

CERTAIN RELATIONS BETWEEN p -REGULAR CLASS SIZES AND THE p -STRUCTURE OF p -SOLVABLE GROUPS

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Abstract

Let G be a finite p -solvable group for a fixed prime p . We study how certain arithmetical conditions on the set of p -regular conjugacy class sizes of G influence the p -structure of G . In particular, the structure of the p -complements of G is described when this set is $\{1, m, n\}$ for arbitrary coprime integers $m, n > 1$. The structure of G is determined when the noncentral p -regular class lengths are consecutive numbers and when all of them are prime powers.

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1. Introduction

The influence of the conjugacy class sizes on the structure of a finite group G has been studied by many authors. For a prime p , we consider the set of p -regular classes in G , that is, of conjugacy classes of p' -elements in G . Several theorems have put forward that certain properties on the sizes of these classes are also reflected on the p -structure of G (see for instance [7] or [8]). In this note we show how imposing some arithmetical conditions on this set of p -regular class sizes in a p -solvable group G yields restrictions on the structure of G or of its p -complements. This shows furthermore, how global information on conjugacy classes of G can provide local information on the p -complements of G .

We use the methods and results developed by the authors in [1] for studying the graph $\Gamma_p(G)$. The vertices of this graph are the noncentral p -regular classes of the p -solvable group G and an edge connects two classes if their sizes are not coprime numbers. It was proved in [1, Theorem 1] that $\Gamma_p(G)$ has at most two connected

components. When $\Gamma_p(G)$ has exactly two components and p divides the size of some p -regular class belonging to the component which does not contain the maximal size classes, then it is shown in [1, Theorem 5] that G has normal p -complement and this complement is quasi-Frobenius with abelian kernel and complement. Following [3] a group G is said to be quasi-Frobenius when $G/Z(G)$ is a Frobenius group and then the inverse images of the kernel and a complement of $G/Z(G)$ are called the kernel and complement of G . There exist examples, which will be provided, showing that if the above divisibility hypothesis on the prime p is eliminated, then G need not be p -nilpotent. However, it can be questioned whether in this case the p -complements of G are still quasi-Frobenius. Our first main result provides an affirmative answer to a particular case in which p may divide the size of the maximal p' -classes in G .

From now on, any group will be a finite group and we will denote by $G_{p'}$ the set of p' -elements of G and by $\text{Con}(G_{p'})$ the set of conjugacy classes in $G_{p'}$.

THEOREM A. *Suppose that G is a p -solvable group. Let m and n be the two maximal sizes in $\text{Con}(G_{p'})$ with $m > n > 1$. Suppose that $(m, n) = 1$ and that p is not a prime divisor of n . Then G is solvable and*

- (a) *the p -regular conjugacy class lengths of G are $\{1, n, m\}$;*
- (b) *a p -complement of G is a quasi-Frobenius group with abelian kernel and complement. Furthermore, its conjugacy class lengths are $\{1, n, m_{p'}\}$.*

On the one hand, Theorem A extends the main result of [5], by taking a prime p not dividing $|G|$. On the other hand, it allows us to obtain the structure of the p -complements of G when G has exactly two noncentral p -regular class sizes which are coprime numbers. We want to mention that the structure of G is completely determined when its p -regular conjugacy class sizes are exactly $\{1, m\}$ for an arbitrary positive integer m [2, Theorem A].

COROLLARY B. *Let G be a p -solvable group and suppose that the set of p -regular conjugacy class sizes of G is $\{1, n, m\}$ with $(m, n) = 1$. Then G is solvable and the p -complements of G are quasi-Frobenius groups with abelian kernel and complement. Moreover, the class sizes of any p -complement are $\{1, n_{p'}, m_{p'}\}$.*

As an application of Corollary B, among other results, we determine the structure of p -solvable groups whose noncentral p -regular class sizes are consecutive integers.

THEOREM C. *Suppose that G is a p -solvable group. If $\{n, n+1, \dots, n+r\}$ is the set of lengths of noncentral classes in $\text{Con}(G_{p'})$, then one of the following holds:*

- (a) *$r = 0$, $n = p^a$, for some a and G has abelian p -complements.*

(b) $r = 0, n = p^a q^b$, for some prime $q \neq p$ and integers $a \geq 0$ and $b \geq 1$, and $G = PQ \times A$, with $P \in \text{Syl}_p(G), Q \in \text{Syl}_q(G)$ and $A \leq Z(G)$. Moreover, if $a = 0$ then $G = P \times Q \times A$.

(c) $r = 1$ and any p -complement of G is a quasi-Frobenius group with abelian kernel and complement. Also, if p does not divide n then G is p -nilpotent and if, in addition, p does not divide $n + 1$, then $G = P \times H$, where H is the p -complement of G .

The next theorem determines the structure of groups having prime powers as p -regular class lengths, so it extends Theorem 2 and Corollary 2.2 of [6]. However, our proof is based on certain properties of the graph $\Gamma_p(G)$, and thus it sufficiently differs from the proof of the mentioned results for ordinary conjugacy classes.

THEOREM D. *Suppose that G is a p -solvable group. Every conjugacy class in $\text{Con}(G_{p'})$ has prime power size if and only if one of the following holds:*

(a) G has abelian p -complements. This occurs if and only if the size of every conjugacy class in $\text{Con}(G_{p'})$ is a power of p .

(b) G is nilpotent with abelian Sylow r -subgroups for all primes r distinct from p and from some prime $q \neq p$. This occurs if and only if the size of every class in $\text{Con}(G_{p'})$ is a power of q .

