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# Fundamental Tone, Concentration of Density, and Conformal Degeneration on Surfaces

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*Abstract.* We study the effect of two types of degeneration of a Riemannian metric on the first eigenvalue of the Laplace operator on surfaces. In both cases we prove that the first eigenvalue of the round sphere is an optimal asymptotic upper bound. The first type of degeneration is concentration of the density to a point within a conformal class. The second is degeneration of the conformal class to the boundary of the moduli space on the torus and on the Klein bottle. In the latter, we follow the outline proposed by N. Nadirashvili in 1996.

## 1 Introduction

Given a Riemannian metric $g$ on a closed surface $\Sigma$, let the spectrum of the Laplace operator $\Delta_g$ acting on smooth functions be the sequence

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq \lambda_k(g) \leq \cdots \nearrow \infty,$$

where each eigenvalue is repeated according to its multiplicity. The first nonzero eigenvalue $\lambda_1(g)$ is called the *fundamental tone* of $(\Sigma, g)$. Let $\mathcal{R}(\Sigma)$ be the space of Riemannian metrics on $\Sigma$ with total area one. We are interested in the asymptotic behavior of the functional $\lambda_1 \colon \mathcal{R}(\Sigma) \to ]0, \infty[$ under two types of degeneration of the Riemannian metric described below.

### 1.1 Concentration to Points

It is expected that a metric maximizing $\lambda_1 \colon \mathcal{R}(\Sigma) \to ]0, \infty[$ has lots of symmetries. For example, on the sphere, the torus, and the projective plane, the $\lambda_1$-maximizing metrics are the standard homogeneous ones. Here we consider the opposite situation where the distribution of mass of a sequence of metrics concentrates to a point, developing a $\delta$-like singularity.

***Definition 1.1.1*** A sequence $(g_n) \subset \mathcal{R}(\Sigma)$ is said to *concentrate to the point* $p \in \Sigma$ if for each neighborhood $\mathcal{O}$ of $p$, $\lim_{n\to\infty} \int_{\mathcal{O}} dg_n = 1$.

***Question*** Does concentration to a point impose any restriction on the asymptotic behavior of the eigenvalues of the Laplace operator $\Delta_{g_n}$ on the surface $\Sigma$?

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Without any further constraints, the answer is no.

***Proposition 1.1.2*** *For any metric $g_0$ and any point $p \in \Sigma$, there exists a sequence $(g_n)$ of pairwise isometric metrics concentrating to $p$. In particular the metrics $(g_n)$ are isospectral.*

Under the additional assumption that the metrics $g_n$ are conformally equivalent, we obtain an optimal asymptotic upper bound on the fundamental tone.

***Theorem 1.1.3*** *Let $[g] = \{\alpha g \mid \alpha \in C^\infty(\Sigma), \alpha > 0\}$ be a conformal class on a closed surface $\Sigma$.*

(i) *For any sequence of metrics $(g_n)$ in the conformal class $[g]$ which concentrates to a point $p \in \Sigma$,*

$$\limsup_{n\to\infty} \lambda_1(g_n) \leq 8\pi.$$

(ii) *For any point $p \in \Sigma$, there exists a sequence $(g_n)$ of metrics of unit area in the conformal class $[g]$ concentrating to $p$ such that*

$$\lim_{n\to\infty} \lambda_1(g_n) = 8\pi.$$

Proposition 1.1.2 and Theorem 1.1.3 will be proved in Section 5.

## 1.2 Conformal Degeneration

Given a conformal class $[g]$ on the torus $T^2$, define

$$\nu([g]) := \sup_{\tilde{g}\in\mathcal{R}(T^2)\cap[g]} \lambda_1(\tilde{g}).$$

This corresponds to the first conformal eigenvalue of Colbois and El Soufi [5]. Let

$$\mathcal{M} := \{a + ib \in \mathbb{C} \mid 0 \leq a \leq 1/2, a^2 + b^2 \geq 1, b > 0\}.$$

Any metric on $T^2$ is conformally equivalent to a flat torus $\mathbb{C}/\Gamma$ for some lattice $\Gamma$ of $\mathbb{C}$ generated by $1 \in \mathbb{C}$ and $a + ib \in \mathcal{M}$. It follows that $\mathcal{M}$ is a natural representation of the moduli space $\mathcal{M}(T^2)$ of conformal classes on the torus (see Figure 1).

***Definition 1.2.1*** A sequence of metrics on the torus $T^2$ is *degenerate* if the corresponding sequence $(a_n + ib_n) \subset \mathcal{M}$ satisfies $\lim_{n\to\infty} b_n = \infty$.

***Theorem 1.2.2*** *If a sequence $(g_n)$ of Riemannian metrics of unit area on the torus is degenerate, then* $\lim_{n\to\infty} \nu([g_n]) = 8\pi$. *In particular,* $\limsup_{n\to\infty} \lambda_1(g_n) \leq 8\pi$.

The proof will be presented in Section 3. Using a detailed version of a concentration lemma (Lemma 3.2.1) implicitly used in [17] and an estimate on the Dirichlet energy of harmonic functions on long cylinders (Lemmas 3.6.1 and 2.1.1), we complete the outline proposed by Nadirashvili in [17].

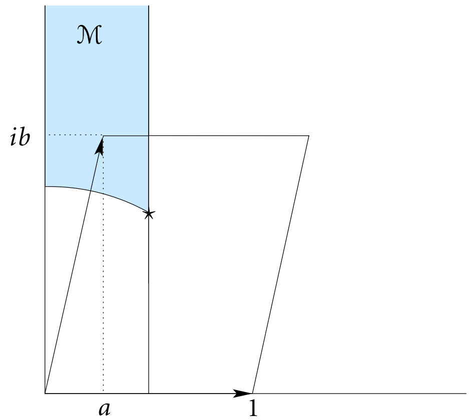


*Figure 1*: Moduli space of conformal structures on $T^2$

### 1.3 Maximization of $\lambda_1$ on Surfaces

One motivation for Theorem 1.2.2 is the role it plays in $\lambda_1$-maximization on the torus. More generally, it is natural to ask which metric (if any) on a closed surface $\Sigma$ maximizes the fundamental tone $\lambda_1$ in the space $\mathcal{R}(\Sigma)$ of Riemannian metrics of unit area. It is well known that $\Lambda(\Sigma) := \sup_{g\in\mathcal{R}(\Sigma)} \lambda_1(g)$ is finite for any closed surface $\Sigma$ [11,18,20], but the explicit value of $\Lambda(\Sigma)$ has been computed for only a few surfaces.

