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GROUPINGS OF METABELIAN GROUPS AND EXTENSION CATEGORIES

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Let G be a metabelian group and R an integral domain of characteristic zero, such that no rational prime divisor of $|G|$ is invertible in R. By RG we denote the group ring of G over R. In this note we shall prove

THEOREM. *If $RG \cong RH$ as R-algebras, then $G \cong H$.*

The question whether this result holds was posed to me by S. K. Sehgal. The result for $R = Z$ is contained in G. Higman's thesis, and he apparently also proved a more general result. At any rate, I think that the methods of the proof are interesting *eo ipso*, since they establish a "Noether-Deuring theorem" for extension categories.

In proving the above result, it is necessary to study closely the category of extensions $(\mathfrak{g}_S, S)$, where the objects are short exact sequences of SG-modules

$$0 \to M \to N \to \mathfrak{g}_S \to 0;$$

$\mathfrak{g}_S$ denotes the augmentation ideal of SG and S is a Dedekind domain, M and N are finitely generated SG-modules, morphisms are homomorphisms over $\mathfrak{g}_S$.

We shall prove the following results which are of interest also for their own sake:

(i) If S is a complete valuation ring, then the decomposition of an object $\mathfrak{E} \in (\mathfrak{g}_S, S)$ as a product of indecomposable objects is unique up to isomorphism. (This is the analogue of the Krull-Schmidt theorem.)

(ii) If S is semilocal and R is an overring of S, which is finitely generated and faithful projective over R, then two exact sequences $\mathfrak{E}_1, \mathfrak{E}_2 \in (\mathfrak{g}_S, S)$ are isomorphic if and only if the extensions $R \otimes_S \mathfrak{E}_1$ and $R \otimes_S \mathfrak{E}_2 \in (\mathfrak{g}_R, R)$ are isomorphic. (This is the analogue of the Noether-Deuring theorem.)

In order to prove the theorem, we first reduce to the case that R is finitely generated, and then by specializing to the case of a ring of integers in an algebraic number field. I am very grateful to W.-D. Geyer for showing me a proof that this can be done.

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1. Extension categories. Let Λ be a noetherian ring with identity; by ${}_{\Lambda}\mathfrak{M}^{\mathrm{f}}$ we denote the category of finitely generated left Λ-modules. Let $B \in {}_{\Lambda}\mathfrak{M}^{\mathrm{f}}$ be a fixed module. By (B, Λ) we denote the following category. Objects are short exact sequences of finitely generated left Λ-modules

$$\mathfrak{E}: 0 \to M' \to M \to B \to 0.$$

A morphism $(\alpha, \beta): \mathfrak{E}_1 \to \mathfrak{E}_2$ is a pair of Λ-homomorphisms making the following diagram commute

$$\begin{array}{ccccccccc} \mathfrak{E}_1 : 0 & \longrightarrow & M_1' & \longrightarrow & M_1 & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ \mathfrak{E}_2 : 0 & \longrightarrow & M_2' & \longrightarrow & M_2 & \longrightarrow & B & \longrightarrow & 0. \end{array}$$

These extension categories have been studied extensively in [**3**] and [**4**]. In particular, it is shown in [**3**, 1.7] that (B, Λ) has finite products. If an extension

$$\mathfrak{E}: 0 \to M_1' \oplus M_2' \to M \to B \to 0$$

is given, and if $\Pi_i: M_1' \oplus M_2' \to M_i'$, $i = 1, 2$ are the projections, then $\mathfrak{E}$ is the product of the extensions $\mathfrak{E}\Pi_1$ and $\mathfrak{E}\Pi_2$, where $\mathfrak{E}\Pi_i$ is a pushout

$$\begin{array}{ccccccccc} \mathfrak{E} : 0 & \longrightarrow & M_1' \oplus M_2' & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\Pi_i} & & | & & \| & & \\ \mathfrak{E}\Pi_i : 0 & \longrightarrow & M_i' & \longrightarrow & M_i & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Moreover, all products arise this way.