(c) $G = P \times H$, where P is a Sylow p -subgroup of G and H is a p -complement of G . Furthermore, H is quasi-Frobenius with abelian kernel and complement and the conjugacy class sizes of H are $\{1, q^s, r^t\}$ for positive integers s, t and for some distinct primes q, r , both distinct from p . This occurs if and only if the p -regular class sizes of G are exactly $\{1, q^s, r^t\}$.

2. Preliminaries

In order to obtain our main theorems, we need some previous results as well as to develop certain properties of the p -regular classes of maximal size. It is worth mentioning that all these results hold for ordinary conjugacy classes (just take a prime p not dividing the order of G). The first lemma is basic when studying p -regular class lengths.

LEMMA 1. *Let G be a p -solvable group and let $B = b^G, C = c^G \in \text{Con}(G_{p'})$ such that $(|B|, |C|) = 1$. Then*

(a) $C_G(b)C_G(c) = G$;

(b) $BC = CB$ is a conjugacy class in $\text{Con}(G_{p'})$ and $|BC|$ divides $|B||C|$.

PROOF. See [8, Lemma 1].

□

LEMMA 2. *Suppose that G is a p -solvable group and let B_0 be a noncentral class in $\text{Con}(G_{p'})$ of maximal length. Then the following properties hold:*

- (a) *Let $C \in \text{Con}(G_{p'})$ such that $(|B_0|, |C|) = 1$. Then $|\langle C^{-1}C \rangle|$ divides $|B_0|$.*
- (b) *Let n and $m = |B_0|$ be the two maximal sizes in $\text{Con}(G_{p'})$. Suppose that $(n, m) = 1$. Let $D \in \text{Con}(G_{p'})$ with $|D| > 1$. If $(|D|, n) = 1$, then $|D| = m$.*

PROOF. (a) If C is a class in $\text{Con}(G_{p'})$ such that $(|B_0|, |C|) = 1$, it follows from Lemma 1 (b) that $CB_0 \in \text{Con}(G_{p'})$ and by maximality $|CB_0| = |B_0|$. By Lemma 1 (b) again, we have $C^{-1}CB_0 = B_0$. Thus, $\langle C^{-1}C \rangle B_0 = B_0$ and consequently, B_0 is a union of some cosets of the normal subgroup $\langle C^{-1}C \rangle$. Then $|\langle C^{-1}C \rangle|$ divides $|B_0|$.

(b) Choose $A \in \text{Con}(G_{p'})$ such that $|A| = n$ and let $D \in \text{Con}(G_{p'})$ such that $|D| > 1$ and $(|D|, n) = 1$. Then DA is a conjugacy class in $\text{Con}(G_{p'})$ by Lemma 1 (b). Since $|DA| \geq n$, then $|DA| = n$ or $|DA| = m$. Suppose first that $|DA| = n$. By Lemma 1 (b) again, $D^{-1}DA$ is also a class in $\text{Con}(G_{p'})$ and $A \subseteq D^{-1}DA$, so $A = D^{-1}DA$. Thus, $A = \langle D^{-1}D \rangle A$ and $1 \neq |\langle D^{-1}D \rangle|$ divides $|A|$. Moreover, since $\langle D^{-1}D \rangle \subseteq \langle AA^{-1} \rangle$, we have that $|\langle D^{-1}D \rangle|$ divides $|\langle AA^{-1} \rangle|$. By (a), $|\langle AA^{-1} \rangle|$ divides $|B_0|$, yielding the contradiction $|\langle D^{-1}D \rangle|$ divides $|B_0|$.

Suppose now that $|DA| = |B_0|$. By Lemma 1 (b), we have that $|B_0|$ divides $|A||D|$, which implies that $|B_0| = |D|$. □

Other properties related to p -regular classes of maximal size were developed in [1] (see Proposition 1 and Theorem 3). In the following theorem we summarize in a somehow different form some of these properties. If m is a positive integer, then $\pi(m)$ will denote the set of prime divisors of m and we will also write $\pi(X)$ to denote the set of primes dividing $|X|$ for any subgroup X of G or any $X \in \text{Con}(G_{p'})$.

THEOREM 3. *Suppose that G is a p -solvable group. Let B_0 be a noncentral class in $\text{Con}(G_{p'})$ of maximal length and write*

$$M = \langle D \in \text{Con}(G_{p'}) \mid (|D|, |B_0|) = 1 \rangle.$$

Then $M = P \times M_{p'}$, where $M_{p'}$ is abelian and P is a Sylow p -subgroup of M . Furthermore, $Z(G)_{p'} \subseteq M_{p'}$, and $\pi(M_{p'}/Z(G)_{p'}) \subseteq \pi(B_0)$. In particular, if $M > Z(G)_{p'}$, then $|B_0|$ cannot be a power of p .

PROOF. We define $N = \langle D^{-1}D \mid D \in \text{Con}(G_{p'}), (|D|, |B_0|) = 1 \rangle$. From the definition of M and N , it is clear that $N = [M, G]$. Let $c \in C$, where C is a class in $\text{Con}(G_{p'})$ with $(|B_0|, |C|) = 1$. By applying Lemma 2 (a), we obtain $\pi(N) \subseteq \pi(B_0)$, whence $(|N|, |C|) = 1$. Since $|N : C_N(c)|$ divides $(|N|, |C|)$, it follows that $N = C_N(c)$, so $N \leq Z(M)$. Since M/N is contained in the centre of G/N we deduce that M is nilpotent, hence we can write $M = P \times M_{p'}$.