The first such result was obtained by Hersch in 1970 [11]. He proved that the round metric $g_{S^2}$ of unit area is the unique maximum of $\lambda_1$ on $\mathcal{R}(S^2)$. The proof relies on the fact that (by Riemann's uniformization theorem) any two metrics on the sphere are conformally equivalent.

In 1973, Berger [2] proved that among flat metrics on the torus, $\lambda_1$ is maximized by the flat equilateral metric $g_{eq}$, that is, the metric induced from the quotient of $\mathbb{C}$ by the lattice generated by 1 and $e^{i\pi/3}$ (indicated by the $\star$ in Figure 1). He conjectured that this metric is a global maximum of $\lambda_1$ over all Riemannian metrics of unit area. In 1996, Nadirashvili [17] proposed a method of proof.

**Nadirashvili's approach** Start with a maximizing sequence $(g_n)$ (*i.e.*, such that $\lambda_1(g_n) \to \Lambda(T^2)$) and show that it admits a subsequence converging to a real analytic metric $\overline{g}$. Nadirashvili [17] proved that a metric maximizing $\lambda_1$ on a surface $\Sigma$ is also $\lambda_1$-minimal. This means that $(\Sigma, \overline{g})$ is minimally immersed in a round sphere by its first eigenfunctions. It was proved by El Soufi and Ilias [7] that for any closed manifold other than the sphere (in particular for the torus), the isometry group of a $\lambda_1$-minimal metric $\overline{g}$ coincides with its group of conformal transformations, (see also [16]). Since the group of conformal transformations of a torus acts transitively, this implies that the curvature of $\overline{g}$ is constant. By the Gauss–Bonnet theorem, the curvature of $\overline{g}$ must therefore be zero. The result of Berger [2] stated above completes the proof.

The first step in showing that $(g_n)$ admits a convergent subsequence is to prove that the associated sequence $([g_n])$ of conformal classes admits a converging subse-

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quence. Since $\lambda_1(g_{eq}) = 8\pi^2/\sqrt{3} > 8\pi$, Theorem 1.2.2 implies that the corresponding sequence $(a_n + ib_n)$ is bounded and therefore admits a convergent subsequence.

*Remark* 1.3.1.

- Explicit $\lambda_1$-maximal metrics are also known for the projective plane [18] and the Klein bottle [6, 14]. There is a conjecture for surfaces of genus two [13].
- The existence of analytic $\lambda_1$-maximal metrics has recently been used in [14] and [6].

## 1.4 Conformal Degeneration on the Klein Bottle

Define two affine transformations $t_b$ and $\tau$ of $\mathbb{C}$ by

$$t_b(x + iy) = x + i(y + b), \quad \tau(x + iy) = x + \pi - iy.$$

Let $G_b$ be the group of transformations generated by $t_b$ and $\tau$.

***Lemma 1.4.1*** *Any Riemannian metric g on the Klein bottle $\mathbb{K}$ is conformally equivalent to one of the standard flat models $K_b := \mathbb{C}/G_b$. In other words, there exists a smooth function $\alpha\colon K_b \to ]0, \infty[$ such that $(\mathbb{K}, g)$ is isometric to $\big(K_b, \alpha(dx^2 + dy^2)\big)$.*

It follows that the moduli space of conformal classes on the Klein bottle is identified with the set of positive real numbers.

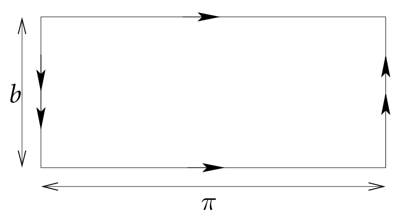


*Figure 2*: Author: Is this figure referred to anywhere?

***Theorem 1.4.2*** *Let $(g_n) \subset \mathcal{R}(\mathbb{K})$ be a sequence of metrics on the Klein bottle.*

(1) *If* $\lim_{n\to\infty} b_n = 0$, *then* $\limsup_{n\to\infty} \lambda_1(g_n) = 8\pi$.
(2) *If* $\lim_{n\to\infty} b_n = \infty$, *then* $\limsup_{n\to\infty} \lambda_1(g_n) \le 12\pi$.

The proof will be presented in Section 4. We follow the outline proposed by Nadirashvili in [17] and in a private communication in 2005. The first case is very similar to the corresponding result for the torus (Theorem 1.2.2). The second case uses the fact that the standard metric on $\mathbb{R}P^2$ is minimally embedded in $S^4$ by its first eigenfunctions. A theorem of Li and Yau [18] on conformal area of minimal surfaces is then used to obtain an estimate on the Dirichlet energy of a test function.

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### 1.5 Friedlander and Nadirashvili Invariant

For a closed manifold $M$ of dimension at least 3, Colbois and Dodziuk [4] proved that the first eigenvalue is unbounded on the set of Riemannian metrics of unit area, that is $\Lambda(M) = +\infty$. On the other hand, it is known that the supremum $\nu(C)$ of $\lambda_1$ restricted to metrics of unit area in any fixed conformal class $C$ is finite [8]. Friedlander and Nadirashvili [9] introduced the differential invariant

$$I(M) := \inf\{\nu(C) \mid C \text{ is a conformal class on } M\}$$

and proved that it satisfies $I(M) \geq \lambda_1(S^n, g_{S^n})$, where $g_{S^n}$ is the round metric of unit area on the sphere $S^n$. It is unknown if this invariant distinguishes nonequivalent differential structures. In fact, it is very difficult to compute $I(\Sigma)$ explicitly, even for surfaces. Since any two metrics on $S^2$ are conformally equivalent, it is obvious that $I(S^2) = 8\pi$. For similar reasons, $I(\mathbb{R}P^2) = 12\pi$. The invariant for the torus and for the Klein bottle are obtained as corollaries to Theorems 1.2.2 and 1.4.2.

***Corollary 1.5.1*** *The Friedlander–Nadirashvili invariants of the torus and of the Klein bottle are* $8\pi$.

This result is in agreement with the following.

***Conjecture 1.5.2*** **(Friedlander–Nadirashvili)** *For any closed surface* $\Sigma$ *other than the projective plane* $\mathbb{R}P^2$, $I(\Sigma) = 8\pi$.

