\sum_{n\le x}^{*}$ means that the summation is over all natural numbers $n \le x$ such that $a(n) \ne 0$.

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Proof Interchanging summation, we see that

$$\begin{aligned}\sideset{}{^*}\sum_{n\le x}\nu(p,n) &= \sideset{}{^*}\sum_{\substack{q^m\le x\\ a(q^m)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m\|n}} 1\\ &= \sideset{}{^*}\sum_{\substack{q\le x\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q\|n}} 1 \;+\; \sideset{}{^*}\sum_{\substack{q^m\le x, m\ge 2\\ a(q^m)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m\|n}} 1.\end{aligned}$$

The contribution of terms q^m with $m \ge 2$ is

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{q^m\le x,\\ m\ge 2\\ a(q^m)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m\|n}} 1 &= \sideset{}{^*}\sum_{\substack{q^m\le x^\epsilon\\ m\ge 2\\ a(q^m)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m\|n}} 1 \;+\; \sideset{}{^*}\sum_{\substack{x^\epsilon\le q^m\le x\\ m\ge 2\\ a(q^m)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q^m\|n}} 1\\ &\ll \sideset{}{^*}\sum_{\substack{q^m\le x^\epsilon\\ m\ge 2\\ a(q^m)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{n\le x/q^m} 1 \;+\; x\sideset{}{^*}\sum_{\substack{x^\epsilon\le q^m\le x\\ m\ge 2}}\frac{1}{q^m}\\ &\ll \frac{x}{(\log x)^{\frac12}}\sideset{}{^*}\sum_{\substack{q^m\le x^\epsilon\\ m\ge 2}}\frac{1}{q^m} + x\int_{x^\epsilon}^{x}\frac{dt}{t^2}, \quad \text{by Proposition 4.5}\\ &\ll \frac{x}{\sqrt{\log x}} + \frac{x}{x^\epsilon} \ll \frac{x}{\sqrt{\log x}}.\end{aligned}$$

Also, we have

$$\sideset{}{^*}\sum_{\substack{q\le x\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q\|n}} 1 = \sideset{}{^*}\sum_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q\|n}} 1 \;+\; \sideset{}{^*}\sum_{\substack{x^{1/\log\log x}\le q\le x\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q\|n}} 1. \tag{7.1}$$

We show that the second double sum on the right of (7.1) contributes a negligible amount. Indeed, consider first the quantity

$$\sideset{}{^*}\sum_{\substack{x^\epsilon\le q\le x\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}}\ \sideset{}{^*}\sum_{\substack{n\le x\\ q\|n}} 1. \tag{7.2}$$

This is majorized by

$$\sideset{}{^*}\sum_{n\le x}\ \sideset{}{^*}\sum_{\substack{x^\epsilon\le q\le x\\ q\|n}} 1.$$

The inner sum is bounded and so by Proposition 4.5, we see that (7.2) is

$$\ll x/\sqrt{\log x}. \tag{7.3}$$

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Now, consider the quantity

$$\sum^{*}_{\substack{x^{1/\log\log x}\le q\le x^{\epsilon}\\ a(q)\equiv 0 \,(\mathrm{mod}\ p)}} \sum^{*}_{\substack{n\le x\\ q||n}} 1. \tag{7.4}$$

By Proposition 4.5, the inner sum is $\ll x/q\sqrt{\log x}$. Since

$$\sum_{x^{1/\log\log x}\le q\le x^{\epsilon}} \frac{1}{q} = \log\log\log x + O(1),$$

it follows that (7.4) is

$$\ll xL_3(x)/\sqrt{\log x}. \tag{7.5}$$

Putting (7.3) and (7.5) together, we deduce that

$$\sum^{*}_{\substack{q\le x\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}} \sum^{*}_{\substack{n\le x\\ q||n}} 1 = \sum^{*}_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}} \sum^{*}_{\substack{n\le x\\ q||n}} 1 \quad + O(xL_3(x)/\sqrt{\log x}).$$

Now by Proposition 4.5, Lemma 4.3 (and the fact that in the sum $a^0(q) = 1$), the sum on the right is equal to

$$(1+o(1))\frac{u_f x}{\sqrt{\pi}} \sum^{*}_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}} \frac{1}{q\sqrt{\log x/q}}\left(1 + O\left(\frac{1}{q}\right) + O\left(\frac{1}{\log x/q}\right)\right) =$$

$$(1+o(1))\frac{u_f x}{\sqrt{\pi}} \sum^{*}_{\substack{q\le x^{1/\log\log x}\\ a(q)\equiv 0\,(\mathrm{mod}\ p)}} \frac{1}{q\sqrt{\log x/q}} + O\left(\frac{x}{(\log x)^{\frac{1}{2}}}\right).$$