THEOREM 1. (Krull-Schmidt theorem) *Let $M' \in {}_{\Lambda}\mathfrak{M}^{\mathrm{f}}$ be such that for all indecomposable direct summands N' of M', the ring of endomorphisms,* $\mathrm{End}_{\Lambda}(N')$ *is a local ring; if $\mathfrak{E} \cong \prod_{i=1}^{n} \mathfrak{E}_i$*

$$\mathfrak{E}_i: 0 \to N_i' \to N_i \to B \to 0,$$

with N_i' indecomposable, then the above decomposition is unique up to isomorphism and numbering.

Proof. It should be noted that the category (B, Λ) does not have biproducts, and so the usual proof of the Krull-Schmidt theorem does not carry over.

Since all modules under consideration are noetherian, we can use induction on n. By the usual proof of the Krull-Schmidt theorem, [**9**, Ch. I, 4.10], we may assume $M' = N_1' \oplus N_2' = L_1' \oplus L_1'$, where N_1' and L_1' are indecomposable; moreover, there exists an automorphism φ of M' which induces isomorphisms

$$\alpha = \varphi|_{N_1'}: N_1' \to L_1' \text{ and } \beta = \varphi|_{N_2'}: N_2' \to L_2'.$$

Now, $\mathfrak{E} = \mathfrak{E}\mu_1 \Pi \mathfrak{E}\mu_2 \cong \mathfrak{E}\nu_1 \Pi \mathfrak{E}\nu_2$, where

$$\mu_i : M \to N_i' \text{ and } \nu_i : M \to L_i',\ i = 1, 2,$$

are the projections. The morphism $\tilde{\varphi} : \mathfrak{E} \to \mathfrak{E}$ induced by φ is given (up to the isomorphism) as the product of the two morphisms

$$\mu_1 \alpha : \mathfrak{E} \to \mathfrak{E}\nu_1 \text{ and } \mu_2 \beta : \mathfrak{E} \to \mathfrak{E}\nu_2.$$

But then

$$(\mathfrak{E}\mu_1 \alpha) \cong (\mathfrak{E}\mu_1)\alpha \cong \mathfrak{E}\nu_1 \text{ and } (\mathfrak{E}\mu_2 \alpha) = \mathfrak{E}\nu_2.$$

In particular, $\mathfrak{E}\mu_1 \cong \mathfrak{E}\nu_1$ and $\mathfrak{E}\mu_2 \cong \mathfrak{E}\nu_2$. Hence we are done by induction.

We assume now that Λ is an R-algebra, where R is noetherian and semilocal, with Λ finitely generated over R. Let J be the Jacobson radical of R and denote by $\hat{R}$ the J-adic completion of R.

THEOREM 2. *Let an extension*

$$\mathfrak{E} : 0 \to M' \to M \to B \to 0 \in (B, \Lambda)$$

be given and assume that $\text{End}_\Lambda(M')$ *is finitely generated over* R *and that there exists some integer* n *with* $J^n \text{Ext}_\Lambda{}^1(B, M') = 0$. *If*

$$\hat{R} \otimes_R \mathfrak{E} \simeq (\hat{R} \otimes_R \mathfrak{E}_1) \Pi \hat{\mathfrak{E}}_2$$

for some $\mathfrak{E}_1 \in (B, \Lambda)$ *and* $\hat{\mathfrak{E}}_2 \in (\hat{R} \otimes_R B, \hat{R} \otimes_R \Lambda)$, *then there exists an extension* $\mathfrak{E}_2 \in (B, \Lambda)$ *with* $\mathfrak{E} \underset{\wedge}{\simeq} \mathfrak{E}_1 \Pi \mathfrak{E}_2$.

Proof. It should be noted that $\hat{R}$ is faithfully flat as R-module, and so, if

$$\mathfrak{E} : 0 \to M' \to M \to B \to 0$$

is an exact sequence then

$$\hat{R} \oplus_R \mathfrak{E} : 0 \to \hat{R} \otimes_R M' \to \hat{R} \otimes_R M \to \hat{R} \otimes_R B \to 0$$

is an exact sequence in $(\hat{R} \otimes_R B, \hat{R} \otimes_R \Lambda)$.