Now, it is trivial that $Z(G)_{p'} \subseteq M$, and consequently $Z(G)_{p'} \subseteq M_{p'}$. Note that $M_{p'} = Z(G)_{p'}$ if and only if $M = Z(G)_{p'}$, for in each case M is generated by central p' -elements. We can assume that $M_{p'} > Z(G)_{p'}$, since otherwise the theorem is trivially true. Let $r \in \pi(M_{p'}/Z(G)_{p'})$ and choose $R \in \text{Syl}_r(M_{p'})$. Notice that $R \leq G$ and that $1 \neq [R, G] \leq [M, G] = N$. Therefore, $r \in \pi(N) \subseteq \pi(B_0)$ and thus $\pi(M_{p'}/Z(G)_{p'}) \subseteq \pi(B_0)$ as required. In particular, if $M > Z(G)_{p'}$ then there exists some prime q distinct from p dividing $|B_0|$.

Finally, if D is a generating class of M , we choose $d \in D$ and if R is the above Sylow subgroup, then $|R : C_R(d)|$ divides $(|R|, |D|) = 1$, so $R = C_R(d)$ and we get $R \leq Z(M)$. Hence $M_{p'}$ is abelian. □

The following result is a generalization for p -regular elements of a lemma of Ito. It will be necessary for determining the structure of the centralizers of p' -elements whose conjugacy class has maximal size. Although it is stated and shown as [2, Lemma 1], we are including its proof here for the seek of completeness.

LEMMA 4. *Let G be a finite group, $x \in G_{p'}$ and $C_G(x) < G$. Assume the following:*

- (1) *If $C_G(a) \leq C_G(x)$ for $a \in G_{p'}$, then $C_G(a) = C_G(x)$.*
- (2) *If $C_G(x) \leq C_G(b)$ for $b \in G_{p'}$, then $C_G(x) = C_G(b)$ or $b \in Z(G)$.*

Then either $C_G(x) = P \times L$, with P a Sylow p -subgroup of $C_G(x)$ and $L \leq Z(C_G(x))$ or $C_G(x) = P Q \times A$, with P a p -Sylow of $C_G(x)$, Q a q -Sylow of $C_G(x)$, for some prime $q \neq p$, and $A \leq Z(G)$.

PROOF. Write $x = x_1 x_2 \cdots x_s$, where the order of each x_i is a power of a prime distinct from p and the x_i commute pairwise. As $x \notin Z(G)$, there exists an i such that $x_i \notin Z(G)$. By applying Hypothesis (2) we have $C_G(x) = C_G(x_i)$, whence there is no loss if we assume that x is a q -element for some prime $q \neq p$.

Suppose that there exists a prime divisor r of $|C_G(x)|$ such that $r \neq p, q$ (in another case the lemma is proved) and take R a Sylow r -subgroup of $C_G(x)$. If $y \in R$, since x and y have coprime orders, then $C_G(yx) = C_G(x) \cap C_G(y)$. By Hypothesis (1), it follows that $C_G(x) = C_G(yx) \subseteq C_G(y)$. Thus $R \leq Z(C_G(x))$, so we can write $C_G(x) = P Q \times A$, for some $P \in \text{Syl}_p(C_G(x))$, $Q \in \text{Syl}_q(C_G(x))$ and $A \leq Z(C_G(x))$. If $A \leq Z(G)$ we have finished. Suppose then that there exists a noncentral $u \in A$. Since u is a $\{p, q\}$ -element which commutes with x , by applying Hypotheses (1) and (2), we obtain $C_G(ux) = C_G(x) = C_G(u)$. Now, take $z \in Q$. Then $C_G(uz) = C_G(u) \cap C_G(z) \subseteq C_G(u) = C_G(x)$. By Hypothesis (1), we get $C_G(uz) = C_G(x)$, so $C_G(x) \subseteq C_G(z)$. Therefore, $z \in Z(C_G(x))$. If we put $L = Q \times A$, then $C_G(x) = P \times L$, with $L \leq Z(C_G(x))$ as required. □

The following result will also be needed for distinct sets of primes.

LEMMA 5. *Let π be a set of primes and suppose that G is a π -separable group. Then the conjugacy class length of any π -element $g \in G$ is a π' -number if and only if G has abelian Hall π -subgroups.*

PROOF. Suppose that any π -element $g \in G$ satisfies that $|g^G|$ is a π' -number, and work by induction on $|G|$ so as to prove that any Hall π -subgroup of G is abelian. Assume first that $O_{\pi'}(G) > 1$ and write $\bar{G} = G/O_{\pi'}(G)$. Notice that any π -element of \bar{G} can be written as \bar{g} , with g a π -element of G , so it is easy to see that the hypothesis is inherited in \bar{G} . We conclude that \bar{G} has abelian Hall π -subgroups and then so has G .

Now assume that $O_{\pi'}(G) = 1$, and thus $O_{\pi}(G) > 1$. The hypothesis implies that for any π -element $g \in G$ there exists a Hall π -subgroup H of G such that $g \in H \subseteq C_G(g)$. Thus $g \in C_G(H) \subseteq C_G(O_{\pi}(G)) \subseteq O_{\pi}(G)$. Consequently, G has a normal Hall π -subgroup, which moreover is abelian. The converse direction in the statement of the lemma is trivial. □

3. Theorem A and Corollary B

PROOF OF THEOREM A. We proceed with a series of steps.