## 2 Analytic Background

Let $\Sigma$ be a closed surface. In dimension two the Dirichlet energy of a function $u \in C^\infty(\Sigma)$, $D(u) = \int_\Sigma |\nabla_g u|^2 dg$ is invariant under conformal diffeomorphisms. In order to estimate the first eigenvalue of the Laplace operator $\Delta_g$, the following variational characterization will be used:

$$\lambda_1(g) = \inf\Big\{ \frac{D(u)}{\int_\Sigma u^2 dg} \;\Big|\; u \in C^\infty(\Sigma), u \neq 0, \int_\Sigma u dg = 0\Big\}. \tag{2.0.1}$$

### 2.1 Dirichlet Energy Estimate on Thin Cylinders

The main technical tool that we use is an estimate on the Dirichlet energy of harmonic functions on long cylinders in terms of their restrictions to the boundary circles.

***Lemma 2.1.1*** *Let* $\Omega = (0, L) \times S^1$ *with* $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. *Consider* $f \in C^\infty(\overline{\Omega})$ *such that* $f(0, \theta) = 0$ *and* $|f(L, \theta)| \leq 1$. *Let* $u(\theta) = f(L, \theta)$. *If* $f$ *is harmonic, then*

$$\int_\Omega |\nabla f|^2 \leq \frac{2\pi}{L} + \coth(L) \int_{\theta=0}^{2\pi} |u'(\theta)|^2 d\theta.$$

The idea is to express the Fourier series of the function $f$ in terms of the Fourier series of $u$. This is similar to Hurwitz's proof of the isoperimetric inequality [12].

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**Proof** Let the Fourier representation of $u$ be

$$u(\theta) = K + \sum_{k=1}^{\infty} \big( a_k \cos(k\theta) + b_k \sin(k\theta) \big) .$$

Direct computation shows that $f$ admits the representation

$$f(x,\theta) = \frac{Kx}{L} + \sum_{k=1}^{\infty} \frac{\sinh(kx)}{\sinh(kL)} \big( a_k \cos(k\theta) + b_k \sin(k\theta) \big) ,$$

and integration by parts leads to

$$\begin{aligned}
\int_\Omega |\nabla f|^2 &= -\int_\Omega f\Delta f + \int_{\partial\Omega} f \frac{\partial f}{\partial \nu} \\
&= \int_{\theta=0}^{2\pi} u(\theta)\partial_x f(x,\theta) d\theta \Big|_{x=L} \\
&= \int_{\theta=0}^{2\pi} \Big( K + \sum_{k=1}^{\infty} a_k \cos(k\theta) + b_k \sin(k\theta) \Big) \\
&\qquad \times \Big( \frac{K}{L} + \sum_{k=1}^{\infty} k \coth(kL) \big( a_k \cos(k\theta) + b_k \sin(k\theta) \big) \Big) d\theta \\
&= 2\pi \frac{K^2}{L} + \pi \sum_{k=1}^{\infty} k \coth(kL)(a_k^2 + b_k^2)
\end{aligned}$$

For $x > 0$,

$$\frac{d}{dx} \coth(x) = -\frac{4}{(e^x - e^{-x})^2} < 0,$$

so that for each $k \geq 1$, $\coth(kL) \leq \coth(L)$. It follows that

$$\begin{aligned}
\int_\Omega |\nabla f|^2 &\leq 2\pi \frac{K^2}{L} + \pi \coth(L) \sum_{k=1}^{\infty} k(a_k^2 + b_k^2) \\
&\leq 2\pi \frac{K^2}{L} + \pi \coth(L) \sum_{k=1}^{\infty} k^2(a_k^2 + b_k^2) \\
&= 2\pi \frac{K^2}{L} + \coth(L) \int_{\theta=0}^{2\pi} |u'(\theta)|^2 d\theta,
\end{aligned}$$

where $|K| = \frac{1}{2\pi} \big| \int_{\theta=0}^{2\pi} u(\theta) d\theta \big| \leq 1$, since $u(\theta) \in [-1,1]$. ■

A simple conformal change of coordinates is used to extend Lemma 2.1.1 to the situation where the boundary circle has arbitrary length.

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***Corollary 2.1.2*** *Suppose the hypothesis of Lemma* 2.1.1 *holds with the circle* $\mathbb{R}/2\pi\mathbb{Z}$ *replaced by* $\mathbb{R}/r\mathbb{Z}$. *For any* $L > 0$,

$$\int_{\Omega} |\nabla f|^2 \leq \frac{r}{L} + \frac{r}{2\pi} \coth(\frac{2\pi L}{r}) \int_{x=0}^{r} |u'(x)|^2 dx.$$

### 2.2 Conformal Renormalization of Centers of Mass

It is possible to conformally move any nonsingular distribution of mass on the sphere $S^n \subset \mathbb{R}^{n+1}$ in such a way that its center of mass becomes the origin of $\mathbb{R}^{n+1}$.

***Lemma 2.2.1*** **(Hersch Lemma)** *Let* $\mu$ *be a measure on the sphere* $S^n$. *If the support of* $\mu$ *is not a point, then there exists a conformal transformation* $\tau$ *of the sphere* $S^n$ *such that* $\int_{S^n} \pi \circ \tau \, d\mu = 0$, *where* $S^n \overset{\pi}{\hookrightarrow} \mathbb{R}^{n+1}$ *is the standard embedding.*

This lemma was obtained by Hersch [11] in 1970, (see also [19]). It is proved using a topological argument similar to the proof of the Brouwer fixed point theorem.

***Corollary 2.2.2*** *Let* $\mu$ *be a measure on a surface* $\Sigma$, *and consider an embedding* $\phi\colon \Sigma \to S^n$. *If the support of* $\mu$ *is not a point, then there exists a conformal transformation* $\tau$ *of* $S^n$ *such that* $\int_{\Sigma} \pi \circ \tau \circ \phi \, d\mu = 0$.

**Proof** The result follows from application of the Hersch Lemma to the push-forward measure $\phi_* \mu$ since $\int_{S^n} f \, d(\phi_* \mu) = \int_{\Sigma} f \circ \phi \, d\mu$ for any smooth function $f$. ■

## 3 Conformal Degeneration on the Torus

The goal of this section is to prove Theorem 1.2.2.