Now applying Lemma 7.1, we see that this is

$$= (1+o(1))\frac{u_f\delta(p)xL_2(x)}{\sqrt{\pi}(\log x)^{\frac{1}{2}}} + O\Big(\frac{xL_3(x)}{(\log x)^{\frac{1}{2}}}\Big).$$

This proves the lemma. ■

Lemma 7.3 *Assume $p < y$. Then*

$$\sum^{*}_{n\le x} \nu(p,n)^2 = (1+o(1))\frac{u_f\delta^2(p)xL_2^2(x)}{\sqrt{\pi}(\log x)^{\frac{1}{2}}} + O\Big(\frac{\delta(p)xL_2(x)L_3(x)}{(\log x)^{\frac{1}{2}}}\Big).$$

Proof The sum in question is equal to

$$\sideset{}{^*}\sum_{\substack{q_1^{m_1}\le x\\ a(q_1^{m_1})\equiv 0 \,(\mathrm{mod}\ p)}} \quad \sideset{}{^*}\sum_{\substack{q_2^{m_2}\le x\\ a(q_2^{m_2})\equiv 0 \,(\mathrm{mod}\ p)}} \quad \sideset{}{^*}\sum_{\substack{n\le x\\ q_1^{m_1}||n,\ q_2^{m_2}||n}} 1.$$

By a small modification to the argument given in the proof of Lemma 7.2, we find that the contribution of terms with $q_1 = q_2$ is

$$\ll \frac{xL_2(x)}{(\log x)^{1/2}}.$$

Next, we consider the contribution S (say) of terms with $q_1^{m_1} q_2^{m_2} > x^{\epsilon}$. For estimating this, we may suppose that $q_1^{m_1} > q_2^{m_2}$. Since $q_2 \ge 2$, we may suppose that $x/2 \ge q_1^{m_1} \ge x^{\epsilon/2} = z$ (say).

Denote by S_1 the contribution of terms for which $z \le q_1^{m_1} \le \sqrt{x/2}$ and by S_2 the contribution of all remaining terms in S. Then by Proposition 4.5, we have

$$\begin{aligned} S_1 &\ll x \sideset{}{^*}\sum_{z\le q_1^{m_1}\le\sqrt{x/2}} \frac{1}{q_1^{m_1}} \sum_{q_2^{m_2}\le q_1^{m_1}} \frac{1}{q_2^{m_2}\sqrt{\log \frac{x}{q_1^{m_1}q_2^{m_2}}}} \\ &\ll x \sum_{z\le q_1^{m_1}\le\sqrt{x/2}} \frac{1}{q_1^{m_1}\sqrt{\log\frac{x}{q_1^{2m_1}}}} \log\log(q_1^{m_1}) \\ &\ll xL_2(x)\int_z^{\sqrt{x/2}} \frac{dt}{t(\log t)\sqrt{\log x/t^2}} \ll \frac{xL_2(x)}{\sqrt{\log x}}. \end{aligned}$$

Next, we observe that

$$S_2 \ll \sum_{\sqrt{x/2}<q_1^{m_1}\le x/2} \quad \sideset{}{^*}\sum_{n\le x/q_1^{m_1}} \nu(p,n)$$

and by Lemma 7.2, this is

$$\ll xL_2(x) \sum_{\sqrt{x/2}<q_1^{m_1}\le x/2} \frac{1}{q_1^{m_1}} \frac{1}{\sqrt{\log x/q_1^{m_1}}} \ll \frac{xL_2(x)}{\sqrt{\log x}}.$$

It remains to estimate

$$\sideset{}{^*}\sum_{\substack{q_1^{m_1}q_2^{m_2}\le x^{\epsilon}\\ a(q_1^{m_1})\equiv 0\,(\mathrm{mod}\ p)\\ a(q_2^{m_2})\equiv 0\,(\mathrm{mod}\ p)}} \quad \sideset{}{^*}\sum_{\substack{n\le x\\ q_1^{m_1}||n,\ q_2^{m_2}||n}} 1 = I + J, \quad \text{say,}$$

