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NIL ALGEBRAS WITH RESTRICTED GROWTH

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Abstract It is shown that over an arbitrary countable field there exists a finitely generated algebra that is nil, infinite dimensional and has Gelfand–Kirillov dimension at most 3.

Keywords: nil algebras; growth of algebras; Gelfand–Kirillov dimension

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1. Introduction

In 1902 William Burnside asked the following question, which later became known as the Burnside Problem: does a finitely generated group whose elements all have finite order need to be finite? An analogous problem for algebras is the Kurosh Problem: if A is a finitely generated algebra over a field K, and every element of A is algebraic over K, does it follow that A is finite dimensional over K? A special case of the Kurosh Problem, sometimes known as Levitski's Problem, concerns nil algebras: if A is a finitely generated algebra over a field K and every element of A is nilpotent, is A finite dimensional over K?

The seminal works of Golod and Shafarevich [**1**, **2**] in 1964 showed that the answer to these famous problems was negative. Their method entailed the construction of a finitely generated nil algebra A which was infinite dimensional; from this algebra the counterexample to the Burnside Problem arises by considering a group whose elements are of the form $1 + n$, for a particular nil algebra A and some $n \in A$.

The groups and the algebras constructed by the Golod–Shafarevich method have exponential growth. Gromov [**3**] proved in 1981 that, under the assumption that the group has polynomial growth, the answer to the Burnside Problem is positive. In fact, he proved that a finitely generated group with polynomial growth has a nilpotent normal subgroup of finite index. As a consequence, if a finitely generated group has polynomial growth and each element has finite order, then the group is finite.

Golod and Shafarevich's work, together with Gromov's result, naturally raises the question as to whether a finitely generated nil algebra with polynomial growth is of necessity finite dimensional (see [**9**]; L. W. Small, personal communication, February 2004). Surprisingly, this is not the case; in [**5**] Lenagan and Smoktunowicz constructed, over any countable field, an infinite-dimensional finitely generated nil algebra with Gelfand–Kirillov dimension at most 20. This result raises the following question: what is the minimal rate of growth for a finitely generated infinite-dimensional nil algebra? In this paper, we make progress on this latter question; by refining the methods of [**5**], we construct, over any countable field, an infinite-dimensional finitely generated nil algebra with Gelfand–Kirillov dimension at most 3. (In fact, our algebra requires only two generators.)

2. Notation

In what follows, K will be a countable field and A will be the free K-algebra in two non-commuting indeterminates x and y. The set of monomials in x, y is denoted by M, and $M(n)$ denotes the set of monomials of degree n for each $n \geqslant 0$. Thus, $M(0) = \{1\}$ and for $n \geqslant 1$ the elements in $M(n)$ are of the form $x_1 \cdots x_n$, where $x_i \in \{x, y\}$. The K-subspace of A spanned by $M(n)$ will be denoted by $H(n)$ and elements of $H(n)$ will be called *homogenous polynomials of degree* n. The *degree*, $\deg f$, of any $f \in A$, is the least $d \geqslant 0$ such that $f \in H(0) + \cdots + H(d)$. Any $f \in A$ can be uniquely presented in the form $f = f_0 + f_1 + \cdots + f_d$, where each $f_i \in H(i)$. The elements f_i are the *homogeneous components* of f. A right ideal I of A is *homogeneous* if for every $f \in I$ all homogeneous components of f are in I. If V is a linear space over K, then $\dim_K V$ denotes the dimension of V over K. The Gelfand–Kirillov dimension of an algebra R is denoted by $\mathrm{GKdim}(R)$. For elementary properties of Gelfand–Kirillov dimension we refer the reader to [**4**].

For any real number k, define $\lfloor k \rfloor$ to be the largest integer not exceeding k.

Throughout the paper, $\bar{A}$ will denote the subalgebra of A consisting of polynomials with constant term equal to 0.

Assume that all logarithms in this paper are of base 2.

The aim of this paper is to present an algebra with the desired properties in the form $\bar{A}/E$ for a suitable ideal E. Roughly speaking, we construct a sequence of linear spaces $U(2^n)$, and then set E to be the largest subset such that, for all $n \geqslant 0$, $AEA \cap H(2^n) \subseteq U(2^n)$. As the sets $U(2^n)$ will be very large in dimension ($\dim_K U(2^n) + 2 = \dim_K H(2^n)$ for most n) and behave like an ideal (that is, $H(2^n)U(2^n) + U(2^n)H(2^n) \subseteq U(2^{n+1})$), the ideal E will be very large and hence $\mathrm{GKdim}\, \bar{A}/E$ will be small. To guarantee that the algebra $\bar{A}/E$ is nil we allow the sets $U(2^n)$ to have a bigger co-dimension at some sparse places.