### 3.1 Moduli Space of Tori

For any $b > 0$, let

$$T_b := \{ [x + iy] \in \mathbb{C}/\mathbb{Z} \mid -b/2 < y < b/2 \}$$

be a cylinder of length $b$. Given $a + ib \in \mathcal{M}$, let $\Gamma_{a,b}$ be the lattice of $\mathbb{C}$ generated by 1 and $a + ib$. Consider the group $G_{a,b}$ of transformations of $T_\infty$ generated by

$$[x + iy] \mapsto [x + a + i(y + b)].$$

The cylinder $T_b$ is a fundamental domain of this action, and the torus $\mathbb{C}/\Gamma_{a,b}$ can also be obtained as $T_\infty / G_{a,b}$.

### 3.2 Concentration on Thin Cylinders

In order to make notation less cumbersome, consider a sequence $(g_n)$ of metrics such that $b_n = n$. The first eigenvalue of the flat torus corresponding to $g_n$,

$$\lambda_1(T_n / G_{a_n,n}) = \frac{4\pi^2}{n^2},$$

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tends to zero with $n$ going to infinity [1]. Imposing a uniform positive lower bound $\lambda_1(g_n) \geq K > 0$ on the first eigenvalues for $g_n$ should therefore imply that $g_n$ is "far from being flat". The next lemma makes this intuitive idea precise by showing that the Riemannian measures $dg_n$ concentrate on relatively thin cylindrical parts in $(T^2, g_n)$.

***Lemma 3.2.1*** *If* $\liminf_{n\to\infty} \lambda_1(g_n) > 0$, *then for any* $\epsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that for any* $n \geq N$,

$$\max\left\{ \int_{T_{3n/4}} dg_n, \int_{T_n \setminus T_{n/4}} dg_n \right\} \geq 1 - \epsilon.$$

Let $A_n^\epsilon$ be the maximizing cylinder: either $T_{3n/4}$ or $T_n \setminus T_{n/4}$. This lemma says that most of the mass (*i.e.*, $1 - \epsilon$) is concentrated on a cylinder whose length is $3/4$ of the total length. Without loss of generality, we will suppose $A_n^\epsilon = T_{3n/4}$.

**Proof** The function $\gamma_n([x + iy]) = \cos(2\pi y/n)$ is a first eigenfunction on the flat torus $T_n/G_{a_n,n}$ corresponding to $g_n$. Let $c_n = \int_{T_n} \gamma_n(x + iy)\, dg_n$ and define $h_n \colon T_n \to$

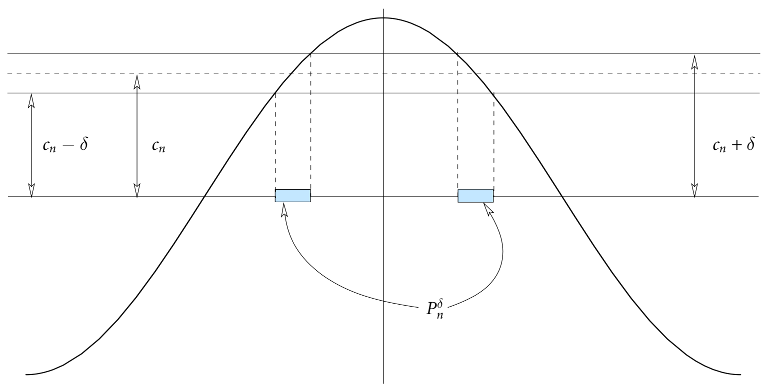


*Figure 3*: Concentration

$\mathbb{R}$ by $h_n = \gamma_n - c_n$, which satisfies $\int_{T_n} h_n\, dg_n = 0$. Given $\delta > 0$, let $P_n^\delta = h_n^{-1}([-\delta, \delta])$ and $Q_n^\delta = T_n \setminus P_n^\delta$. Since $\delta^2 \leq h_n^2$ on $Q_n^\delta$,

$$\delta^2 \int_{Q_n^\delta} dg_n \leq \int_{Q_n^\delta} h_n^2\, dg_n \leq \int_{T_n} h_n^2\, dg_n.$$

Using the variational characterization of $\lambda_1(g_n)$ and the conformal invariance of the Dirichlet energy,

$$\delta^2 \int_{Q_n^\delta} dg_n \leq \frac{\int_{T_n} |\nabla h_n|^2\, dg_n}{\lambda_1(g_n)} \leq \frac{\int_{T_n} |\nabla \gamma_n|^2\, dg_n}{K} = \frac{4\pi^2}{Kn^2} \to 0.$$

So that, for each $\delta > 0$, $\lim_{n\to\infty} \int_{Q_n^\delta} dg_n = 0$ and $\lim_{n\to\infty} \int_{P_n^\delta} dg_n = 1$. Observe that

$$P_n^\delta = \left\{ [x+iy] \;\middle|\; c_n - \delta \le cos(2\pi y/n) \le c_n + \delta \right\}.$$

For $\delta > 0$ small enough, $P_n^\delta$ is the union of two intervals around $\pm\arccos(c_n)$ whose lengths are at most $\frac{n}{10}$. The set $P_n^\delta$ will either be included in $T_{3n/4}$ or in $T_n \setminus T_{n/4}$, (see Figure 3). ■

### 3.3 Transplantation to the Sphere

Let $\sigma\colon \mathbb{C} \to S^2$ be the stereographic parametrization of the sphere by its equatorial plane

$$\sigma(u+iv) = \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1) \tag{3.3.1}$$

and define the conformal equivalence $\phi\colon \mathbb{C}/\mathbb{Z} \to S^2 \setminus \{\text{poles}\}$ by

$$\phi([z]) = \sigma(e^{-2\pi i z}).$$

### 3.4 Renormalization of the Centers of Mass

It follows from Corollary 2.2.2 that there exists conformal transformations $\tau_n$ such that

$$\int_{A_n^\epsilon} \pi \circ \tau_n \circ \phi \, dg_n = 0, \tag{3.4.1}$$

where $S^2 \overset{\pi}{\hookrightarrow} \mathbb{R}^3$ is the standard embedding. For each $n \in \mathbb{N}$, since $\pi_1^2+\pi_2^2+\pi_3^2 = 1$ on $S^2$, there exists an indice $i = i(n) \in \{1,2,3\}$ such that the function $u_n = \pi_i \circ \tau_n \circ \phi$ satisfies

$$\int_{A_n^\epsilon} u_n^2 \, dg_n \ge \frac{1}{3} \int_{A_n^\epsilon} dg_n \ge \frac{1-\epsilon}{3}. \tag{3.4.2}$$

### 3.5 Test Functions

The function $u_n$ will be extended and perturbed to a function $f_n$ defined on $T_n$ and admissible for the variational characterization (2.0.1) of $\lambda_1(g_n)$.