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where in I we have the terms with $m_1 > 1$ or $m_2 > 1$, and in J we have the terms with $m_1 = m_2 = 1$. In order to estimate I, suppose without loss of generality that $m_1 \geq 2$. Then by Proposition 4.5, we have

$$\begin{aligned} I &\ll x \sideset{}{^*}\sum_{\substack{q_1^{m_1} \\ m_1 \geq 2}} \frac{1}{q_1^{m_1}} \sideset{}{^*}\sum_{\substack{q_2^{m_2} \\ q_1^{m_1} q_2^{m_2} \leq x^{\epsilon}}} \frac{1}{q_2^{m_2} \sqrt{\log \frac{x}{q_1^{m_1} q_2^{m_2}}}} \\ &\ll \frac{x}{\sqrt{\log x}} \sideset{}{^*}\sum_{\substack{q_1^{m_1} \\ m_1 \geq 2}} \frac{1}{q_1^{m_1}} \Bigl(\sum_{q_2 \leq x^{\epsilon}} \frac{1}{q_2} + \sum_{\substack{q_2 \\ m_2 \geq 2}} \frac{1}{q_2^{m_2}} \Bigr) \\ &\ll \frac{x L_2(x)}{\sqrt{\log x}}. \end{aligned}$$

Next, we consider

$$J = \sideset{}{^*}\sum_{\substack{q_1 q_2 \leq x^{\epsilon} \\ a(q_1) \equiv 0 \ (\mathrm{mod}\ p) \\ a(q_2) \equiv 0 \ (\mathrm{mod}\ p)}} \sideset{}{^*}\sum_{\substack{n \leq x \\ q_1 \| n,\ q_2 \| n}} 1$$

By Propositions 4.5 and 4.8, we have

$$\begin{aligned} J &= (1 + o(1)) \frac{u_f x}{\sqrt{\pi \log x}} \sideset{}{^*}\sum_{\substack{q_1 q_2 \leq x^{\epsilon} \\ a(q_1) \equiv 0 \ (\mathrm{mod}\ p) \\ a(q_2) \equiv 0 \ (\mathrm{mod}\ p) \\ q_1 \neq q_2}} \frac{1}{q_1 q_2} + O\Bigl(\frac{x L_2(x)}{\sqrt{\log x}} \Bigr) \\ &= (1 + o(1)) \frac{u_f x}{\sqrt{\pi \log x}} \Bigl(\sideset{}{^*}\sum_{\substack{q \leq x \\ a(q) \equiv 0 \ (\mathrm{mod}\ p)}} \frac{1}{q} \Bigr)^2 + O\Bigl(\frac{x L_2(x)}{\sqrt{\log x}} \Bigr) \\ &= (1 + o(1)) \frac{u_f x}{\sqrt{\pi \log x}} \bigl(\delta(p) L_2(x) + O(L_3(x)) \bigr)^2 + O\Bigl(\frac{x L_2(x)}{\sqrt{\log x}} \Bigr) \\ &= (1 + o(1)) \frac{u_f \delta^2(p) x L_2^2(x)}{\sqrt{\pi} (\log x)^{\frac{1}{2}}} + O\Bigl(\delta(p) \frac{x L_2(x) L_3(x)}{\sqrt{\log x}} \Bigr). \end{aligned}$$

This proves the lemma. ■

Lemma 7.4 *Suppose $p < y$, then*

$$\sideset{}{^*}\sum_{n \leq x} \bigl(\nu(p, n) - \delta(p) L_2(x) \bigr)^2 \ll \frac{\delta(p) x}{(\log x)^{\frac{1}{2}}} L_2(x) L_3(x).$$

Proof This follows from Lemmas 7.2 and 7.3. ■

Lemma 7.5 *Assume $p < y$, then*

$$\#\{ n \leq x \mid \nu(p, n) = 0 \} \ll \frac{x L_3(x)}{\delta(p) (\log x)^{\frac{1}{2}} L_2(x)}.$$

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Proof By Lemma 7.4, this is

$$\ll \frac{1}{\delta^2(p)L_2^2(x)}\Big\{\delta(p)\frac{x}{(\log x)^{\frac{1}{2}}}L_2(x)L_3(x)\Big\} = \frac{xL_3(x)}{\delta(p)(\log x)^{\frac{1}{2}}L_2(x)}. \qquad \blacksquare$$