3. Enumerating elements

We start with the following lemma.

Lemma 3.1. *Let K be a countable field and let $\bar{A}$ be as above. Then there exists a subset $Z \subseteq \mathbb{N}$, with all $i \in Z$ being greater than or equal to 5, and an enumeration*

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$\{f_i\}_{i\in Z}$ *of* $\bar{A}$ *such that* $\lfloor\log i\rfloor > 6^{6\deg f_i}$. *Moreover, the set* Z *has the following property: if* $i > j$ *and* $i, j \in Z$, *then* $i > 2^{2^{2^{2j}}}$.

Proof. As $\bar{A}$ is a finitely generated algebra over a countable field, it is itself countable. Let $\bar{A} = \{g_1, g_2, \dots\}$ be an arbitrary enumeration. We now inductively define an increasing function $\theta\colon \mathbb{N} \to \mathbb{N}$ as follows: first set $\theta(1) := \min\{i \in \mathbb{N} \mid i > 4,\ \lfloor\log i\rfloor > 6^{6\deg g_1}\}$.

As an inductive hypothesis, suppose that θ is defined over $\{1, \dots, n\}$ such that $\lfloor\log(\theta(i))\rfloor > 6^{6\deg g_i}$ for each $i \leqslant n$. Then set

$$\theta(n+1) = \min\Big\{s \in \mathbb{N} \mid \lfloor\log s\rfloor > 6^{6\deg g_{n+1}},\ s > 2^{2^{2^{2\theta(n)}}}\Big\}.$$

If we now rename the elements of $\bar{A}$ by setting $f_{\theta(s)} = g_s$, then we have a listing of the elements of $\bar{A}$ with the required properties. □

Theorem 3.2. *Let* Z *and* $\{f_i\}_{i\in Z}$ *be as in Lemma 3.1. Let* $i \in Z$ *and let* I *be the two-sided ideal generated by* $f_i^{10w_i}$, *where* $w_i = 4\cdot 2^{2^i-\lfloor\log i\rfloor}$. *There is a linear* K*-space* $F_i \subseteq H(2^{2^i-\lfloor\log i\rfloor})$ *such that* $I \subseteq \sum_{k=0}^{\infty} H(k(2^{2^i-\lfloor\log i\rfloor}))F_iA$ *and* $\dim_K(F_i) < 2^{2^i} - 2$.

Proof. Note that $6^{6\deg(f_i)} < \lfloor\log i\rfloor$ by Lemma 3.1. Apply [**7**, Theorem 2] with $f = f_i$, $r = 2^{2^i-\lfloor\log i\rfloor}$, $w = w_i = 4\cdot 2^{2^i-\lfloor\log i\rfloor}$ and put $F_i = \operatorname{span}_K F$, where F is the corresponding set F of the conclusion of Lemma 3.1. Note that these choices of f, r, w satisfy the hypotheses of [**7**, Theorem 2]. Although the algebra A in [**7**, Theorem 2] is generated by three elements rather than two, this does not influence the proof. □

4. Definition of $U(2^n)$ and $V(2^n)$

In this section we shall define the sets $U(2^n)$, for $n = 1, 2, \dots$, mentioned in § 2.

For each $i \in Z$, set $S_i = [2^i - i - \lfloor\log i\rfloor, 2^i - \lfloor\log i\rfloor - 1]$ and set $S = \bigcup_{i\in Z} S_i$. Note that the S_i are pairwise disjoint.

Theorem 4.1. *Let* Z *and* F_i *be as in Theorem 3.2. There are* K*-linear subspaces* $U(2^n)$ *and* $V(2^n)$ *of* $H(2^n)$ *such that, for all* $n > 0$:

(i) $\dim_K V(2^n) = 2$ *if* $n \notin S$;

(ii) $\dim_K V(2^{2^i-i-\lfloor\log(i)\rfloor+j}) = 2^{2^j}$ *for all* $1 < i \in Z$ *and all* $0 \leqslant j \leqslant i-1$;

(iii) $V(2^n)$ *is spanned by monomials;*

(iv) $F_i \subseteq U(2^{2^i-\lfloor\log(i)\rfloor})$ *for every* $i \in Z$;

(v) $V(2^n) \oplus U(2^n) = H(2^n)$;

(vi) $H(2^n)U(2^n) + U(2^n)H(2^n) \subseteq U(2^{n+1})$;

(vii) $V(2^{n+1}) \subseteq V(2^n)V(2^n)$;

(viii) *if* $n \notin S$, *then there are monomials* $m_1, m_2 \in V(2^n)$ *such that* $V(2^n) = Km_1 + Km_2$ *and* $m_2H(2^n) \subseteq U(2^{n+1})$.