Let $I_n^- = [-7n/16, -6n/16]$ and $I_n^+ = [6n/16, 7n/16]$.

Given $\alpha_n^- \in I_n^-$ and $\alpha_n^+ \in I_n^+$, define cylinders (see Figure 5)

$$B(\alpha_n^-) = \left\{ [x+iy] \in T_n \mid y \le \alpha_n^- \right\},$$
$$B(\alpha_n^+) = \left\{ [x+iy] \in T_n \mid \alpha_n^+ \le y \right\}.$$

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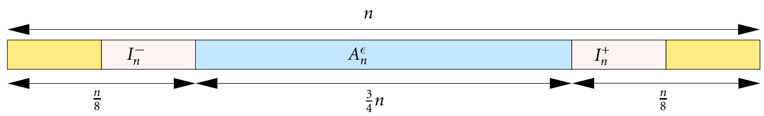


*Figure 4*: Concentration.

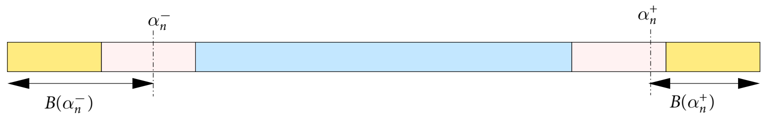


*Figure 5*: The cylinders $B(\alpha_n^{\pm})$

Their lengths are at least $n/16$. Define $w_n\colon T_n \to \mathbb{R}$ by the following differential problem:

$$\begin{cases} \Delta w_n = 0 & \text{on } B(\alpha_n^-) \cup B(\alpha_n^+), \\ w_n = 0 & \text{on } \partial T_n = \left\{ [x+iy] \in \mathbb{C}/\mathbb{Z} \mid |y| = n/2 \right\}, \\ w_n = u_n & \text{on } T_n \setminus (B(\alpha_n^-) \cup B(\alpha_n^+)). \end{cases}$$

Since the continuous function $w_n$ is piecewise smooth and satisfies $w_n = 0$ on the boundary of $T_n$, it is compatible with the identification of the boundary and induces a piecewise smooth function on the torus.

Let $\delta_n = \int_{T_n} w_n \, dg_n$. Since $\int_{A_n^\epsilon} w_n \, dg_n = 0$, it follows from concentration of the measures $dg_n$ on $A_n^\epsilon$ and from the maximum principle that

$$|\delta_n| = \Big| \int_{T_n \setminus A_n^\epsilon} w_n \, dg_n \Big| \leq \max_{x \in T_n \setminus A_n^\epsilon} |w_n(x)| \int_{T_n \setminus A_n^\epsilon} dg_n \leq \varepsilon.$$

This means that $w_n$ is almost admissible for the variational characterization of $\lambda_1(g_n)$. Define $f_n\colon T_n \to \mathbb{R}$ by

(3.5.1) $$f_n = w_n - \delta_n,$$

so that $\int_{T_n} f_n \, dg_n = 0$. From (3.4.1) and (3.4.2) it follows that

(3.5.2) $$\int_{T_n} f_n^2 \, dg_n \geq \int_{A_n^\epsilon} (u_n - \delta_n)^2 \, dg_n = \int_{A_n^\epsilon} u_n^2 \, dg_n + \delta_n^2 \int_{A_n^\epsilon} dg_n \geq \frac{\mathcal{A}(A_n^\epsilon, g_n)}{3} \geq \frac{1-\epsilon}{3}.$$

Using the variational characterization (2.0.1) of $\lambda_1(g_n)$ and conformal invariance of

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the Dirichlet energy leads to

$$
\begin{aligned}
(3.5.3)\quad \lambda_1(g_n) &\leq \frac{3\int_{T_n}|\nabla f_n|^2\,dg_n}{1-\varepsilon}\\
&= \frac{3}{1-\varepsilon}\Big(\int_{T_n\setminus(B(\alpha_n^-)\cup B(\alpha_n^+))}|\nabla w_n|^2\,dg_n + \int_{B(\alpha_n^-)\cup B(\alpha_n^+)}|\nabla w_n|^2\,dg_n\Big)\\
&\leq \frac{3}{1-\varepsilon}\Big(\int_{S^2}|\nabla \pi_i|^2\,dg_{S^2} + \int_{B(\alpha_n^-)\cup B(\alpha_n^+)}|\nabla w_n|^2\,dg_n\Big)\\
&= \frac{8\pi}{1-\varepsilon} + \frac{3}{1-\varepsilon}\int_{B(\alpha_n^-)\cup B(\alpha_n^+)}|\nabla w_n|^2\,dg_n.
\end{aligned}
$$

## 3.6 Energy Estimate

On a long flat cylinder like $B(\alpha_n^{\pm})$, the Dirichlet energy of a harmonic function is controlled by the Dirichlet energy of its restriction to the boundary circles. Corollary 2.1.2 is applied to the function $f = w_n$ on the cylinders $\Omega = B(\alpha_n^{\pm})$. Their lengths are at least $n/16$. For any $x$, $u(x) := f(x, \alpha_n^{\pm}) \in [-1, 1]$ since it is a coordinate function on the sphere.

The next lemma shows that the numbers $\alpha_n^{\pm} \in I_n^{\pm}$ can be chosen to make the Dirichlet energy of $u_n$ on the boundary of $B(\alpha_n^{\pm})$ small.

