MULTIPLICITY OF POSITIVE PERIODIC SOLUTIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS

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In this paper, we study the existence of positive periodic solutions to the equation \( x'' = f(t, x) \). It is proved that such an equation has more than one positive periodic solution when the nonlinearity changes sign. The proof relies on a fixed point theorem in cones.

1. INTRODUCTION

In this paper, we are concerned with the existence of single and multiple (strictly) positive 1-periodic solutions to the equation

\[(1.1) \quad x'' = f(t, x),\]

where \( f(t, x) : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) is continuous and 1-periodic in the first variable. By a positive periodic solution of (1.1) we understand a function \( x \in C([0, 1], (0, \infty)) \) satisfying (1.1) and the periodic boundary condition

\[(1.2) \quad x(0) = x(1), \quad x'(0) = x'(1).\]

The existence of positive periodic solutions to equation (1.1) has been extensively studied in the literature (see, for example, [1, 2, 3] and the references therein). In these papers, the two most common techniques to establish existence are the theory of upper and lower solutions [4] and topological degree theory [5]. On the other hand, some fixed point theorems in cones for completely continuous operators have been extensively employed in studying the existence of positive solutions to boundary value problems [6]. However, for the periodic problem, a theory using cones has only recently [7] been applied. One of the difficulties involved in discussing the periodic problem is the sign of the Green’s functions for the corresponding linear periodic problem. In [7], Torres succeeded in overcoming this difficulty by using a new \( L^p \)-anti-maximum principle and obtained some new existence results for problem (1.1)-(1.2) by a well known fixed point theorem of compression and expansion of cones.

The aim of this paper is to use some of the basic results in [7] together with a new fixed point theorem in cones to obtain the existence of single and multiple positive periodic solutions to (1.1). The results we obtain are new.

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2. Preliminaries

Let \( a(t) \) be a 1-periodic function and \( a \in L^1(0,1) \). Now we consider the linear equation

\[
x'' + a(t)x = 0
\]
with the periodic boundary condition (1.2). In this section, we assume conditions under which the only solution to equation (2.1)-(1.2) is the trivial one. As a result, the nonhomogeneous problem

\[
x'' + a(t)x = h(t), \quad x(0) = x(1), \quad x'(0) = x'(1)
\]
has a unique solution given by

\[
x(t) = (Ch)(t) := \int_0^1 G(t,s)h(s)ds.
\]
Here \( G(t,s) \) is the Green function. Let us define

\[
\Lambda^- = \{ a < 0 \}, \quad \Lambda^+ = \{ a > 0, \|a\|_p < K(2q) \text{ for some } 1 \leq p \leq +\infty \}.
\]
Here the notation \( a > 0 \) means that \( a(t) > 0 \) for all \( t \in [0,1] \) and \( a(t) > 0 \) for \( t \) in a subset of positive measure, \( a < 0 \) means that \( -a > 0 \) and \( \| \cdot \|_p \) denotes the usual \( L^p \)-norm over \( (0,1) \) for any given exponent \( p \in [1,\infty] \). The conjugate exponent of \( p \) is denoted by \( q \): \( 1/p + 1/q = 1 \). The explicit formula for \( K(q) \) is

\[
K(q) = \begin{cases} 
\frac{2\pi}{q} \left( \frac{2}{2+q} \right)^{1-2/q} \left( \frac{\Gamma(1/q)}{\Gamma(1/2+1/q)} \right)^2 & \text{if } 1 \leq q < \infty, \\
4 & \text{if } q = \infty,
\end{cases}
\]
where \( \Gamma \) is the Gamma function. Now we present two basic results which were established by Torres in [7].

**Lemma 2.1.** ([7]) Assume that \( a(t) \in \Lambda^- \), then \( G(t,s) < 0 \) for all \( (t,s) \in [0,1] \times [0,1] \).

**Lemma 2.2.** ([7]) Assume that \( a(t) \in \Lambda^+ \), then \( G(t,s) > 0 \) for all \( (t,s) \in [0,1] \times [0,1] \).