According to the hypotheses, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\hat{R} \otimes_R \mathfrak{E} : 0 & \longrightarrow & \hat{R} \otimes_R M' & \longrightarrow & \hat{R} \otimes_R M & \longrightarrow & \hat{R} \otimes_R B & \longrightarrow & 0 \\
\downarrow & & \downarrow \hat{\alpha} & & \downarrow \hat{\beta} & & \| & & \\
\hat{R} \otimes_R \mathfrak{E}_1 : 0 & \longrightarrow & \hat{R} \otimes_R M_1' & \longrightarrow & \hat{R} \otimes_R M_1 & \longrightarrow & \hat{R} \otimes_R B & \longrightarrow & 0
\end{array}$$

where $\hat{\alpha}$ is a split epimorphism. By [**8**] we can approximate $\hat{\alpha}$ by $1 \otimes \alpha$ modulo J^n in such a way that $\alpha : M' \to M_1'$ is a split epimorphism. It

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remains to show $\mathfrak{E}\alpha \simeq \mathfrak{E}_1$. Since

$$(\hat{R} \otimes_R \mathfrak{E})(1 \otimes \alpha) - \hat{R} \otimes_R \mathfrak{E}_1 \simeq (\hat{R} \otimes_R \mathfrak{E})(1 \otimes \alpha - \hat{\alpha})$$

is an element in $J^n \mathrm{Ext}^1_{\hat{R} \otimes_R \Lambda}(\hat{R} \otimes_R B, \hat{R} \otimes_R M_1') = 0$, we conclude

$$(\hat{R} \otimes_R \mathfrak{E})(1 \otimes \alpha) \simeq \hat{R} \otimes_R \mathfrak{E}_1.$$

However

$$\mathrm{Ext}^1_{\hat{R} \otimes_R \Lambda}(\hat{R} \otimes_R B, \hat{R} \otimes_R M_1') \simeq \hat{R} \otimes_R \mathrm{Ext}_\Lambda{}^1(B, M_1') = \mathrm{Ext}_\Lambda{}^1 (B, M_1')$$

since J^n annihilates this group. Thus two sequences are isomorphic if and only if their completions are isomorphic. Hence $\mathfrak{E}_1 \simeq \mathfrak{E}\alpha$ and so $\mathfrak{E} \simeq \mathfrak{E}_1 \amalg \mathfrak{E}_2$.

The proof gives rise to the following

COROLLARY. *Under the hypotheses of Theorem 2, two sequences $\mathfrak{E}_1$ and $\mathfrak{E}_2$ in (B, Λ) are isomorphic if and only if their completions are isomorphic.*

Let now, in addition, S be a commutative R-algebra such that $\hat{S} = \hat{R} \otimes_R S$ is a faithful projective $\hat{R}$-module of finite type. We make the following assumptions: For

$$\mathfrak{E}: 0 \rightarrow M' \rightarrow M \rightarrow B \rightarrow 0,$$

$\mathrm{End}_\Lambda(M')$ is a finitely generated R-module, and there exists an integer n such that $J^n \mathrm{Ext}_\Lambda{}^1(B, M') = 0$.

THEOREM 3. (Noether-Deuring) *$\mathfrak{E}$ is isomorphic to $\mathfrak{E}_1$ if and only if $S \otimes_R \mathfrak{E}$ is isomorphic to $S \otimes_R \mathfrak{E}_1$.*

Remark. This result holds more generally if one replaces isomorphism by being a direct factor.