***Lemma 3.6.1*** *There exist $\alpha_n^- \in I_n^-$ and $\alpha_n^+ \in I_n^+$ such that*

$$\int_{x=0}^{1}|\partial_x u_n(x+i\alpha_n^{\pm})|^2 dx \leq \frac{128\pi}{3n}.$$

**Proof** Let $E_n^{\pm} = \{[x+iy] \mid y \in I_n^{\pm}\}$. Since the width of $I_n^{\pm}$ is at least $n/16$, the mean-value theorem implies

$$\iint_{E_n^{\pm}}|\nabla u_n|^2\,dxdy \geq \frac{n}{16}\min\Big\{\int_{x=0}^{1}|\nabla u_n(x+i\alpha_n^{\pm})|^2 dx \;\Big|\; \alpha_n^{\pm}\in I_n^{\pm}\Big\}$$

In particular, there exists $\alpha_n^{\pm} \in I_n^{\pm}$ such that

$$\int_{x=0}^{1}|\partial_x u_n(x+i\alpha_n^{\pm})|^2 dx \leq \frac{16}{n}\iint_{E_n^{\pm}}|\nabla u_n|^2\,dxdy.$$

By conformal invariance of the Dirichlet energy, this is bounded above by

$$\frac{16}{n}\int_{\tau_n\circ\phi(E_n^{\pm})}|\nabla\pi_i|^2\,dg_{S^2} \leq \frac{16}{n}\int_{S^2}|\nabla\pi_i|^2\,dg_{S^2} = \frac{16}{n}.\frac{8\pi}{3} = \frac{128\pi}{3n}. \qquad \blacksquare$$

**Proof of Theorem 1.2.2** Let $f_n$ be the function given by (3.5.1). Using the estimate on boundary derivative (Lemma 3.6.1), the Dirichlet energy estimate on cylinders

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(Lemma 2.1.1) implies $\lim_{n\to\infty}\int_{B(\alpha_n^{\pm})}|\nabla f_n|^2\,dg_n = 0$. Using inequality (3.5.3), obtained by the variational characterization of $\lambda_1(g_n)$, it follows that $\limsup_n \lambda_1(g_n) \leq \frac{8\pi}{1-\epsilon}$. Since $\epsilon > 0$ is arbitrary, it follows that $\limsup_{n\to\infty}\lambda_1(g_n) \leq 8\pi$.

Finally, the lower bound follows from the result of Friedlander and Nadirashvili [9] stated in the introduction: for any conformal class $C$ on a closed surface, $\nu(C) \geq 8\pi$. ∎

## 4 Conformal Degeneration on the Klein Bottle

The goal of this section is to prove Theorem 1.4.2. Let $S^k(r)$ be the $k$-dimensional sphere of radius $r$ with its standard metric $g_{S^k(r)}$. Let $\mathbb{R}P^k(r)$ be the associated projective space with standard metric $g_{\mathbb{R}P^k(r)}$. Recall from the introduction that any Klein bottle is conformally equivalent to a unique $K_b = \mathbb{C}/G_b$, where $G_b$ is the group of transformations of $\mathbb{C}$ generated by $t_b(x+iy) = x+i(y+b)$ and $\tau(x+iy) = x+\pi-iy$. The rectangle $[0,\pi]\times[-b/2,b/2]$ is a fundamental domain for the action of $G_b$ on $\mathbb{C}$. Reversing and identifying the opposite vertical sides of this rectangle, we obtain a Möbius strip $M_b = \big([0,\pi]\times[-b/2,b/2]\big)/\tau$.

### 4.1 Transplantation to the Sphere $S^4$ Via Projective Space

In this paragraph we exhibit a conformal embedding of the infinite Möbius strip $M_\infty$ in the sphere $S^4$. We start with a lemma which will be used to embed a Möbius strip conformally in $\mathbb{R}P^2$.

***Lemma 4.1.1*** *The conformal application $\phi\colon \mathbb{C}\to S^2\subset\mathbb{R}^3$ defined by*

$$\phi(x+iy) = \frac{1}{e^{2y}+1}(2e^y\cos(x), 2e^y\sin(x), e^{2y}-1)$$

*satisfies*

$$\phi(x+2\pi+iy) = \phi(x+iy), \quad \phi(\tau(x+iy)) = -\phi(x+iy).$$

*It induces a conformal equivalence $\phi\colon M_\infty \to \mathbb{R}P^2\setminus\{[0:0:1]\}$.*

The Veronese map $v$ is a well known minimal isometric embedding of $\mathbb{R}P^2(\sqrt{3})$ in the sphere $S^4$ by its first eigenfunctions. This means that the components of $v$ are eigenfunctions for $\lambda_1(g_{\mathbb{R}P^2(\sqrt{3})}) = 2$. For details, see [1] and [15].

It follows that the composition $v\circ\phi$ is a conformal embedding of the Möbius strip $M_\infty$ in $S^4$.

### 4.2 Concentration on Möbius Strips

Without loss of generality, consider a sequence $b_n = n$.

***Lemma 4.2.1*** *If $\liminf_{n\to\infty}\lambda_1(g_n) > 0$, then for any $\epsilon > 0$, there exists $N\in\mathbb{N}$ such that for any $n \geq N$,*

$$\max\Big\{\int_{M_{3n/4}} dg_n, \int_{M_n\setminus M_{3n/4}} dg_n\Big\} \geq 1-\epsilon.$$

**Proof** Since the function $u_n([x+iy]) = \cos(\frac{2\pi y}{n})$ used in the proof of Lemma 3.2.1 is even, it induces a first eigenfunction on the flat Klein bottle $K_n$. The cylinders constructed in this proof are compatible with the identification on the Möbius strip, also because the function $u_n$ is even. ∎

***Notation*** Without loss of generality, we suppose the maximum is reached by $A_n^\epsilon := M_{3n/4}$. See Lemma 3.2.1 for details.

### 4.3 Renormalization of the Centers of Mass

It follows from Corollary 2.2.2 that there exist conformal transformations $\tau_n$ of $S^4$ such that $\int_{A_n^\epsilon} \pi \circ \tau_n \circ \nu \circ \phi \, dg_n = 0$, where $\pi\colon S^4 \hookrightarrow \mathbb{R}^5$ is the standard embedding. For $1 \le i \le 5$, let

$$u_n^i = \pi_i \circ \tau_n \circ \nu \circ \phi. \tag{4.3.1}$$

### 4.4 Test Functions

The numbers

$$\alpha_n \in I_n := \Big[\frac{6n}{16}, \frac{7n}{16}\Big]$$

will be chosen later (see Lemma 3.6.1). For each $1 \le i \le 5$, define $w_n^i\colon M_n \to \mathbb{R}$ by the following differential problem:

$$\begin{cases} \Delta w_n^i = 0 & \text{on } M_n \setminus M_{\alpha_n}, \\ w_n^i = 0 & \text{on } \partial M_n = \big\{ [(x,y)] \in M_n \bigm| |y| = n/2 \big\}, \\ w_n^i = u_n^i & \text{on } M_{\alpha_n}, \end{cases}$$

where $u_n^i$ is defined by (4.3.1). Since the continuous function $w_n^i$ is piecewise smooth and satisfies $w_n^i = 0$ on the boundary of $M_n$, it is compatible with the identification of the boundary and induces a piecewise smooth function on the Klein bottle $K_n$.

