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Time-periodic generalised solitary waves with a hydraulic fall

Josh Shelton†

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

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In an open channel flow, deviations to the lower topography can induce abrupt changes in the wave height, known as hydraulic jumps. This phenomenon occurs when the flow switches from subcritical to supercritical (or *vice versa*), and is commonly observed in rivers, flumes and weirs. Theoretical insight is typically sought through the study of reduced models such as the forced Korteweg–de Vries equation, in which previous work has predominantly focused on either stationary formulations or the initial transient behaviour caused by perturbations. In a joint theoretical and numerical study of the free-surface Euler equations, Keeler & Blyth (*J. Fluid Mech.*, vol. 993, 2024, A9) have detected a new class of unsteady solutions to this problem. These emerge from an unstable steady solution, and feature large-amplitude time-periodic ripples emitted from a sudden decrease in the water depth forced by topography, known as a hydraulic fall.

Key words: solitary waves, topographic effects

1. Introduction

The influence of submerged objects and topography on the motion of surface water waves has long been an object of study due to the fascinating wave patterns that can emerge, and the application of these in engineering. Well-known examples of this behaviour include Kelvin wake patterns that form in a wedge behind moving disturbances in a fluid, such as ships (Reed & Milgram 2002), and von Kármán vortex streets generated behind bluff bodies (Thompson, Leweke & Hourigan 2021). Similar phenomena are also induced by topography, such as sea or river beds which confine the flowing fluid from below. For instance, at larger scales topography is known to enhance wave amplitudes and promote breaking (Peregrine 1983), and at smaller scales the effect of confinement and vibration at different frequencies excites Faraday wave patterns (Westra, Binks & Van De Water 2003). In rivers, tidal forcing can produce tidal bores, where a travelling surface wave propagates

† Email address for correspondence: j.shelton@bath.ac.uk

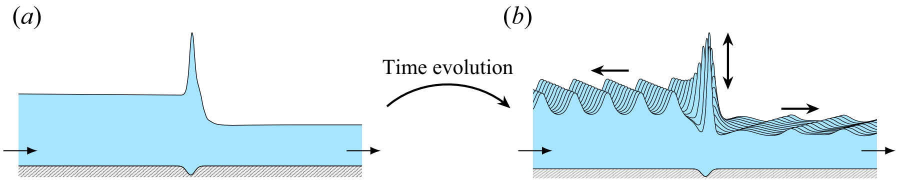

Figure 1. Numerical solutions for the flow of an irrotational fluid past indented topography. A steady unstable solution, denoted a hydraulic fall, is shown in (*a*). The transient behaviour from the instability eventually approaches a time-periodic solution, shown in (*b*). Figures produced from numerical results of Keeler & Blyth (2024).

upstream against the river current. This is an example of a moving hydraulic jump, in which the mean fluid levels upstream and downstream of the wave differ (Baines 2022).

Surface waves may also exhibit a hydraulic jump induced by fluid flow past a static deformation in the topography. This is known as a hydraulic fall if the upstream depth is greater than that downstream (Shen 1992). Here, the upstream fluid speed is slower than the surface wavespeed (subcritical flow), and the downstream fluid speed is faster than the wavespeed (supercritical flow). Thus, hydraulic-fall solutions permit surface waves of different forms to propagate upstream and downstream. They are typically associated with near-critical flows in which the upstream Froude number, $Fr = U/\sqrt{gH}$, is close to but smaller than 1. The non-dimensional Froude number characterises the balance between inertia (the upstream fluid speed U) and gravity (measured with the gravitational constant g and the upstream fluid depth H).