Step 1. Let $d \in G_{p'}$ such that $|d^G| = m$. If $x \in Z(C_G(d))_{p'}$, then either $x \in Z(G)$ or $C_G(x) = C_G(d)$.

It is clear that $C_G(d) \subseteq C_G(x)$, so $(|x^G|, n) = 1$. If $x \notin Z(G)$ then, by Lemma 2 (b), we have $|x^G| = m$ and $C_G(x) = C_G(d)$.

Step 2. Let $d \in G_{p'}$ such that $|d^G| = m$. Then either $C_G(d) = P_d Q_d \times L$ where $L \leq Z(G)$, $P_d \in \text{Syl}_p(C_G(d))$ and $Q_d \in \text{Syl}_q(C_G(d))$ for some prime $q \neq p$ or $C_G(d) = P_d \times L_d$ with $P_d \in \text{Syl}_p(C_G(d))$ and $L_d \leq Z(C_G(d))$. In addition, if there exists some $d \in G_{p'}$ such that $|d^G| = m$ and $C_G(d) = P_d Q_d \times L$, with $L \leq Z(G)$, then $C_G(y) = P_y Q_y \times L$ for any $y \in G_{p'}$ such that $|y^G| = m$. In this case n is a power of q .

To prove the first assertion it is enough to verify the hypotheses of Lemma 4 for d . First suppose that $C_G(x) \leq C_G(d)$ for some $x \in G_{p'}$. Then $C_G(x) = C_G(d)$ by the maximality of m . Now, if $C_G(d) \leq C_G(x)$ for some noncentral $x \in G_{p'}$, then $|x^G| = m$ by applying Lemma 2 (b). Hence $C_G(d) = C_G(x)$ and the first part of the step follows.

Assume now that there exists some $d \in G_{p'}$ such that $|d^G| = m$ and $C_G(d) = P_d Q_d \times L$ with $L \leq Z(G)$. Let $y \in G_{p'}$ such that $|y^G| = m$. Since $|C_G(d)| = |C_G(y)|$ and $L \leq C_G(y)$, we can trivially write $C_G(y) = P_y Q_y \times L$ for some P_y and Q_y Sylow p - and q -subgroups of $C_G(y)$, respectively. Also, in this case $|C_G(d) : Z(G)_{p'}|$ is a $\{p, q\}$ -number. As n is a p' -number dividing $|G : Z(G)_{p'}| = |G : C_G(d)||C_G(d) : Z(G)_{p'}|$, we deduce that n is a q -number.

Step 3. There exists some $d \in G_{p'}$ of prime power order with $|d^G| = m$. If d is such an element whose order is an r -power for some prime $r \neq p$, then r does not divide m .

In order to prove the existence of such an element, choose d to be any p -regular element such that $|d^G| = m$. Observe that we can factor $d = d_r d_{r'}$ for some prime $r \neq p$, where d_r and $d_{r'}$ are the r -part and r' -part of d respectively, and such that d_r is noncentral. As $C_G(d) \subseteq C_G(d_r)$ we have $(|d_r^G|, n) = 1$. Applying Lemma 2 (b), we obtain $C_G(d) = C_G(d_r)$ and then $|d_r^G| = m$. Thus the first assertion is proved.

Now suppose that d is an r -element with $|d^G| = m$. By Step 2, we distinguish two possibilities: either $C_G(d) = P_d R_d \times L$, with $L \leq Z(G)$, $P_d \in \text{Syl}_p(C_G(d))$ and $R_d \in \text{Syl}_r(C_G(d))$ (note that the fact that d is a noncentral element implies that the prime fixed in Step 2 is necessarily r) or $C_G(d) = P_d \times L_d$, with $L_d \leq Z(C_G(d))$. In the first case, we know by Step 2 that n is an r -power, so r cannot divide m and we have finished the proof.

Suppose now that $C_G(d)$ is of the second type and let a be a p -regular element such that $|a^G| = n$. By Lemma 1 (b), $G = C_G(a)C_G(d)$ and so

$$n = |G : C_G(a)| = |C_G(d) : C_G(d) \cap C_G(a)|.$$

Assume that r divides m and we will derive a contradiction. In this case r does not divide $|C_G(d) : C_G(d) \cap C_G(a)|$, and as $\langle d \rangle$ is a normal r -subgroup of $C_G(d)$, then $d \in C_G(d) \cap C_G(a)$ and $a \in C_G(d)$. Now, Step 1 yields the contradiction $C_G(d) = C_G(a)$ and the step is proved.

For the rest of the proof we are fixing two p -regular elements a and b such that $|a^G| = n$ and $|b^G| = m$. Additionally, by Step 3 we can choose b to be a q -element for some prime $q \neq p$, so that q does not divide m . We also notice that, in accordance with Step 2, there exist two types for the structure of $C_G(b)$, but if the first one occurs, then actually $C_G(b) = P_b Q_b \times L$, with $Q_b \in \text{Syl}_p(C_G(b))$, because b is a noncentral q -element.

In the following steps we are going to make use of the subgroup M defined in Theorem 3 as well as of the notation and conclusions in that theorem without mentioning them. Recall that M is the subgroup generated by all the conjugacy classes in $\text{Con}(G_{p'})$ whose size is coprime to m . Notice that certainly $Z(G)_{p'} < M$.

Step 4. We can assume that a is a q' -element.