8 Proof of Theorem 1.2

For a prime p, let

$$G_p(x) = \#\{n \le x \mid p|n, (n, a(n)) = 1, q|n \Rightarrow q \ge p\}$$

and $G(x) = \sum_{p\le x} G_p(x) = A_1 + A_2 + A_3$, where

$$A_1 = \sum_{p\le L_2^{\frac{1}{2}-\epsilon}(x)} G_p(x), \quad A_2 = \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} G_p(x), \quad A_3 = \sum_{p\ge L_2^{1+\epsilon}(x)} G_p(x).$$

Now, using Lemma 7.5, we have

$$\begin{aligned} A_1 &\le \sum_{p\le L_2^{\frac{1}{2}-\epsilon}(x)} \#\{n \le x \mid p|n, (n, a(n)) = 1\} \\ &\ll \frac{xL_3(x)}{(\log x)^{\frac{1}{2}}L_2(x)} \sum_{p\le L_2^{\frac{1}{2}-\epsilon}(x)} \frac{1}{\delta(p)} \\ &\ll \frac{xL_3(x)}{(\log x)^{\frac{1}{2}}L_2(x)} \sum_{1\ll p\le L_2^{\frac{1}{2}-\epsilon}(x)} p, \quad \text{as } \delta(p) \gg \frac{1}{p} \\ &\ll \frac{x}{(\log x)^{\frac{1}{2}}L_2^{\epsilon}(x)} = o\Big(\frac{x}{(L_3(x)\log x)^{\frac{1}{2}}}\Big). \end{aligned}$$

Moreover, by Lemma 5.2, we have

$$\begin{aligned} A_2 &\le \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} \#\{n \le x \mid p|n, a(n) \ne 0, q|n \Rightarrow q \ge p\} \\ &\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} \frac{1}{p} \prod_{\substack{l\le p \\ l \\ \text{prime}}} \Big(1-\frac{1}{l}\Big) \\ &\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sum_{L_2^{\frac{1}{2}-\epsilon}(x)<p<L_2^{1+\epsilon}(x)} \frac{1}{p\log p} \\ &\ll \frac{x}{L_3(x)(\log x)^{\frac{1}{2}}} = o\Big(\frac{x}{(L_3(x)\log x)^{\frac{1}{2}}}\Big). \end{aligned}$$

Let $y_1 = L_2(x)^{1+\epsilon}$ and as in (5.1), $N_{y_1}(x) = \#\{n \leq x \mid q|n \Rightarrow q \geq y_1,\ a(n) \neq 0\}$. Then

$$N_{y_1}(x) - \sideset{}{^*}\sum_{\substack{y_1 \leq q_1^m, q_2 \leq x \\ a(q_1^m) \equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sideset{}{^{**}}\sum_{\substack{n \leq x \\ q_1^m || n,\ q_2 | n}} 1 \leq A_3 \leq N_{y_1}(x),$$

where $\sum^{**}_{n \leq x}$ means that the summation is over all natural numbers $n \leq x$ such that $a(n) \neq 0$ and $q|n$ implies that $q > y_1$.

By Lemma 5.1, to prove the theorem, it suffices to show that

$$\sideset{}{^*}\sum_{\substack{y_1 \leq q_1^m,\ q_2 \leq x \\ a(q_1^m) \equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sideset{}{^{**}}\sum_{\substack{n \leq x \\ q_1^m || n,\ q_2 | n}} 1 = o\Big(\frac{x}{(L_3(x) \log x)^{\frac{1}{2}}} \Big). \tag{8.1}$$

In order to prove (8.1), let us write

$$\begin{aligned} \sideset{}{^*}\sum_{\substack{y_1 \leq q_1^m,\ q_2 \leq x \\ a(q_1^m) \equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sideset{}{^{**}}\sum_{\substack{n \leq x \\ q_1^m || n,\ q_2 | n}} 1 &= \sideset{}{^*}\sum_{\substack{y_1 \leq q_1^m,\ q_2 \leq x,\ m \geq 2 \\ a(q_1^m) \equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sideset{}{^{**}}\sum_{\substack{n \leq x, \\ q_1^m || n,\ q_2 | n}} 1 \\ &\quad + \sideset{}{^*}\sum_{\substack{y_1 \leq q_1,\ q_2 \leq x \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sideset{}{^{**}}\sum_{\substack{n \leq x \\ q_1 || n,\ q_2 | n}} 1 \\ &= B_1 + B_2. \end{aligned}$$