Given a specified topography, the aim is to solve for the fluid velocity and the wave surface. This is known as the forward problem, in contrast to the inverse problem, where one wishes to predict the topography given surface measurements. The modelling assumptions of inviscid, irrotational, and two-dimensional (2-D) flow are usually used, which have been supported by experimental evidence (Fadda & Raad 1997). This results in a nonlinear free-boundary problem for Laplace's equation posed on a 2-D domain. Due to the lack of exact solutions, progress is made numerically. Typical numerical approaches to solve this formulation include boundary-integral methods and conformal mapping, which yield a 1-D formulation for the wave surface (Ambrose *et al.* 2022), and the finite-element methods applied to the full 2-D problem. To make analytical progress, many authors additionally use weakly nonlinear and long-wave assumptions, which yield a forced Korteweg–de Vries (KdV) equation for the wave height defined on one spatial domain (Dias & Vanden-Broeck 2002).

2. The numerical findings of Keeler & Blyth (2024)

In their recent work, Keeler & Blyth (2024) considered the free-surface flow of a 2-D fluid satisfying the nonlinear Euler equations bounded from below by topography. Numerical solutions were obtained with the finite-element method on a truncated domain. Their choice for this topography, $y(x) = a\exp(-b^2x^2)$, represents a 2-D 'bump' for $a > 0$, and a 'dip' for $a < 0$, both of which are localised near $x = 0$. Here, a and b are specified constants. The key result of their work is the discovery of time-periodic solitary waves in this formulation, which emerge from linearly unstable solutions of the steady formulation. An example of this process is shown in figure 1.

Steady free-surface flow over protruding topography, analogous to $a > 0$, has been well studied in the potential flow literature for a plethora of geometries and submerged objects,

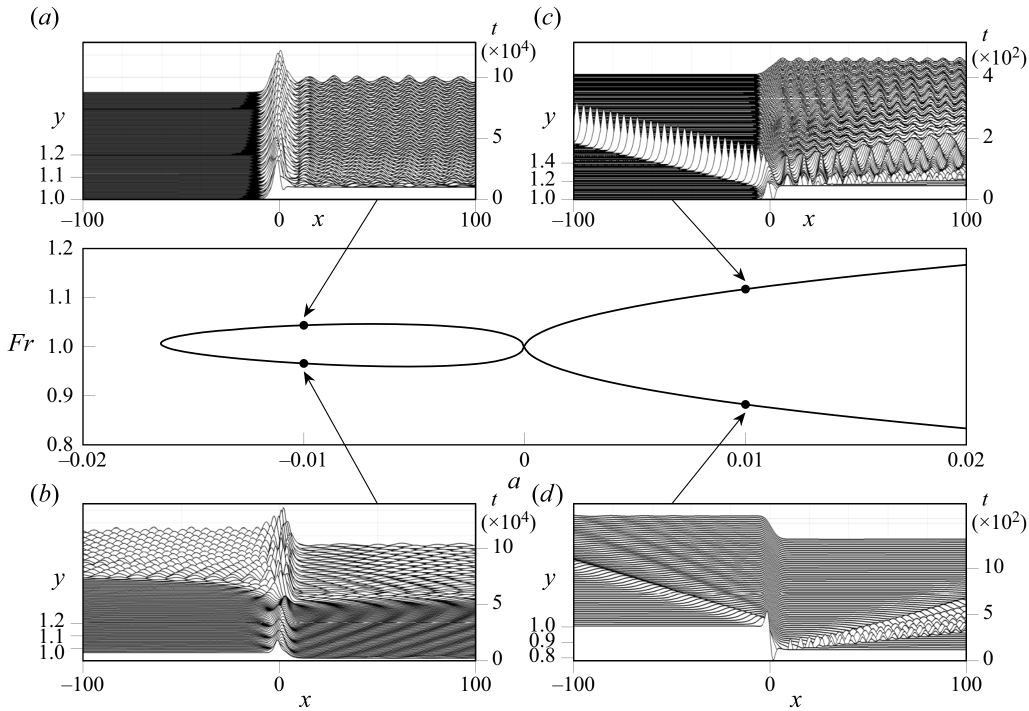

Figure 2. Parameter values of steady solutions calculated by Keeler & Blyth (2024) are shown in the (a, Fr) plane. Solutions with $Fr < 1$ are hydraulic falls, where the downstream ($x \to \infty$) fluid level is lower than that of the upstream ($x \to -\infty$) level. For the converse case with $Fr > 1$, the solutions are hydraulic rises. The qualitative stability properties of these solutions differ depending on the sign of the topography amplitude, a. Time-evolution plots for four different solutions are shown in (*a–d*). The solutions with indented topography ($a < 0$) are unstable, and time evolution ultimately reveals that these settle into time-periodic behaviour. Figure adapted from Keeler & Blyth (2024).