Since $(|a^G|, m) = 1$, then $a \in M$, or more precisely, $a \in M_{p'} - Z(G)_{p'}$. Write $a = a_q a_{q'}$, where a_q and $a_{q'}$ are the q -part and the q' -part of a , respectively, and notice that if $a_q \notin Z(G)$ then $q \in \pi(M_{p'}/Z(G)_{p'}) \subseteq \pi(m)$, a contradiction. Thus $a_q \in Z(G)$ and $C_G(a) = C_G(a_{q'})$, so we can replace a by $a_{q'}$ to assume that a is a q' -element.

Step 5. $C_G(a)_q = Z(G)_q$.

Suppose that there exists some noncentral q -element $y \in C_G(a)$. By Step 4, a and y

have coprime orders, so $C_G(ay) = C_G(a) \cap C_G(y) \subseteq C_G(a)$. The maximality of n implies that $C_G(ay) = C_G(a)$, whence $C_G(a) \subseteq C_G(y)$. Consequently, $(|y^G|, m) = 1$ and then $y \in M_{p'} - Z(G)_{p'}$, which implies the contradiction $q \in \pi(M_{p'}/Z(G)_{p'}) \subseteq \pi(m)$.

Step 6. $(C_G(a) \cap C_G(b))_{p'} = Z(G)_{p'}$.

Assume that there exists some noncentral p -regular $x \in C_G(a) \cap C_G(b)$. By considering Step 2, we have two possibilities describing the structure of $C_G(b)$. Suppose first that $C_G(b) = P_b \times L_b$, with L_b abelian. Then $x \in L_b$ and, by applying Step 1, we get $C_G(b) = C_G(x)$. Therefore, $a \in C_G(b)$, whence $C_G(b) \subseteq C_G(a)$, a contradiction. In this case the step is proved.

Suppose now that $C_G(b) = P_b Q_b \times L$ with $L \leq Z(G)$. Since x is a p' -element in $C_G(b)$, we can clearly factor $x = x_q x_{q'}$, where the q' -part $x_{q'} \in Z(G)$ and consequently $x_q \notin Z(G)$. But observe that x_q centralizes a , a contradiction to Step 5.

Step 7. For any noncentral q -element $x \in G$ it holds $|x^G| = m$.

As q is not a divisor of m , we can assume without loss of generality that $x \in Q - Z(G)_q$ for some Sylow q -subgroup Q of G contained in $C_G(b)$.

We distinguish the two possibilities for $C_G(b)$, provided by Step 2. If $C_G(b) = P_b \times L_b$, with L_b abelian, then $x \in L_b$. Since $x \notin Z(G)$, by Step 1 we obtain $C_G(x) = C_G(b)$, so $|x^G| = m$ and this case is finished.

Therefore, for the rest of the proof of this step we will assume that $C_G(b) = P_b Q \times L$, with $L \leq Z(G)$, and we also know that $n = |G : C_G(a)|$ is a q -power. On the other hand, we know that $M_{p'}$ is abelian and $Z(G)_q \subseteq M_{p'} \subseteq C_G(a)$, and taking into account Step 5, we can write $M_{p'} = S \times Z(G)_q$, where q does not divide $|S|$. Notice that $\langle x \rangle$ acts coprimely on the abelian subgroup S , so S can be factorized by coprime action properties

$$S = [S, \langle x \rangle] \times C_S(x).$$

Denote by $U = [S, \langle x \rangle]$. Observe that $a \in S$ and factor $a = uw$ with $u \in U$ and $w \in C_S(x)$. Consider the element $g = xw$ and, since w is a noncentral q' -element centralizing x , then

$$C_G(g) = C_G(x) \cap C_G(w) \subseteq C_G(x).$$

If q is not a divisor of $|x^G|$, then $(|x^G|, n) = 1$. Consequently, $|x^G| = m$ by Lemma 2 (b) and the proof is finished. Therefore, we can assume that q does divide $|x^G|$, whence q divides $|g^G|$ and, in particular, $|g^G| \neq m$. Now, if $|g^G| = n$ then $g \in M_{p'}$, but this implies that $x \in M_{p'} \cap Q \subseteq C_G(a) \cap Q = Z(G)_q$ by Step 5, which is a contradiction. Thus, from the maximality of m and n we deduce that $|g^G| < n$ and we will show that this fact provides a contradiction too. Observe that

$$(1) \quad |SQ : C_{SQ}(g)| \leq |g^G| < n = |G : C_G(a)| = |Q : C_G(a) \cap Q| = |Q : Z(G)_q|.$$

Moreover, since $S \trianglelefteq G$, S is abelian, $S \cap Q = 1$, and $S \subseteq C_G(w)$ we have

$$(2) \quad \begin{aligned} C_{SQ}(g) &= C_{SQ}(w) \cap C_{SQ}(x) = SC_Q(w) \cap C_S(x)C_Q(x) \\ &= C_S(x)[SC_Q(w) \cap C_Q(x)] = C_S(x)[C_Q(w) \cap C_Q(x)]. \end{aligned}$$

We denote by $D = C_Q(w) \cap C_Q(x)$. By combining (1) and (2), we obtain

$$\frac{|Q|}{|Z(G)_q|} > \frac{|S||Q|}{|C_S(x)||D|}.$$

This implies that $|D : Z(G)_q| > |S : C_S(x)| = |U|$.

On the other hand, as $D \subseteq C_G(x)$ and S is normal in G , then D acts on U by conjugation and we claim that $C_D(u) = Z(G)_q$. If $l \in C_D(u)$, since $D \subseteq C_Q(w)$, then $l \in C_G(a) \cap Q = Z(G)_q$ by Step 5. The other containment is trivial, so the claim is proved. We conclude that

$$|D : C_D(u)| = |D : Z(G)_q| > |U|,$$

that is, the size of the orbit of u under the action of D is greater than $|U|$, which is a contradiction.