**Remark 2.3.** If \( p = 1 \), condition \( \|a\|_p < K(2q) \) can be weakened to \( \|a\|_1 \leq K(\infty) = 4 \) by the celebrated stability criterion of Lyapunov. In case \( p = \infty \), condition \( \|a\|_p < K(2q) \) reads as \( \|a\|_\infty < K(2) = \pi^2 \), which is a well known criterion for the anti-maximum principle used in related literature. In this case, \( \|a\|_p < K(2q) \) can be weakened to \( a(t) < \pi^2 \).

In the following, we always denote

\[
m = \min_{0 \leq s,t \leq 1} G(t,s), \quad M = \max_{0 \leq s,t \leq 1} G(t,s), \quad \sigma = m/M \text{ if } a(t) \in \Lambda^+ \text{ and } \sigma = M/m \text{ if } a(t) \in \Lambda^-.
\]
Thus \( M > m > 0 \) if \( a(t) \in \Lambda^+ \) and \( m < M < 0 \) if \( a(t) \in \Lambda^- \). In either case, we have \( 0 < \sigma < 1 \).

In this paper we shall establish the existence of positive periodic solutions to equation (1.1), using the following well known fixed point theorem in cones [8].

**Theorem 2.4.** Let \( X \) be a Banach space and \( K \) be a cone in \( X \). Assume \( \Omega_1, \Omega_2 \) are open subsets of \( X \) with \( 0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2 \). Let

\[
\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K
\]

be a continuous and completely continuous operator such that

(i) \( \|\Phi x\| \leq \|x\| \) for \( x \in K \cap \partial \Omega_1 \),

(ii) there exists \( \psi \in K \setminus \{0\} \) such that \( x \neq \Phi x + \lambda \psi \) for \( x \in K \cap \partial \Omega_2 \) and \( \lambda > 0 \).

Then \( \Phi \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

**Remark 2.5.** In Theorem 2.4, if (i) and (ii) are replaced by

(i) \( \|\Phi x\| \leq \|x\| \) for \( x \in K \cap \partial \Omega_2 \), and

(ii) there exists \( \psi \in K \setminus \{0\} \) such that \( x \neq \Phi x + \lambda \psi \) for \( x \in K \cap \partial \Omega_1 \) and \( \lambda > 0 \),

then \( \Phi \) has also a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).

In applications below, we take \( X = C[0,1] \) with the supremum norm \( \| \cdot \| \) and define

\[
K = \left\{ x \in X : x(t) \geq 0 \text{ for all } t \text{ and } \min_{0 \leq t \leq 1} x(t) \geq \sigma \|x\| \right\},
\]

where \( \sigma \) is as in (2.6).

One may readily verify that \( K \) is a cone in \( X \). Suppose now that \( F : [0,1] \times \mathbb{R} \to [0, \infty) \) is a continuous function and define an operator \( T : X \to X \) by

\[
(Tx)(t) = \int_0^1 G(t,s)F(s,x(s)) \, ds
\]

for \( x \in X \) and \( t \in [0,1] \). It is easy to prove:

**Lemma 2.6.** \( T \) is well defined and maps \( X \) into \( K \). Moreover, \( T \) is continuous and completely continuous.

### 3. Main results

In this section we establish the existence and multiplicity of positive periodic solutions to (1.1).

**Theorem 3.1.** Suppose that there exist \( a \in \Lambda^+ \) and \( 0 < r < R \) such that

\[
f(t, x) + a(t)x \geq 0, \quad \forall \ x \in [\sigma r, R].
\]

Then Equation (1.1) has at least one positive solution if one of the following two conditions holds
(I) \( f(t, x) \geq 0, \quad \forall x \in [\sigma r, r] \) and \( f(t, x) \leq 0, \quad \forall x \in [\sigma R, R] \);

(II) \( f(t, x) \leq 0, \quad \forall x \in [\sigma r, r] \) and \( f(t, x) \geq 0, \quad \forall x \in [\sigma R, R] \).

**Proof:** The existence is established using Theorem 2.4 and Remark 2.5. To do so, let us write equation (1.1) as

(3.1) \( x'' + a(t)x = f(t, x) + a(t)x \).

Define the open sets

\( \Omega_r = \{ x \in C[0, 1] : \|x\| < r \} \) and \( \Omega_R = \{ x \in C[0, 1] : \|x\| < R \} \).