Let us consider B_1 first. The terms for which $q_1^m q_2 \geq (\log x) x^{1/2} y_1^2$ contribute an amount that is

$$\begin{aligned} &\ll \frac{\sqrt{x}}{\log x} \sum_{q_2 \leq x} \frac{1}{q_2} \sum_{\substack{q_1^m \geq y_1 \\ m \geq 2}} \frac{1}{q_1^m} \\ &\ll \frac{\sqrt{x}}{y_1 \log x} L_2(x) \ll \frac{x}{L_2^{\epsilon}(x) \log x}. \end{aligned}$$

For the remaining terms, $q_1^m q_2 \leq (\log x) x^{1/2} y_1^2$. We use Proposition 4.5 to see that the remaining terms in B_1 are

$$\begin{aligned} &\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sum_{y_1 \leq q_2 \leq x} \frac{1}{q_2} \sum_{\substack{y_1 \leq q_1^m \\ m \geq 2}} \frac{1}{q_1^m} \\ &\ll \frac{x}{y_1 (\log x)^{\frac{1}{2}}} \sum_{y_1 \leq q_2 \leq x} \frac{1}{q_2} \\ &\ll \frac{x L_2(x)}{y_1 (\log x)^{\frac{1}{2}}} = \frac{x}{(\log x)^{\frac{1}{2}} L_2^{\epsilon}(x)}. \end{aligned}$$

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For B_2, we observe that if $a(q_1) \neq 0$ and $a(q_1) \equiv 0 \pmod{q_2}$, then $q_2 \leq |a(q_1)| \leq 2\sqrt{q_1}$. Hence $q_1 \geq q_2^2/4$ and so $q_1 q_2 \geq q_2^3/4$. Hence for the inner sum in B_2 to be nonempty, we need $q_2 \leq (4x)^{1/3}$. Thus

$$
\begin{aligned}
B_2 &= \sideset{}{^*}\sum_{\substack{y_1 \leq q_1 \leq x \\ y_1 \leq q_2 \leq (4x)^{1/3} \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \sideset{}{^{**}}\sum_{\substack{n \leq x \\ q_1 || n,\ q_2 | n}} 1 \\
&= \sideset{}{^*}\sum_{\substack{y_1 \leq q_1 \leq \sqrt{x} \\ y_1 \leq q_2 \leq 2\sqrt{q_1} \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \sideset{}{^{**}}\sum_{\substack{n \leq x \\ q_1 || n,\ q_2 | n}} 1 + \sideset{}{^*}\sum_{\substack{\sqrt{x} \leq q_1 \leq x \\ y_1 \leq q_2 \leq 2\sqrt{q_1} \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \sideset{}{^{**}}\sum_{\substack{n \leq x \\ q_1 || n,\ q_2 | n}} 1 \\
&= D_1 + D_2.
\end{aligned}
$$

Then by Proposition 4.5 and the fact that $q_1 q_2 \ll x^{3/4}$, we have

$$
\begin{aligned}
D_1 &\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sideset{}{^*}\sum_{\substack{y_1 \leq q_1 \leq \sqrt{x} \\ y_1 \leq q_2 \leq 2\sqrt{q_1} \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \frac{1}{q_1 q_2} \\
&= \frac{x}{(\log x)^{\frac{1}{2}}} \Biggl\{ \sideset{}{^*}\sum_{\substack{y_1 \leq q_2 \leq 2x^{1/4} \\ \frac{1}{4} q_2^2 \leq q_1 \leq q_2^2 \log q_2 \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \frac{1}{q_1 q_2} + \sideset{}{^*}\sum_{\substack{y_1 \leq q_2 \leq 2x^{1/4} \\ q_2^2 \log q_2 \leq q_1 \leq \sqrt{x} \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \frac{1}{q_1 q_2} \Biggr\}.
\end{aligned}
$$

By Proposition 2.3, the second sum is

$$
\ll \frac{x L_2(x)}{(\log x)^{\frac{1}{2}}} \sum_{y_1 \leq q_2 \leq 2x^{1/4}} \frac{1}{q_2^2} \ll \frac{x L_2(x)}{y_1 (\log x)^{\frac{1}{2}}} = \frac{x}{(\log x)^{\frac{1}{2}} L_2^{\epsilon}(x)}.
$$