Step 8. If d and t are two noncentral p -regular elements such that $|t^G| \neq m = |d^G|$, then $(C_G(d) \cap C_G(t))_{p'} = Z(G)_{p'}$ and $|t^G| = n$. Consequently, the only p -regular conjugacy class lengths of G are $\{1, n, m\}$.

To prove the first equality we distinguish again the two cases provided by Step 2 for $C_G(d)$. Suppose first that $C_G(d) = P_d \times L_d$ with $L_d \leq Z(C_G(d))$, and suppose that there exists a noncentral p -regular element $y \in C_G(d) \cap C_G(t)$. As $y \in Z(C_G(d))$ but y is noncentral, we deduce, by Step 1, that $C_G(y) = C_G(d)$. Thus $t \in C_G(d)$ so, in particular, $t \in Z(C_G(d))$ and $C_G(t) = C_G(d)$ again by Step 1, contradicting our hypotheses. Therefore, in this case the equality $(C_G(d) \cap C_G(t))_{p'} = Z(G)_{p'}$ is proved.

Now we assume that $C_G(d) = P_d Q_d \times L$, with $L \leq Z(G)$ (note that here we are applying the second part of Step 2, for if $C_G(d)$ has the first type structure given in that step, then $C_G(b)$ has the same structure, so the prime q is fixed). In this case we also know that n is a q -number. We prove first that t can be assumed to be a q' -element. Let t_q be the q -part of t and notice that $C_G(t) \subseteq C_G(t_q) \subseteq G$. If $t_q \notin Z(G)$, by Step 7 we get $|t_q^G| = m$, and accordingly $|t^G| = m$, against the hypothesis of this step. Therefore, $t_q \in Z(G)$, so $C_G(t) = C_G(t_{q'})$, where $t_{q'}$ is the q' -part of t , and thus there is no loss if we assume that t is a q' -element, as wanted.

Now, suppose that there exists some noncentral p -regular $y \in C_G(d) \cap C_G(t)$ and factor $y = y_q y_{q'}$, where y_q and $y_{q'}$ are the q -part and the q' -part of y , respectively. In view of the structure of $C_G(d)$, it is clear that $y_{q'} \in Z(G)$, so $y_q \notin Z(G)$. Again by

Step 7, we get $|y_q^G| = m$. Moreover,

$$C_G(ty) = C_G(ty_q) = C_G(t) \cap C_G(y_q) \subseteq C_G(y_q)$$

and the maximality of m implies that $|(ty)^G| = m$. As $C_G(ty) \subseteq C_G(t)$ then $(|t^G|, n) = 1$, and by Lemma 2 (b), we obtain the contradiction $|t^G| = m$. Thus, we also conclude in this case that $(C_G(d) \cap C_G(t))_{p'} = Z(G)_{p'}$.

We show finally that $|t^G| = n$. Notice that

$$|t^G| = |G : C_G(t)| \geq |C_G(d) : C_G(d) \cap C_G(t)| \geq |C_G(d) : Z(G)|_{p'}.$$

On the other hand, by applying the above property to a and d , we have $(C_G(d) \cap C_G(a))_{p'} = Z(G)_{p'}$. In addition, $C_G(a)C_G(d) = G$ by Lemma 1 (b), and since n is a p' -number, we obtain

$$n = |G : C_G(a)| = |C_G(d) : C_G(d) \cap C_G(a)| = |C_G(d) : Z(G)|_{p'}.$$

Therefore, $|t^G| \leq n$, and since $|t^G| \neq m$, we conclude that $|t^G| = n$ and the step is proved.

Step 9. $C_G(a)_{p'} = M_{p'}$.

Let K be a p -complement of $C_G(a)$ and choose R a Sylow r -subgroup of $C_G(a)$ with $R \subseteq K$ for any prime $r \neq p$. If $R \leq Z(G)$ we certainly have $R \subseteq M_{p'}$. We can assume that R is noncentral, so that r divides $|K : Z(G)_{p'}|$. Step 6 and the fact that $G = C_G(a)C_G(b)$ imply that

$$|K : Z(G)_{p'}| = |C_G(a) : C_G(a) \cap C_G(b)|_{p'} = |G : C_G(b)|_{p'} = m_{p'}.$$

Consequently, r divides m . Now, for any $y \in R$ we have $|y^G| \neq m$ by Step 3, so $|y^G| = n$ or 1 in view of Step 8. By definition of M , it follows that $y \in M_{p'}$ and thus $R \subseteq M_{p'}$. Consequently, $K \subseteq M_{p'}$ and the other containment is obvious.

Step 10. G is solvable and any p -complement of G is quasi-Frobenius with abelian kernel and complement. Furthermore, the conjugacy class sizes of any p -complement of G are $\{1, n, m_{p'}\}$.

Let H be a p -complement of G . First we are going to prove that, up to conjugacy, $H = M_{p'}T$ where T is a p -complement of $C_G(b)$, and that T is abelian.

In accordance with Step 2, we assume first that $C_G(b) = P_b \times L_b$, with L_b abelian. The fact that $G = C_G(a)C_G(b)$ together with Step 9 implies that $M_{p'}L_b$ is a p -complement of G satisfying the above stated conditions. In particular, notice that $G/M_{p'}$ is the product of two nilpotent groups, and this forces G to be solvable by applying Kegel-Wielandt's theorem.