Proof. For a Λ-module M we write $\hat{M} = \hat{R} \otimes_\Lambda M$. Observe that the hypotheses on S imply that tensoring with S is an exact functor. We have the following implications:

$$S \otimes_R \mathfrak{E} \simeq S \otimes_R \mathfrak{E}_1 \Leftrightarrow \hat{S} \otimes_R \mathfrak{E} = \hat{S} \otimes_R \mathfrak{E}_1,$$

the latter being an isomorphism of $\hat{S} \otimes_R \Lambda$-modules. Let $J = \bigcap_{i=1}^s m_i$, where m_i are the maximal ideals in R. Then $\hat{R} = \prod_{i=1}^s \hat{R}_i$, where $\hat{R}_i$ is a complete local ring. Since we have assumed that $\hat{S}$ is a faithful projective R-module of finite type, we have

$$\hat{S} \simeq_R \bigoplus_{i=1}^s \hat{R}_i{}^{(n_i)}, \; n_i > 0, \; 1 \leqq i \leqq s.$$

We denote by $(\hat{B}, \Lambda)_{\hat{S}}$ the extensions and morphisms defined as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{N}' & \longrightarrow & \hat{N} & \longrightarrow & \hat{B} & \longrightarrow & 0 \\
 & & \alpha\downarrow & & \beta\downarrow & & \| & & \\
0 & \longrightarrow & \hat{N}_1' & \longrightarrow & \hat{N}_1 & \longrightarrow & \hat{B} & \longrightarrow & 0
\end{array}$$

where the modules are $\hat{\Lambda}$-modules; in addition $\hat{N}'$ and $\hat{N}_1'$ are $\hat{S} \otimes_{\hat{R}} \hat{\Lambda}$-modules. Morphisms are $\hat{\Lambda}$-morphisms; in addition α is an $\hat{S} \otimes_{\hat{R}} \Lambda$-homomorphism. It is shown in [**4**] that the categories $(\hat{B}, \hat{\Lambda})_S$ and $(\hat{S} \otimes_{\hat{R}} \hat{B}, \hat{S} \otimes_{\hat{R}} \hat{\Lambda})$ are equivalent. Under this equivalence, the isomorphism $\hat{S} \otimes_R \xi \simeq \hat{S} \otimes_{\hat{R}} \xi_1$ will be carried into an isomorphism of exactsequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{S} \otimes_R M' & \longrightarrow & X & \longrightarrow & \hat{R} \otimes_R B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \hat{S} \otimes_R N' & \longrightarrow & Y & \longrightarrow & \hat{R} \otimes_R B & \longrightarrow & 0.
\end{array}$$

We view these sequences as objects in $(\hat{B}, \hat{\Lambda})$ and observe that

$$\hat{S} \otimes_{\hat{R}} M' \simeq \bigoplus_{i=1}^{s} (\hat{R} \otimes_R M')^{(n_i)} \text{ as } \hat{R} \otimes_R \Lambda\text{-modules.}$$

Now we apply the Krull-Schmidt theorem (I) and (observing $n_i > 0$) we conclude that the sequences $\hat{R} \otimes_R \mathfrak{E}$ and $\hat{R} \otimes_R \mathfrak{E}_1$ are isomorphic; by Theorem 2, the sequences $\mathfrak{E}$ and $\mathfrak{E}_1$ are isomorphic.

2. Grouprings of metabelian groups. Let R be an integral domain of characteristic zero; G and H are finite groups such that no rational prime divisor of $|G|$ is a unit in R.

We assume that we have an isomorphism of R-algebras

$$\varphi: RG \rightarrow RH.$$

1. *Reduction to the case where R is finitely generated.* Under the above isomorphism we have

$$\varphi: g \rightarrow \sum_i r_{ij} h_j,\ g_i \in G,\ h_j \in H,\ r_{ij} \in R$$
$$\varphi^{-1}: h_k \rightarrow \sum_k s_{kl} g_1,\ g_1 \in G,\ h_j \in H,\ s_{kl} \in R.$$

Let R_0 be the subring of R generated by $\{r_{ij}, s_{k1}, 1\}$. Then R_0 satisfies the same hypotheses as R, and φ induces an isomorphism of R-algebras

$$\varphi_0: R_0G \rightarrow R_0H;$$

moreover, R_0 is finitely generated.

Hence we assume from now on that R is finitely generated.