The first sum is

$$
\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sum_{\frac{1}{4} y_1^2 \leq q_1 \leq x} \frac{1}{q_1} \sideset{}{^*}\sum_{\substack{\sqrt{\frac{q_1}{\log q_1}} \leq q_2 \leq 2\sqrt{q_1} \\ a(q_1) \equiv 0 \,(\mathrm{mod}\ q_2)}} \frac{1}{q_2}.
$$

We note that the inner sum over q_2 is bounded. In fact with $0 < |a(q_1)| \leq 2\sqrt{q_1}$, there exists at most one $q_2 \geq \sqrt{q_1/\log q_1}$ that divides $a(q_1)$. Thus, the right-hand side is

$$
\ll \frac{x}{(\log x)^{\frac{1}{2}}} \sum_{y_1 \leq q_1 \leq x} \frac{\sqrt{\log q_1}}{q_1^{3/2}} \ll \frac{x}{(L_2(x) \log x)^{\frac{1}{2}}}.
$$

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In order to estimate D_2, we write

$$\begin{aligned} D_2 = {} & \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ \sqrt{x}\le q_1\le \frac{x}{2q_2}\\ a(q_1)\equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sum^{**}_{\substack{n\le x\\ q_1||n,\ q_2|n}} 1 \quad + \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ \frac{x}{2q_2}\le q_1\le \frac{x}{q_2}\\ a(q_1)\equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sum^{**}_{\substack{n\le x\\ q_1||n, q_2|n}} 1 \\ & + \sum^{*}_{\substack{e^{\sqrt{\log x}}\le q_2\le \left(\frac{x}{\log x}\right)^{1/3}\\ \sqrt{x}\le q_1\le \frac{x}{q_2}\\ a(q_1)\equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sum^{**}_{\substack{n\le x\\ q_1||n,\ q_2|n}} 1 + \sum^{*}_{\substack{\left(\frac{x}{\log x}\right)^{1/3}\le q_2\le x\\ \sqrt{x}\le q_1\le \frac{x}{q_2}\\ a(q_1)\equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sum^{**}_{\substack{n\le x\\ q_1||n,\ q_2|n}} 1 \\ = {} & J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Here

$$\begin{aligned} J_4 &\ll x \sum^{*}_{\sqrt{x}\le q_1\le x} \frac{1}{q_1} \sum^{*}_{\left(\frac{x}{\log x}\right)^{1/3}\le q_2\le 2^{2/3}x^{1/3}} \frac{1}{q_2} \\ &\ll x^{2/3}(\log x)^{1/3}\pi((4x)^{1/3}) \sum^{*}_{\sqrt{x}\le q_1\le x} \frac{1}{q_1}, \end{aligned}$$

where $\pi(t)$ denotes the number of primes $\le t$. Thus

$$J_4 \ll \frac{x}{(\log x)^{2/3}} \sum^{*}_{\sqrt{x}\le q_1\le x} \frac{1}{q_1} \ll \frac{xL_2(x)}{(\log x)^{2/3}}$$

and

$$\begin{aligned} J_3 &\ll x \sum^{*}_{\sqrt{x}\le q_1\le x} \frac{1}{q_1} \sum^{*}_{\substack{q_2|a(q_1)\\ q_2\ge e^{\sqrt{\log x}}}} \frac{1}{q_2} \\ &\ll \frac{x}{e^{\sqrt{\log x}}} \sum^{*}_{\sqrt{x}\le q_1\le x} \frac{1}{q_1} \# \left\{ q_2 \mid q_2 \ge e^{\sqrt{\log x}}, q_2|a(q_1), 0 \ne a(q_1) \le 2\sqrt{x} \right\} \\ &\ll \frac{x\sqrt{\log x}}{e^{\sqrt{\log x}}} \sum_{q_1\le x} \frac{1}{q_1} \ll \frac{x\sqrt{\log x}\, L_2(x)}{e^{\sqrt{\log x}}}. \end{aligned}$$

In order to estimate J_1 and J_2, we write

$$J_1 = \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ \sqrt{x}\le q_1\le \frac{x}{2q_2}\\ a(q_1)\equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sum^{**}_{\substack{n\le x\\ q_1||n,\ q_2||n}} 1 + \sum^{*}_{\substack{y_1\le q_2\le e^{\sqrt{\log x}}\\ \sqrt{x}\le q_1\le \frac{x}{2q_2}\\ a(q_1)\equiv 0 \,(\mathrm{mod}\ q_2)}} \ \sum^{**}_{\substack{n\le x\\ q_1||n\\ q_2^m||n, m\ge 2}} 1 = J_{11} + J_{12},$$