Let \( K \) be a cone defined by (2.7) and define an operator on \( K \) by

(3.2) \((\Phi x)(t) = \int_0^1 G(t, s) \left[ f(s, x(s)) + a(t)x \right] ds \).

Clearly, \( \Phi : K \cap (\Omega_R \setminus \Omega_r) \rightarrow C[0, 1] \) is continuous and completely continuous since \( f : [0, 1] \times [\sigma r, R] \rightarrow R \) is continuous. Also we have \( \Phi(K) \subset K \).

Let us suppose that condition (I) holds (the proof for condition (II) is similar).

By the first inequality of condition (I), we have \( f(t, x) + a(t)x \leq a(t)x, \forall x \in [\sigma r, r] \).

Let \( \psi \equiv 1 \), so \( \psi \in K \). Now we prove that

(3.3) \( x \not= \Phi x + \lambda \psi, \forall x \in K \cap \partial \Omega_r \) and \( \lambda > 0 \).

Suppose not, that is, suppose there exist \( x_0 \in K \cap \partial \Omega_r \) and \( \lambda_0 > 0 \) such that \( x_0 = \Phi x_0 + \lambda_0 \psi \). Now since \( x_0 \in K \cap \partial \Omega_r \), then \( x_0(t) \geq \sigma \|x_0\| = \sigma r \). Let \( \mu = \min_{t \in [0, 1]} x_0(t) \).

Then we have

\[
x_0(t) = (\Phi x_0)(t) + \lambda_0 = \int_0^1 G(t, s) \left[ f(s, x_0(s)) + a(t)x_0(s) \right] ds + \lambda_0 \\
\geq \int_0^1 G(t, s)a(s)x_0(s) ds + \lambda_0 \geq \mu \int_0^1 G(t, s)a(s) ds + \lambda_0 = \mu + \lambda_0.
\]

This implies \( \mu \geq \mu + \lambda_0 \), a contradiction. Therefore, (3.3) holds.

On the other hand, by the second inequality of condition (I), we have

\( f(t, x) + a(t)x \leq a(t)x, \forall x \in [\sigma R, R] \).

Now we prove that

(3.4) \( \|\Phi x\| \leq \|x\|, \forall x \in K \cap \partial \Omega_R \).

In fact, for any \( x \in K \cap \partial \Omega_R \), we have

\[
(\Phi x)(t) = \int_0^1 G(t, s) \left[ f(s, x(s)) + a(t)x \right] ds \leq \int_0^1 G(t, s)a(s)x(s) ds \\
\leq \int_0^1 G(t, s)a(s) ds \cdot \max_{t \in [0, 1]} x(t) = \|x\|.
\]
Therefore, $\|\Phi x\| \leq \|x\|$, that is, (3.4) holds.

It follows from Remark 2.5, (3.3) and (3.4) that $\Phi$ has a fixed point $x \in K \cap (\overline{\Omega}_R \setminus \Omega_r)$. Clearly, this fixed point is a positive solution of (1.1) satisfying $r \leq \|x\| \leq R$.

REMARK 3.2. In [7, Theorem 3.2], it is proved that equation (1.1) has at least one positive periodic solution provided one of the following two conditions holds for some $a(t) \in \Lambda^+$ and $0 < r < R$:

(I)* $f(t, x) + a(t)x \geq (M/m^2)x \quad \forall x \in [(m/M)r, r]; \quad f(t, x) + a(t)x \leq 1/M, \quad \forall x \in [R, (M/m)R];$

(II)* $f(t, x) + a(t)x \leq 1/M, \quad \forall x \in [(m/M)r, r]; \quad f(t, x) + a(t)x \geq (M/m^2)x, \quad \forall x \in [R, (M/m)R].$

Theorem 3.1 improves the above result since we only need the sign of $f(t, x)$ in (I) and (II).

The following multiplicity result follows immediately from Theorem 3.1.