2. *Reduction to the case where R is finitely generated as Z_0-module, $Z_0 = \cap Z_p$, where the intersection is taken over all localizations of Z at*

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those primes which divide $|G|$. This will be done by using the following unpublished result of W.-D. Geyer:

THEOREM 4. *Let* A *be an excellent Dedekind domain with a Hilbert field of fractions* K, *and* R *an integral domain finitely generated over* A. *Let* $\mathfrak{P}$ *be a finite set of prime ideals in* A *with* $\mathfrak{p}R \neq R$ *for* $\mathfrak{p} \in \mathfrak{P}$. *Then there exists a homomorphism*

$$\varphi: R \rightarrow \tilde{K},$$

where $\tilde{K}$ *is the algebraic closure of* K *such that every* $\mathfrak{p} \in \mathfrak{P}$ *has prime divisors in* $\operatorname{Im} \varphi$.

The proof is done in several steps.

Step 1. *Localization at* $A \backslash \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{p}$. Let A' be the localization of A at $A \bigcup_{\mathfrak{p} \in \mathfrak{P}} \mathfrak{p}$. Then A' is a semi-local principal ideal domain with

$$\mathfrak{P}' = \max (A') = \{\mathfrak{p}' : \mathfrak{p}' = \mathfrak{p}A', \mathfrak{p} \in \mathfrak{P}\}.$$

For $R' = R \otimes_A A'$ we still have $R'\mathfrak{p}' \neq R'$ for $\mathfrak{p}' \in \mathfrak{P}'$. Thus if the result is proved for the pair (A', R'), then it is also true for (A, R).

We may therefore assume in the sequel that A is a semi-local principal ideal domain with $\max (A) = \{p_1 A, \ldots, p_n A\} = \mathfrak{P}$.

Step 2. *Normalization of* R. It is clear that R can be replaced by a finite integral extension of R. Now, A is excellent, and so the integral closure $\tilde{R}$ of R in its field of fractions is a finite integral extension. Thus we may assume from now on that R is integrally closed in its field of fractions.

Step 3. *Preparation for the case that* R *has degree of transcendency* 1, $\operatorname{tr deg} (R/A) = 1$. Assume $\operatorname{tr deg} (R/A) = 1$. Let $\mathfrak{p}_i$ be minimal prime ideals in R, lying above p_i, then $R/\mathfrak{p}_i$ is transcendent over $A/p_i A$ [**7**, 35.6]. Let $f_i \in R$ be such that $f_i + \mathfrak{p}_i \in R/\mathfrak{p}_i$ is transcendent over $A/p_i A$. Choose $e_i \in A$ with

$$e_i \equiv \begin{cases} 0 \bmod (p_j), & j \neq i \\ 1 \bmod (p_i). \end{cases}$$

If we put $f = \sum_{i=1}^{n} e_i f_i$, then $f + \mathfrak{p}_i \in R/\mathfrak{p}_i$ is transcendent over A/p_i, $1 \leqq i \leqq n$. We choose a system of generators for R over A,

$$R = A[x_1, \ldots, x_t] \text{ with } x_1 = f.$$

If we put $S = R \otimes_A K$, then $\{x_1, \ldots, x_t\}$ generate the K-algebra S. Since A is infinite, K being a Hilbert field, we can apply Noether's normalizing lemma and find elements $a_j \in A$ such that for $x = x_1 + \sum_{j=2}^{t} a_j x_j$, the algebra S is integral over $K[x]$. We even may choose

$a_j \in \bigcap_{i=1}^{r} p_i A$, and hence $x + \mathfrak{p}_i$ is transcendent over A/p_i. We then have

(1) $\mathfrak{p}_i \cap A[x] = p_i[x]$, $1 \leqq i \leqq n$.

Step 4. tr deg $(R/A) = 1$. Let B be the integral closure of $A[x]$ in the field of fractions F of R. Because of formula (1) the valuation rings $R_{\mathfrak{p}_i}$ are also localizations of B. Thus we have a representation

(2) $R = B[b_1/d, \ldots, b/d]$,
$b_i \in B$, $d \in B \backslash \bigcup_{i=1}^{n} \mathfrak{p}_i$.