We can assume then that $C_G(b) = P_b Q_b \times L$, with $L \leq Z(G)$, so $L \leq M_{p'}$. As above, it is clear that $M_{p'} Q_b$ is a p -complement of G , so we only have to show that Q_b

is abelian. To see this, we will check the hypotheses of Lemma 5. For any noncentral $x \in Q_b$, we know by Step 8 that $|x^G| = m$, which is a q' -number. On the other hand, $G/M_{p'}$ is a $\{p, q\}$ -group and $M_{p'}$ is abelian, so G is solvable too. Therefore, we can apply Lemma 5 with $\pi = \{q\}$ to conclude that Q_b is abelian, and we also get the required structure for H .

In order to show that H is quasi-Frobenius, we first prove that $Z(H) = Z(G)_{p'}$. Certainly, $Z(G)_{p'} \subseteq Z(H)$. Conversely, suppose that $x \in Z(H)$ and notice that $|x^G|$ is a p -power. By Step 8, we know that $|x^G| \in \{1, n, m\}$, but the two last possibilities cannot happen since n is not divisible by p and m is not a p -power by Theorem 3. Hence, $x \in Z(G)$ and the equality holds.

Since $M_{p'} \subseteq C_G(a)$ it follows, by applying Step 6, that $M_{p'} \cap T = Z(G)_{p'}$. Now, we denote by $\bar{T} = T/Z(G)_{p'}$ and by $\bar{M}_{p'} = M_{p'}/Z(G)_{p'}$, and in general we will use bars to work in $G/Z(G)_{p'}$. We will show that $(|\bar{T}|, |\bar{M}_{p'}|) = 1$. To see this, we apply the definition of T , the fact that $G = C_G(a)C_G(b)$ and Step 6 to obtain

$$\begin{aligned} |\bar{T}| &= |C_G(b) : Z(G)_{p'}|_{p'} = |C_G(b)|_{p'} / |C_G(a) \cap C_G(b)|_{p'} \\ &= |G : C_G(a)|_{p'} = n_{p'} = n \end{aligned}$$

and analogously, but now using Step 9, we deduce that

$$\begin{aligned} |\bar{M}_{p'}| &= |C_G(a) : Z(G)_{p'}|_{p'} = |C_G(a)|_{p'} / |C_G(a) \cap C_G(b)|_{p'} \\ &= |G : C_G(b)|_{p'} = m_{p'}. \end{aligned}$$

Accordingly $(|\bar{T}|, |\bar{M}_{p'}|) = 1$, as required.

In order to prove that $H/Z(G)_{p'}$ is a Frobenius group (with kernel $\bar{M}_{p'}$ and complement \bar{T}), we will show that $C_{\bar{T}}(\bar{g}) = 1$ for all $1 \neq \bar{g} \in \bar{M}_{p'}$. Suppose on the contrary that $1 \neq \bar{t} \in C_{\bar{T}}(\bar{g})$, for some $1 \neq \bar{g} \in \bar{M}_{p'}$, and we will provide a contradiction. Since we have $(o(\bar{t}), o(\bar{g})) = 1$, then $(\bar{g}\bar{t})^{o(\bar{g})} = \bar{t}^{o(\bar{g})} \neq 1$. Hence,

$$(\bar{g}\bar{t})^{o(\bar{g})} = \bar{t}^{o(\bar{g})} \in \overline{C_G(\bar{t})} \cap \overline{C_G(\bar{g}\bar{t})} = \overline{C_G(\bar{t})} \cap \overline{C_G(\bar{g}\bar{t})} = \overline{C_G(\bar{g})} \cap \overline{C_G(\bar{t})},$$

so, in particular, $C_G(\bar{g}) \cap C_G(\bar{t})$ contains a noncentral p -regular element.

If we show that $(C_G(\bar{g}) \cap C_G(\bar{t}))_{p'} = Z(G)_{p'}$, we will get the desired contradiction. As $\bar{T} \cap \bar{M}_{p'} = 1$ and $\bar{t} \notin Z(G)$ then $\bar{t} \notin M_{p'}$, and consequently $|\bar{t}^G| = m$ by applying Step 8. On the other hand, we have $M_{p'} \subseteq C_G(\bar{g})$ and $|G : M_{p'}| = p^k n$, for some natural number k from Step 9. This implies that $|g^G|$ divides $p^k n$, but $|g^G| \in \{n, m\}$ by Step 8, and m is not a p -power by Theorem 3. We conclude that $|g^G| = n$ and, by Step 8, the equality $(C_G(\bar{g}) \cap C_G(\bar{t}))_{p'} = Z(G)_{p'}$ holds.

Thus, we have shown that H is quasi-Frobenius. Furthermore, the kernel of H is $M_{p'}$, which is abelian, and a complement of H , T , is abelian too (this has been proved at the beginning of this step).

Finally, it remains to prove the last assertion in the statement of this step. It is easy to see that in a quasi-Frobenius group H with abelian kernel and complement, the conjugacy class sizes are exactly $\{1, n_1, m_1\}$, n_1 and m_1 being the orders of the kernel and a complement of \overline{H} , respectively. Since we have shown formerly that these orders are n and $m_{p'}$, the proof is finished. \square

PROOF OF COROLLARY B. If p does not divide n , then the result is immediate from Theorem A. We may assume then that p divides n and consequently, that p is not a divisor of m . In this case, [1, Theorem 5] asserts that G is p -nilpotent and the p -complement is quasi-Frobenius with abelian kernel and complement. In particular, it also follows that G is solvable.