Since $\int_{A_n^\epsilon} w_n^i \, dg_n = 0$, it follows from concentration of the measures $dg_n$ on $A_n^\epsilon$ and from the maximum principle that $\delta_n^i := \int_{M_n} w_n^i \, dg_n$ satisfies $|\delta_n^i| \le \epsilon$. This means that $w_n^i$ is almost admissible for the variational characterization of $\lambda_1(g_n)$. Thus, it is natural to define $f_n^i\colon M_n \to \mathbb{R}$ by $f_n^i = w_n^i - \delta_n^i$, so that for each $i$, $\int_{M_n} f_n^i \, dg_n = 0$ and similarly to inequality (3.5.2)

$$\sum_{i=1}^{5} \int_{M_n} (f_n^i)^2 \, dg_n \ge \int_{A_n^\epsilon} dg_n \ge 1 - \epsilon. \tag{4.4.1}$$

It follows from inequality (4.4.1) and from the variational characterization of

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$\lambda_1(g_n)$ that

$$\lambda_1(g_n)(1-\epsilon) \le \lambda_1(g_n)\Big(\sum_{i=1}^{5}\int_{M_n}(f_n^i)^2\,dg_n\Big) \le \sum_{i=1}^{5}\int_{M_n}|\nabla f_n^i|^2\,dg_n = \sum_{i=1}^{5}\Big(\int_{M_{\alpha_n}}|\nabla w_n^i|^2\,dg_n + \int_{M_n\setminus M_{\alpha_n}}|\nabla w_n^i|^2\,dg_n\Big). \tag{4.4.2}$$

## 4.5 Energy Estimate

*First step: bounding* $\sum_{i=1}^{5}\int_{M_{\alpha_n}}|\nabla w_n^i|^2\,dg_n$.

Recall that the Veronese embedding $v\colon \mathbb{R}P^2(\sqrt{3}) \to S^4$ is isometric and minimal. On $\Sigma_n := \tau_n \circ v(\mathbb{R}P^2(\sqrt{3}))$ we consider the metric induced by $g_{S^4}$. Proposition 1 of [18] says that if a compact surface is minimally immersed in a sphere, then its area cannot be increased by conformal transformations of the sphere. In our particular case this leads to the following proposition.

***Proposition 4.5.1*** *For each $n \in \mathbb{N}$,*

$$\int_{\Sigma_n} dg_{S^4} \le \text{Area of } \mathbb{R}P^2(\sqrt{3}) = 6\pi.$$

It follows from conformal invariance of the Dirichlet energy that

$$\sum_{i=1}^{5}\int_{M_{\alpha_n}}|\nabla w_n^i|^2\,dg_n \le \sum_{i=1}^{5}\int_{\Sigma_n}|\nabla \pi_i|^2\,dg_{S^4}.$$

It is proved on page 146 of [19] that $\Sigma_n$ being isometrically immersed in $S^4$ implies the pointwise identity $\sum_{i=1}^{5}|\nabla\pi_i|^2 = 2$. Whence

$$\sum_{i=1}^{5}\int_{M_{\alpha_n}}|\nabla w_n^i|^2\,dg_n \le 2\int_{\Sigma_n} dg_{S^4} \le 12\pi. \tag{4.5.1}$$

*Second step: bounding* $\sum_{i=1}^{5}\int_{M_n\setminus M_{\alpha_n}}|\nabla w_n^i|^2\,dg_n$.

The function $w_n$ is harmonic on the set $M_n \setminus M_{\alpha_n}$. This is a cylinder of length $L_n := n - \alpha_n \ge 9/16n$ and of width $2\pi$. The next lemma shows that the Dirichlet energy of $w_n$ on the boundary of these cylinders can be controlled by appropriate choice of $\alpha_n$.

***Lemma 4.5.2*** *The number $\alpha_n \in I_n = [6n/16, 7n/16]$ can be chosen such that*

$$\sum_{i=1}^{5}\int_{x=0}^{\pi}|\partial_x w_n^i(x \pm i\alpha_n)|^2 dx \le 192\pi/n.$$

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**Proof** We argue as in Lemma 3.6.1. Let $E_n^\pm = \{[x+iy] \mid y \in I_n\}$. It follows from the mean-value theorem that

$$\frac{n}{16}\min_{\alpha_n^\pm \in I_n^\pm} \sum_{i=1}^{5} \int_{x=0}^{2\pi} |\nabla w_n^i(x,\alpha_n^\pm)|^2 dx \le \sum_{i=1}^{5} \iint_{E_n^\pm} |\nabla w_n^i(x+iy)|^2\, dxdy \le 12\pi. \quad \blacksquare$$

Since $M_n \setminus M_{\alpha_n}$ has length $L_n$ at least $n/16$ and is of width $2\pi$, Lemma 2.1.1 implies

$$\text{(4.5.2)} \quad \sum_{i=1}^{5} \int_{M_n\setminus M_{\alpha_n}} |\nabla w_n^i|^2\, dg_n \le \sum_{i=1}^{5} \Big( \frac{2\pi}{L_n} + \coth(L_n) \int_{x=0}^{2\pi} |\partial_x w_n^i(x \pm i\alpha_n)|^2\, dx \Big)$$
$$\le \frac{\pi}{n}\Big(160 + 5 \times 192 \coth(\frac{n}{16})\Big).$$

**Proof of Theorem 1.4.2** Substitution of inequality (4.5.1) and inequality (4.5.2) in the variational characterization (4.4.2) leads to

$$\limsup_{n\to\infty} \lambda_1(g_n)(1-\epsilon) \le 12\pi + \limsup_{n\to\infty} \frac{\pi}{n}\big(160 + 960\coth(n/16)\big) = 12\pi.$$

Since $\epsilon > 0$ is arbitrary, this completes the proof of Theorem 1.4.2. ■

## 5 Concentration to Points

The main goal of this section is to prove Theorem 1.1.3. We start by proving that concentration to a point has no influence on the spectrum.