Let

(3) $$f(d, x) = d^s + \sum_{j=0}^{s-1} a_j(x) d^j = 0, \quad a_j(x) \in A[x]$$

the minimal polynomial of d over $K(x)$. Now, $d \notin \mathfrak{p}_i$, $1 \leqq i \leqq n$ and so for at least one coefficient $a_{j(i)}(x)$ of f we have

$$a_{j(i)}(x) = \sum\nolimits_k a_{jk} x^k = a_{jk} \in A,$$

where not all a_{jk} are divisible by p_i. Hence we can find an element γ_i which is integral over A with

$$a_{j(i)}(\gamma_i) + 0 \bmod (\tilde{p}_i),$$

where $\tilde{p}_i$ is a fixed extension of p_i to $\tilde{K}$. Because of Hilbert's irreducibility theorem, there exists ξ integral over A such that

(4) $\xi \equiv \gamma_i \bmod (\tilde{p}_i)$, $1 \leqq i \leqq n$, and

(5) $$f(Y, \xi) = Y^s + \sum_{j=0}^{s-1} a_j(\xi) Y^j$$

is irreducible over $K(\xi)$.

That one can combine the irreducibility equation (5) with the finitely many congruences (4) can be found in the classical situation $A = \mathbf{Z}$ in Hilbert [**5**], for arbitrary Hilbert fields in [**1**, 3.4]. Now we define the K-homomorphism

$$\varphi: K[x] \to \tilde{K} \text{ by}$$
$$x \mapsto \xi,$$

and extend φ to the integral closure $S = KB$ of $K[x]$. Because of (4) it follows that the equation

$$f(Y, \xi) = 0$$

does not reduce mod $(\tilde{p}_i)$ to $Y^n = 0$; i.e., it has roots which are units modulo $\tilde{p}_i$. Because of (5) all roots are conjugate over $K(\xi)$, and so to every root there exists a valuation which is conjugate to $\tilde{p}_i$ such that this

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root is a unit. Because of (3) $\varphi(d)$ is such a root. Thus $\varphi(d)$ is a unit for all $\tilde{p}_i$, if one conjugates the extensions $\tilde{p}_i$ suitably. Because of the representation (2) it follows that in $\varphi(R)$ no p_i is invertible.

Step 5. *General case.* We use induction tr deg (F/K). Choose an extension

$$K \subset F_1 \subset F$$

such that tr deg $(F/F_1) = 1$; as in step 3 let $\mathfrak{p}_i$ be minimal prime ideals of R with $p_i \in \mathfrak{p}_i$. We put

$$A_1 = F_1 \cap (\cap_{i=1}^{n} R_{\mathfrak{p}_i});$$

then A_1 is an excellent semilocal principal ideal domain. Moreover, $R_1 = RA_1$ is finitely generated over A_1 and tr deg $(R_1/A_1) = 1$, and so the results of step 4 are applicable. Thus there exists a homomorphism $\varphi: R_1 \rightarrow \tilde{F}_1$, which is the identity on A_1, and thus tr deg $[\varphi R_1/A) <$ tr deg (R/A), and no p_i is a unit in $\varphi(R_1)$. Now we can apply the induction hypotheses to $\varphi(R)$. Hence we obtain the desired homomorphism.

This completes the proof of Theorem 4.

We now come to the second reduction: Let R_0 be the image of R under the homomorphism σ of Theorem 4. The isomorphism φ $RG \rightarrow RH$ then induces an isomorphism $\varphi_0: R_0H \rightarrow R_0H$. We now localize at the primes dividing $|G|$, and obtain the claimed result.

Hence we assume from now on that R is a Dedekind domain, which is free over Z_0 on a finite basis. We shall now turn to the actual proof of the theorem.

Since G is metabelian, there is an exact sequence

$$1 \rightarrow N \overset{\alpha}{\rightarrow} G \overset{\beta}{\rightarrow} \bar{G} \rightarrow 1$$

with N and $\bar{G}$ abelian. We may assume that the isomorphism

$$\varphi: RG \rightarrow RH$$

is augmented.