Let $H \leq G$ be the p -complement of G . For any $x \in H$ we have

$$|x^G| = |G : C_G(x)| = |G : C_G(x)H| |H : C_H(x)|,$$

so $|H : C_H(x)| = |x^G|_{p'}$. It certainly follows that $\{1, n_{p'}, m\}$ is the set of conjugacy class sizes of H . \square

REMARK. Corollary B has been proved conditioning on whether p divides n or not. When p divides n , then the result is obtained from [1, Theorem 5], but actually the fact that a p -complement is quasi-Frobenius follows from that G has normal p -complement (see [1, Proposition 3]). We want to illustrate by an example that if p divides m , under the same hypotheses of Corollary B, then G need not be p -nilpotent. Let G be the dihedral group of order $2pq$ where p and q are two distinct odd primes. Then the p -regular class sizes of G are $\{1, 2, pq\}$, whereas G does not possess normal p -complement.

4. Theorems C and D

Theorems C and D describe the structure of G when we impose certain arithmetical conditions on the set of the p -regular class sizes in two particular cases. In the proof of Theorem C, we will also make use of the next two results. The first one extends a theorem of Ito describing groups with exactly two conjugacy class lengths.

THEOREM 6. *Suppose that G is a p -solvable group and that $\{1, m\}$ are the p -regular conjugacy class sizes of G . Then $m = p^a q^b$, with q a prime distinct from p and $a, b \geq 0$. If $b = 0$, then G has abelian p -complement. If $b \neq 0$, then $G = PQ \times A$, with $P \in \text{Syl}_p(G)$, $Q \in \text{Syl}_q(G)$ and $A \leq Z(G)$. Furthermore, if $a = 0$, then $G = P \times Q \times A$.*

PROOF. See [2, Theorem A]. \square

The following determines the structure of those p -solvable groups whose p -regular class sizes are p' -numbers.

LEMMA 7. *Suppose that G is a p -solvable group and that p is not a divisor of the lengths of p -regular conjugacy classes. Then $G = P \times H$, where P is a Sylow p -subgroup and H is a p -complement of G .*

PROOF. See for instance [1, Proposition 2]. □

Now we are able to prove Theorems C and D.

PROOF OF THEOREM C. If $r = 0$, then the structure of G is given by Theorem 6 and thus we obtain cases (a) and (b).

If $r = 1$, then the first assertion in (c) follows from Corollary B. The second assertion in (c) is obtained by applying [1, Theorem 5] and Lemma 7. Suppose then that $r \geq 2$. Since $(n + r, n + r - 1) = 1$, we can apply Lemma 2 (b) to obtain that $n + r$ is the only p -regular class size greater than 1 which is coprime to $n + r - 1$. Since $n > 1$, this is a contradiction with the fact that $1 \neq n + r - 2$ is coprime to $n + r - 1$. □

We want to remark that [4, Theorem 2] describes the structure of those groups whose noncentral conjugacy class sizes are consecutive numbers. However, its proof can be considerably simplified by following the same arguments as in the proof of Theorem C, taking into account that Lemma 2 can be applied for ordinary conjugacy classes.

PROOF OF THEOREM D. It is easy to see that if G is as described in the conclusions of the theorem, then the lengths of all classes in $\text{Con}(G_{p'})$ are prime powers and these powers are exactly as indicated in each one of the cases. Note that for case (a), it suffices to apply Lemma 5 in the converse direction with $\pi = \{p\}'$.

Conversely, suppose that any p -regular class length is a prime power. We first notice that such a group G is solvable. To prove this, it is sufficient to use induction on $|G|$ and Burnside's Theorem which establishes the non-existence of prime power size classes in a nonabelian simple group.

Assume now that the size of every class in $\text{Con}(G_{p'})$ is a power of p . This implies that G has abelian p -complements by Lemma 5.

Suppose now that the size of every conjugacy class of $\text{Con}(G_{p'})$ is a power of some prime q , distinct from p . By Lemma 7, we have $G = P \times H$, with H a p -complement of G . Then every noncentral class length of H is a q -power. It is well known (see for instance [6, Proposition 4]) that if a prime r does not divide any conjugacy class size of H , then H has a central Sylow r -subgroup. Therefore, for any

prime $r \neq q$ and $R \in \text{Syl}_r(H)$ we have $R \leq Z(H)$. As $G = P \times H$, we obtain that $R \leq Z(G)$. Therefore, G is nilpotent with at most two nonabelian Sylow subgroups (for the primes p and q).

Finally, we assume that there exist at least two distinct prime divisors of the p -regular class sizes. By [1, Theorem 1], we deduce that there must be exactly two distinct primes, say q and r . Although q or r could initially be equal to p , notice that by Theorem 3 the maximal p -regular class size is not a p -power and thus, by [1, Theorem 5], G is p -nilpotent and the p -complement H of G is a quasi-Frobenius group with abelian kernel and complement. In such a group the noncentral conjugacy class sizes are exactly two coprime numbers. Moreover, since any conjugacy class size in H divides some p -regular conjugacy class size in G , it follows that the set of the conjugacy class sizes of H is $\{1, r^t, q^s\}$, where t and s are positive integers. Also, q and r must be distinct from p , since H is a p' -group. Therefore, p does not divide any p -regular class size of G , so by Lemma 7, we obtain $G = P \times H$. Consequently, it follows that $\{1, r^t, q^s\}$ is also the set of the p -regular class sizes of G . \square

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