**Proof of Proposition 1.1.2** There exists a sequence of diffeomorphisms $\phi_n$ such that $\lim_{n\to\infty} \phi_n(x) = p$ $dg_0$-almost everywhere. Indeed, let $f\colon M \to \mathbb{R}$ be any Morse function having $p$ as its unique local minimum and consider $\phi\colon \mathbb{R} \times M \to M$ to be its negative gradient flow with respect to $g_0$. Since the stable manifolds of any critical point other than $p$ are of codimension strictly greater than 1, they are $dg_0$-negligible, hence the sequence $\phi_n$ satisfies the required property and $g_n = \phi_n^* g_0$ concentrates to $p$. ■

We now proceed with the proof of Theorem 1.1.3.

### 5.1 Construction of a Neighborhood System

Let $(g_n) \subset \mathcal{R}(\Sigma)$ be a sequence of metrics concentrating to $p \in \Sigma$. Let $\mathbb{D}$ be the unit open disk in $\mathbb{C}$. Let $\eta\colon D \to \mathbb{D}$ be a conformal chart around $p$ such that $\eta(p) = 0$. Observe that since the metrics $g_n$ are all in the same conformal class, the same chart $\eta$ will be conformal for each of them.

***Lemma 5.1.1*** *There exists a conformal equivalence $\psi\colon D \setminus \{p\} \to (0,\infty) \times S^1$ and a family $\mathcal{U}_n \subset D$ of neighborhood of $p$ such that $\lim_{n\to\infty} \int_{\mathcal{U}_n} dg_n = 1$ and $\psi(D \setminus \mathcal{U}_n) = (0, L_n) \times S^1$ with $L_n \to \infty$ and $\psi(\partial D) = \{0\} \times S^1$.*

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**Proof** For $0 \le \epsilon \le 1$, let $\mathcal{U}(\epsilon) = \eta^{-1}B(0,\epsilon)$. Define $\epsilon_n$ by $\int_{\mathcal{U}(\epsilon_n)} dg_n = 1 - \epsilon_n$. It follows from concentration that $\lim_{n\to\infty} \epsilon_n = 0$ so that $\lim_{n\to\infty} \int_{\mathcal{U}(\epsilon_n)} dg_n = 1$. Define the conformal equivalence $\alpha\colon \mathbb{R}\times S^1 \to \mathbb{C}^*$ by $\alpha(x,y) = e^{-x}\big(\cos(y), \sin(y)\big)$. The composition $\psi = \alpha^{-1} \circ \eta$ has the required property since $\psi(D \setminus \mathcal{U}(\epsilon_n)) = \big(0, -\ln(\epsilon_n)\big) \times S^1$. ∎

## 5.2 Renormalization of the Centers of Mass

Recall that $\sigma$, as defined in (3.3.1), is the stereographic parameterization of the sphere $S^2$ by its equatorial plane. Let $H \subset S^2$ be the southern hemisphere of $S^2$. The map $\phi = \sigma \circ \eta\colon D \to H$ is a conformal equivalence such that $\phi(p)$ is the south pole. It follows from Corollary 2.2.2 that there exist conformal transformations $\tau_n$ of the sphere such that $\int_{\mathcal{U}_n} \pi\circ\tau_n\circ\phi \, dg_n = 0$, where $\pi\colon S^2 \hookrightarrow \mathbb{R}^3$ is the standard embedding. For each $n \in \mathbb{N}$, since $\pi_1^2+\pi_2^2+\pi_3^2 = 1$ on $S^2$, there exists an indice $i = i(n) \in \{1,2,3\}$ such that the function $u_n = \pi_i \circ \tau_n \circ \phi$ satisfies $\int_{\mathcal{U}_n} u_n^2 \, dg_n \ge \frac{1}{3}\int_{\mathcal{U}_n} dg_n$.

## 5.3 Test Functions

Consider $\alpha_n \in [L_n/2, L_n]$ to be chosen later. Define $w_n\colon \Sigma \to \mathbb{R}$ as the unique solution of

$$\begin{cases} w_n = 0 & \text{on } \Sigma \setminus D,\\ w_n = \pi_i \circ \tau_n \circ \phi & \text{on } \mathcal{U}_n \cup \psi^{-1}\big((\alpha_n, L_n) \times S^1\big),\\ \Delta w_n = 0 & \text{on } \psi^{-1}\big((0,\alpha_n) \times S^1\big).\end{cases}$$

By the maximum principle, $\delta_n := \int_\Sigma w_n \, dg_n \le \int_{\Sigma\setminus\mathcal{U}_n} dg_n$. Define $f_n = w_n - \delta_n$ so that $\int_\Sigma f_n = 0$ and $f_n$ is admissible for the variational characterisation of $\lambda_1(g_n)$:

$$\lambda_1(g_n) \le \frac{\int_\Sigma |\nabla f_n|^2 \, dg_n}{\int_\Sigma f_n^2 \, dg_n} \le \frac{\int_D |\nabla w_n|^2 \, dg_{S^2}}{\int_\Sigma (w_n - \delta_n)^2 \, dg_n}.$$

The denominator satisfies

$$\int_\Sigma (w_n - \delta_n)^2 \, dg_n \ge \int_{\mathcal{U}_n} (w_n^2 - 2\delta_n w_n + \delta_n^2) \, dg_n \ge \int_{\mathcal{U}_n} w_n^2 \, dg_n \ge \frac{1}{3}\int_{\mathcal{U}_n} dg_n.$$

Whence,

$$\begin{aligned}\frac{\lambda_1(g_n)}{3}\int_{\mathcal{U}_n} dg_n &\le \int_D |\nabla w_n|^2 \, dg_n\\ &\le \int_{\mathcal{U}_n\cup\psi_n^{-1}(0,\alpha_n)\times S^1} |\nabla w_n|^2 \, dg_n + \int_{\psi_n^{-1}(\alpha_n, L_n)\times S^1} |\nabla w_n|^2 \, dg_n\\ &\le \frac{8\pi}{3} + \int_{\psi_n^{-1}(\alpha_n, L_n)} |\nabla w_n|^2 \, dg_n.\end{aligned}$$

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**Proof of Theorem 1.1.3** The set $\psi^{-1}\big((0, \alpha_n) \times S^1\big)$, where $w_n$ is harmonic is conformally equivalent to a cylinder of length $\alpha_n \geq \frac{L_n}{2}$ which becomes infinite as $n$ goes to infinity. The proof is completed by choosing appropriate $\alpha_n$ as it was done in Lemma 3.6.1 and Lemma 4.5.2 and then applying Lemma 2.1.1 to bound the Dirichlet energy. ■

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