LEMMA 1. *There exists an exact sequence of groups*

$$1 \rightarrow M \overset{\gamma}{\rightarrow} H \overset{\delta}{\rightarrow} \bar{H} \rightarrow 1$$

with M and $\bar{H}$ abelian, such that φ induces the following commutative diagram with exact rows and columns:

$$D: \quad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n}_R{}^G & \overset{\tilde{\alpha}}{\longrightarrow} & RG & \overset{\tilde{\beta}}{\longrightarrow} & R\bar{G} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} & & \\ 0 & \longrightarrow & \mathfrak{m}_R{}^H & \overset{\tilde{\gamma}}{\longrightarrow} & RH & \overset{\tilde{\delta}}{\longrightarrow} & R\bar{H} & \longrightarrow & 0, \end{array}$$

where $\mathfrak{n}_R$ *and* $\mathfrak{m}_R$ *denote the augmentation ideals of* N *and* M *respectively:* $\mathfrak{n}_R{}^G$ *and* $\mathfrak{m}_R{}^H$ *are the respective induced modules.*

Proof. Let $e = \sum_{n \in N} n$; then e is mapped under φ to e', and one shows as in [**10**] (using the fact that no prime divisor of $|G|$ is a unit in R) that $e' = \sum_{m \in M} m$, where M is a normal subgroup in H. Since $R\bar{G} = RG(e/(|N|))$, and $R\bar{H} = RH(e'/(|M|))$, it follows that the above diagram is commutative. Moreover from the commutativity of the diagram D it follows that M and $\bar{H}$ are abelian.

LEMMA 2. *In the above diagram* D *we may assume that* $\bar{G} = \bar{H}$ *and* $\varphi'' = \text{id}$.

Proof. Since $\bar{G}$ and $\bar{H}$ are abelian, and since φ'' is augmented, it follows from [**6**] that φ'' is induced from a group homomorphism $\sigma: \bar{G} \rightarrow \bar{H}$; i.e.,

$$\varphi: R\bar{G} \rightarrow R\bar{H}$$
$$\sum r_{\bar{g}}\bar{g} \rightarrow \sum r_{\bar{g}}\bar{g}^{\sigma}.$$

In the exact sequence

$$1 \rightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} \bar{G} \rightarrow 1$$

we replace β by $\beta\sigma$ and get the exact sequence

$$1 \rightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta\sigma} \bar{H} \rightarrow 1,$$

which gives rise to the exact sequence

$$0 \rightarrow \mathfrak{n}_R{}^G \rightarrow RG \rightarrow R\bar{H} \rightarrow 0,$$

and the map $RG \rightarrow R\bar{H}$ is the composite of $\tilde{\beta}$ and $\tilde{\varphi}''$.

We now have the following situation:

$$\mathbf{E}_1: 1 \rightarrow N \rightarrow G \rightarrow \bar{H} \rightarrow 1$$
$$\mathbf{E}_2: 1 \rightarrow M \rightarrow H \rightarrow \bar{H} \rightarrow 1.$$

These sequences give rise to the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\mathfrak{E}_1' : 0 & \longrightarrow & \mathfrak{n}_R{}^G & \longrightarrow & RG & \longrightarrow & R\bar{H} & \longrightarrow & 0 \\
 & & \downarrow \varphi' & & \downarrow \varphi & & \| & & \\
\mathfrak{E}_2' : 0 & \longrightarrow & \mathfrak{m}_R{}^H & \longrightarrow & RH & \longrightarrow & R\bar{H} & \longrightarrow & 0.
\end{array}$$

Moreover, since the homomorphisms are augmented, we get an induced

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commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{n}_R{}^G & \longrightarrow & \mathfrak{g}_R & \longrightarrow & \bar{\mathfrak{h}}_R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathfrak{m}_R{}^H & \longrightarrow & \mathfrak{h}_R & \longrightarrow & \bar{\mathfrak{h}}_R & \longrightarrow & 0. \end{array}$$