THEOREM 3.3. Suppose that there exist $a \in \Lambda^+$ and $0 < r < p < R$ such that

$$f(t, x) + a(t)x \geq 0, \quad \forall x \in [\sigma r, R].$$

Then Equation (1.1) has at least two positive periodic solutions if one of the following two conditions holds

(I) $f(t, x) \geq 0, \quad \forall x \in [\sigma r, r]; \quad f(t, x) < 0, \quad \forall x \in [\sigma p, p]; \quad f(t, x) \geq 0, \quad \forall x \in [R, (M/m)R];$

(II) $f(t, x) \leq 0, \quad \forall x \in [\sigma r, r]; \quad f(t, x) > 0, \quad \forall x \in [\sigma p, p]; \quad f(t, x) \leq 0, \quad \forall x \in [R, (M/m)R].$

PROOF: We only prove the result when condition (I) holds. Define $\Omega_r$, $\Omega_R$, $K$ and $\Phi$ as in Theorem 3.1 and define $\Omega_p = \{x \in C[0, 1] : \|x\| < p\}$.

Essentially the same reasoning as in the proof of Theorem 3.1 guarantees that

(3.5) $x \neq \Phi x + \lambda \psi$ for $\forall x \in K \cap \partial \Omega_r$ and $\lambda > 0$;

(3.6) $x \neq \Phi x + \lambda \psi$ for $\forall x \in K \cap \partial \Omega_R$ and $\lambda > 0$;

(3.7) $\|\Phi x\| < \|x\|$ for $\forall x \in K \cap \partial \Omega_p$.

Thus we can obtain the existence of two positive solutions $x_1$ and $x_2$ by using Theorem 2.4 and Remark 2.5 once, respectively. It is easy to see that $r \leq \|x_1\| < p < \|x_2\| \leq R$ since (3.7) holds.

Next we consider the case of $a(t) \in \Lambda^-$. Here we only state the results and omit their proofs since they can be proved in a similar way to that of Theorems 3.1 and 3.3.

THEOREM 3.4. Suppose that there exist $a \in \Lambda^-$ and $0 < r < R$ such that

$$f(t, x) + a(t)x \leq 0, \quad \forall x \in [\sigma r, R].$$
Then Equation (1.1) has at least one positive periodic solution if one of the following two conditions holds

(I) \( f(t, x) > 0, \forall x \in [\sigma r, r] \) and \( f(t, x) \leq 0, \forall x \in [\sigma R, R] \);

(II) \( f(t, x) \leq 0, \forall x \in [\sigma r, r] \) and \( f(t, x) \geq 0, \forall x \in [\sigma R, R] \).

**Theorem 3.5.** Suppose that there exist \( a \in \Lambda^- \) and \( 0 < r < p < R \) such that

\[ f(t, x) + a(t)x \leq 0, \forall x \in [\sigma r, R]. \]

Then Equation (1.1) has at least two positive periodic solutions if one of the following two conditions holds

(I) \( f(t, x) > 0, \forall x \in [\sigma r, r]; f(t, x) < 0, \forall x \in [\sigma p, p]; f(t, x) \geq 0, \forall x \in [\sigma R, R] \);

(II) \( f(t, x) \leq 0, \forall x \in [\sigma r, r]; f(t, x) > 0, \forall x \in [\sigma p, p]; f(t, x) \leq 0, \forall x \in [\sigma R, R] \).

**Remark 3.6** In fact, we can obtain the existence of more than two positive periodic solutions of equation (1.1) provided \( f(t, x) \) satisfies the required inequalities.

**Example 3.7.** Let us consider the following nonsingular equation

\[ x'' + a(t)x = \mu b(t)(x^\alpha + x^\beta), \]

where \( 0 < \alpha < 1 < \beta, a \in C[0, 1], b \in C[0, 1] \) is a positive function, \( a(t) \in \Lambda^+ \) and \( \mu > 0 \) is a positive parameter. Then equation (3.8) has at least two positive periodic solutions for each \( 0 < \mu < \mu^* \), where \( \mu^* \) is a positive constant described below.