and significant progress has been made through both numerical (Dias & Vanden-Broeck 2002) and asymptotic (Lustri, McCue & Binder 2012) approaches. However, fewer works have investigated time-dependent effects beyond the initial disturbances that can arise. In their work, Keeler & Blyth (2024) begin by considering steady solutions of this formulation with $a > 0$ and $Fr < 1$. They isolate branches of these solutions, confirm with a formal linear stability analysis that these are temporally stable to small perturbations and also show through time evolution that large perturbations result in convergence to the original steady state. Steady solutions for $a > 0$ and $Fr > 1$ are found to be unstable; perturbations to these solutions induce a solitary wave that travels upstream. Time-evolution plots with $a = 0.01$ for both of these examples, demonstrating the nonlinear stability of these solutions, are shown in figure 2(*c*,*d*).

However, the time-dependent behaviour for free-surface flows over indented topography has been discovered by Keeler & Blyth (2024) to be significantly different than that for protruding topography. Here, the steady solutions (with $a < 0$) are linearly unstable, meaning that small-amplitude perturbations from the steady state initially grow in time. By investigating the long-time behaviour of these perturbations, Keeler & Blyth (2024) discovered that the fluid surface eventually approaches motion which is periodic in time. Time-evolution plots demonstrating this behaviour are shown in figure 2(*a*,*b*). Solution (*a*), with $Fr > 1$, contains non-decaying oscillatory waves downstream, whereas solution (*b*), with $Fr < 1$, contains oscillatory behaviour in both the upstream and downstream directions. For this latter case, the oscillatory behaviour is emitted from the central wave crest; large-amplitude travelling waves propagate upstream, and smaller-amplitude waves propagate downstream. The result is a fluid surface that repeats the same unsteady motion after an interval in time, with the oscillatory behaviour extending to spatial infinity. They are denoted generalised solitary waves, since the amplitude of the oscillations does not decay in the far field. These results emphasise the importance of considering solution stability, and the long-time trends of unstable configurations for free-boundary problems.

3. Future directions

The time-periodic surface waves discovered by Keeler & Blyth (2024) in the fluid flow past an indented topography raise a number of interesting questions that warrant additional investigation.

Firstly, it is possible that additional time-periodic solutions exist, which are not associated with the parameter values of the unstable solutions initially calculated by Keeler & Blyth (2024). To investigate this, one can alter the initial value problem with the requirement that the solution returns to its original profile after a period of time. Such an approach has recently been used by Wilkening & Zhao (2023) to calculate time-periodic water waves with spatially periodic topography. Secondly, analytical results have been obtained for hydraulic-fall solutions with protruding topography by Kamchatnov *et al.* (2013), who applied Whitham modulation theory to the forced KdV equation which models shallow water flows. An earlier numerical study of this transient phenomenon in the forced KdV equation was performed by Wu (1987). It would be very interesting to see if such an analysis can be adapted to predict the time-periodic solutions uncovered by Keeler & Blyth (2024). Lastly, it is unknown if this time-periodic behaviour can manifest physically, or if some other neglected effect is required in the model, such as the third spatial dimension from which transverse instabilities can emerge. Experimental results exist for hydraulic falls over protruding topography (Tam *et al.* 2015), confirming the stability of these solutions (analogous to the solution in figure 2*d*), but not for indented topography. Evidence for the physical existence of these time-periodic solutions would continue to highlight the value of numerical and theoretical investigations into simplified formulations of similar problems across fluid dynamics.

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Author ORCID.
Josh Shelton https://orcid.org/0000-0001-6257-5190.

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