Since the submodule $\mathfrak{g}_R\mathfrak{n}$ is a characteristic submodule of $\mathfrak{n}_R{}^G$, we can construct the following commutative diagram with exact rows

$$\begin{array}{cccccccccc} \mathfrak{E}_1: & 0 & \longrightarrow & \mathfrak{n}_R{}^G/(\mathfrak{g}_R\mathfrak{n}_R) & \longrightarrow & \mathfrak{g}_R/(\mathfrak{g}_R\mathfrak{n}_R) & \longrightarrow & \bar{\mathfrak{h}}_R & \longrightarrow & 0 \\ & & & \downarrow \bar{\varphi} & & \downarrow \bar{\varphi} & & \| & & \\ \mathfrak{E}_2: & 0 & \longrightarrow & \mathfrak{m}_R{}^H/(\mathfrak{h}_R\mathfrak{m}_R) & \longrightarrow & \mathfrak{h}_R/(\mathfrak{h}_R\mathfrak{m}_R) & \longrightarrow & \bar{\mathfrak{h}}_R & \longrightarrow & 0. \end{array}$$

However, $\mathfrak{n}_R{}^G/(\mathfrak{g}_R\mathfrak{n}_R) \simeq R \otimes_Z N$ and $m_R{}^H/(\mathfrak{h}_R\mathfrak{m}_R) \simeq R \otimes_Z M$ as $R\bar{H}$-modules [**2**]. Thus, finally we have the commutative diagram with exact rows:

$$\begin{array}{cccccccccc} \mathfrak{E}_1: & 0 & \longrightarrow & R \otimes_Z N & \longrightarrow & R \otimes_Z X & \longrightarrow & R \otimes_Z \bar{\mathfrak{h}} & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \| & & \\ \mathfrak{E}_2: & 0 & \longrightarrow & R \otimes_Z M & \longrightarrow & R \otimes_Z Y & \longrightarrow & R \otimes_Z \bar{\mathfrak{h}} & \longrightarrow & 0, \end{array}$$

where all modules are $R\bar{H}$-modules and the morphisms are $R\bar{H}$-homomorphisms. We now apply Theorem 3 to conclude that we have an isomorphism of exact sequences over $Z_0 \otimes_Z \bar{\mathfrak{h}}$.

$$\begin{array}{cccccccccc} \mathfrak{E}_1'': & 0 & \longrightarrow & Z_0 \otimes_Z N & \longrightarrow & Z_0 \otimes_Z X & \longrightarrow & Z_0 \otimes_Z \bar{\mathfrak{h}} & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \| & & \\ \mathfrak{E}_2'': & 0 & \longrightarrow & Z_0 \otimes_Z M & \longrightarrow & Z_0 \otimes_Z Y & \longrightarrow & Z_0 \otimes_Z \bar{\mathfrak{h}} & \longrightarrow & 0. \end{array}$$

Now M and N are finite abelian groups, and no prime divisor of $|N|$ and $|M|$ is a unit in $Z_0 \otimes_Z N \simeq N$ and $Z_0 \otimes_Z M \cong M$; i.e., $N = M$. In addition

$$\operatorname{Ext}_{ZG}{}^1(\mathfrak{h}, N) = Z_0 \otimes_Z \operatorname{Ext}_{ZG}{}^1(\mathfrak{h}, N) = \operatorname{Ext}_{Z_0G}{}^1(Z_0 \otimes_Z \mathfrak{h}, Z_0 \otimes_Z N).$$

Consequently the extensions

$$\begin{array}{cccccccccc} \mathfrak{E}_3: & 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & \bar{\mathfrak{h}} & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \| & & \\ \mathfrak{E}_4: & 0 & \longrightarrow & M & \longrightarrow & Y & \longrightarrow & \bar{\mathfrak{h}} & \longrightarrow & 0 \end{array}$$

are isomorphic over $\bar{\mathfrak{h}}$. But this means that the group extensions

$$1 \to N \to G \to \bar{H} \to 1$$

and

$$1 \to M \to H \to \bar{H} \to 1$$

are isomorphic by [**3**, 2]. Hence $G \simeq H$.

This proves the desired result.

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