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$$J_2 = \sideset{}{^*}\sum_{\substack{y_1 \le q_2 \le e^{\sqrt{\log x}} \\ \frac{x}{2q_2} \le q_1 \le \frac{x}{q_2} \\ a(q_1) \equiv 0 \ (\mathrm{mod}\ q_2)}} \ \sideset{}{^{**}}\sum_{\substack{n \le x \\ q_1 || n, q_2 || n}} 1 + \sideset{}{^*}\sum_{\substack{y_1 \le q_2 \le e^{\sqrt{\log x}} \\ \frac{x}{2q_2} \le q_1 \le \frac{x}{q_2} \\ a(q_1) \equiv 0 \ (\mathrm{mod}\ q_2)}} \ \sideset{}{^{**}}\sum_{\substack{n \le x \\ q_1 || n \\ q_2^m || n, m \ge 2}} 1 = J_{21} + J_{22}.$$

We show that J_{11} and J_{21} are $o(x/L_3(x)(\log x)^{1/2})$. Similarly, one can show that J_{12} and J_{22} are $o(x/L_3(x)(\log x)^{1/2})$. We can write

$$\begin{aligned} J_{11} &\ll x \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2} \sideset{}{^*}\sum_{\substack{\sqrt{x} \le q_1 \le \frac{x}{2q_2} \\ a(q_1) \equiv 0 \ (\mathrm{mod}\ q_2)}} \frac{1}{q_1 \left(\log \frac{x}{q_1 q_2}\right)^{1/2}} \\ &\ll x \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2} \int_{\sqrt{x}}^{x/2q_2} \frac{d\pi^*(t, q_2)}{t \left(\log \frac{x}{q_2 t}\right)^{1/2}} \\ &\ll x \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2} \left[\left\{ \frac{\pi^*(t, q_2)}{t \left(\log \frac{x}{q_2 t}\right)^{1/2}} \right\}_{t=\sqrt{x}}^{t=x/2q_2} + \int_{\sqrt{x}}^{x/2q_2} \frac{\pi^*(t, q_2)\, dt}{t^2 \left(\log \frac{x}{q_2 t}\right)^{1/2}} \right]. \end{aligned}$$

Then by using Theorem 6.1, we have

$$\begin{aligned} J_{11} &\ll x \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2^2} \left[\left\{ \frac{1}{\log t \left(\log \frac{x}{q_2 t}\right)^{1/2}} \right\}_{t=\sqrt{x}}^{t=x/2q_2} + \int_{\sqrt{x}}^{x/2q_2} \frac{dt}{t \log t \left(\log \frac{x}{q_2 t}\right)^{1/2}} \right] \\ &\ll \frac{x}{\log x} \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2^2} \left[1 + \int_{\sqrt{x}}^{x/2q_2} \frac{dt}{t \left(\log \frac{x}{q_2 t}\right)^{1/2}} \right] \\ &\ll \frac{x}{(\log x)^{1/2}} \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2^2} \ll \frac{x}{y_1 (\log x)^{1/2}}. \end{aligned}$$

Since for each pair of primes q_1, q_2 with $y_1 \le q_2 \le e^{\sqrt{\log x}}, x/2q_2 \le q_1 \le x/q_2$, there are at most two $n \le x$ with $q_1 q_2 \mid n$, we have

$$\begin{aligned} J_{21} &\ll \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \ \sideset{}{^*}\sum_{\substack{\frac{x}{2q_2} \le q_1 \le \frac{x}{q_2} \\ a(q_1) \equiv 0 \ (\mathrm{mod}\ q_2)}} 1 \\ &\ll \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \pi^*(x/q_2,\ q_2) \ll \frac{x}{\log x} \sideset{}{^*}\sum_{y_1 \le q_2 \le e^{\sqrt{\log x}}} \frac{1}{q_2^2}, \qquad \text{by Theorem 6.1} \\ &\ll \frac{x}{y_1 \log x}. \end{aligned}$$

Hence

$$B_1 + B_2 = o\left(\frac{x}{(\log x)^{\frac{1}{2}} L_3(x)} \right).$$

This completes the proof. ∎

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