To show this we shall apply Theorem 3.3 with \( f(t, x) = \mu b(t)(x^\alpha + x^\beta) - a(t)x \). It is easy to see that

\[ \lim_{x \to 0^+} \frac{f(t, x)}{x} = +\infty \quad \text{and} \quad \lim_{x \to +\infty} \frac{f(t, x)}{x} = +\infty \]
since \( 0 < \alpha < 1 < \beta \). Set

\[ T(x) = \frac{x}{x^\alpha + x^\beta}, \quad x > 0. \]

Then \( T(0^+) = T(\infty) = 0 \) and

\[ T(x) \leq T(p) = \sup_{x \in (0, \infty)} T(x), \quad \text{where} \quad p = \left(1 - \frac{\alpha}{\beta - 1}\right)^{1/(\beta - \alpha)}. \]

Let \( \mu^* = \sigma T(p)e^{-1}, \) where \( e = \max_{t \in [0, 1]} (b(t))/(a(t)) \). Then for \( x \in [\sigma p, p] \), we have

\[ f(t, x) = \mu b(t)(x^\alpha + x^\beta) - a(t)x < \mu^* a(t)(p^\alpha + p^\beta) \max_{t \in [0, 1]} \frac{b(t)}{a(t)} - a(t)\sigma p \]

\[ = \sigma T(p)a(t)(p^\alpha + p^\beta) - a(t)\sigma p = 0. \]
(3.9) and (3.10) imply that condition (I) of Theorem 3.3 is satisfied, so the existence is guaranteed.

EXAMPLE 3.8. Let us consider the following singular repulsive equation [7]

\[ x'' - \frac{a}{x^{\lambda}} + k^2 x = e(t) \]  

with \( a > 0, \ k \in (0, \pi), \ \lambda > 0 \) and \( e \in C[0,1] \). Let \( e^* = \max_{t \in [0,1]} e(t) \) and \( e_* = \min_{t \in [0,1]} e(t) \).

Then

(i) Equation (3.11) has at least one positive periodic solution for each \( e(t) \) with \( e_* \geq 0 \); and

(ii) Equation (3.11) has at least one positive periodic solution for each \( e(t) \) with \( e_* < 0 \) and satisfying the following inequality:

\[ e^* \leq \frac{e_*}{\cos^{\lambda}(k/2)} + k^2 \left( \frac{a}{|e_*|} \right)^{1/\lambda} \cos(k/2). \]  

If \( k \in (0, \pi) \), then \( k \in \Lambda^+ \) and we can obtain the following explicit values (see [7])

\[ m = \frac{1}{2k} \cot \left( \frac{k}{2} \right), \quad M = \frac{1}{2k \sin(k/2)} \quad \text{and} \quad \sigma = \cos \left( \frac{k}{2} \right). \]

Now (i) is a direct result of Theorem 3.1 since \( f(t,x) = a/(x^{\lambda}) + e(t) - k^2 x \to +\infty \) as \( x \to 0 \) and \( f(t,x) \to -\infty \) as \( x \to +\infty \).

Next we prove (ii). Condition (I)of Theorem 3.1 reduces to finding \( R > 0 \) such that

\[ \frac{a}{x^{\lambda}} + e_* \geq 0, \quad \forall x \in (0, R] \]  

and

\[ \frac{a}{x^{\lambda}} + e^* \leq k^2 x, \quad \forall x \in \left[ R \cos \left( \frac{k}{2} \right), R \right] \]

Now, we fix \( R = (a/|e_*|)^{1/\lambda} \), then inequality (3.13) is satisfied. By using the monotonocity of \( k^2 x - (a/x^{\lambda}) \), then (3.14) holds if

\[ e^* \leq \frac{e_*}{\cos^{\lambda}(k/2)} + k^2 \left( \frac{a}{|e_*|} \right)^{1/\lambda} \cos(k/2). \]

REMARK 3.9. In [7], it is proved that equation (3.11) has at least one positive periodic solution if \( k \in (0, \pi), \ e \in L^\infty[0,1], \ e_* < 0 \) and the following inequality holds:

\[ e^* \leq \frac{e_*}{\cos^{\lambda}(k/2)} + k(a/|e_*|)^{1/\lambda} \sin(k). \]

It is easy to see that our condition (3.12) is weaker than condition (3.15) since

\[ k^2 \left( \frac{a}{|e_*|} \right)^{1/\lambda} \cos \left( \frac{k}{2} \right) \geq k \left( \frac{a}{|e_*|} \right)^{1/\lambda} \sin(k), \quad \forall k \in (0, \pi). \]
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