Proof. The proof of properties (i)–(vii) is very similar to the proof of [**5**, Theorem 3], and the proof of property (viii) is similar to the proof of [**8**, Theorem 10(8)]. We construct the sets $U(2^n)$ and $V(2^n)$ inductively. Set $V(2^0) = Kx + Ky$ and $U(2^0) = 0$. Assume that we have defined $V(2^m)$ and $U(2^m)$ for $m \leqslant n$ in such a way that conditions (i)–(v) hold for all $m \leqslant n$ and conditions (vi)–(viii) hold for all $m < n$. Then we define $V(2^{n+1})$ and $U(2^{n+1})$ inductively, in the following way. Consider the following three cases.

Case 1 ($n \in S$ and $n+1 \in S$). If $n \in S$ and $n+1 \in S$, define $U(2^{n+1}) = H(2^n)U(2^n) + U(2^n)H(2^n)$ and $V(2^{n+1}) = V(2^n)V(2^n)$. Conditions (vi) and (vii) certainly hold. If, by induction, conditions (v) and (iii) hold for $U(2^n)$ and $V(2^n)$, they hold for $U(2^{n+1})$ and $V(2^{n+1})$ as well. Moreover, $\dim_K V(2^{n+1}) = (\dim_K V(2^n))^2$, inductively satisfying condition (ii).

Case 2 ($n \notin S$). Suppose that $n \notin S$. Then $\dim_K V(2^n) = 2$, and $V(2^n)$ is generated by monomials, by the inductive hypothesis. Let m_1, m_2 be the distinct monomials that generate $V(2^n)$. Then

$$V(2^n)V(2^n) = Km_1m_1 + Km_1m_2 + Km_2m_1 + Km_2m_2.$$

Set $V(2^{n+1}) = Km_1m_1 + Km_1m_2$, so that conditions (i), (iii), (vii) and (viii) hold.

Set

$$U(2^{n+1}) = H(2^n)U(2^n) + U(2^n)H(2^n) + m_2V(2^n).$$

Using this definition, condition (vi) holds and

$$\begin{aligned} H(2^{n+1}) &= H(2^n)H(2^n) \\ &= U(2^n)U(2^n) \oplus U(2^n)V(2^n) \oplus V(2^n)U(2^n) \oplus m_1V(2^n) \oplus m_2V(2^n) \\ &= U(2^{n+1}) \oplus V(2^{n+1}). \end{aligned}$$

Thus, condition (v) holds.

Case 3 ($n \in S$ and $n+1 \notin S$). Suppose that $n \in S$ while $n+1 \notin S$. Then $n = 2^i - \lfloor \log(i) \rfloor - 1$ for some $i \in Z$. By induction on condition (ii),

$$\dim_K V(2^n) = \dim_K V(2^{2^i - i - \lfloor \log(i) \rfloor + i - 1}) = 2^{2^{i-1}}$$

and

$$\dim_K V(2^n)V(2^n) = 2^{2^{i-1}} 2^{2^{i-1}} = 2^{2^i}.$$

By induction on condition (v),

$$H(2^{n+1}) = U(2^n)U(2^n) \oplus U(2^n)V(2^n) \oplus V(2^n)U(2^n) \oplus V(2^n)V(2^n).$$

We know that F_i has a basis $\{f_1, \dots, f_s\}$ for some $f_1, \dots, f_s \in H(2^{2^i} - \lfloor \log(i) \rfloor)$ and $s < 2^{2^i} - 2$. Each f_j can be uniquely decomposed into $\bar{f}_j + g_j$ with $\bar{f}_j \in V(2^n)V(2^n)$ and $g_j \in V(2^n)U(2^n) + U(2^n)U(2^n) + U(2^n)V(2^n)$. Let P be the subspace spanned by $\bar{f}_1, \dots, \bar{f}_s$.

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Since $\dim_K P \leqslant s = \dim F_i < 2^{2^i} - 2 < \dim_K V(2^n)V(2^n) - 2$, there must exist at least two monomials $m_1, m_2 \in V(2^n)V(2^n)$ such that the space $Km_1 + Km_2$ is disjoint from P. Define $V(2^{n+1})$ as this space; this satisfies conditions (i), (iii) and (vii).

As P is disjoint from $Km_1 + Km_2$, there must exist a space $Q \supseteq P$ such that $V(2^n)V(2^n) = Q \oplus (Km_1, Km_2)$. Set

$$U(2^{n+1}) = U(2^n)U(2^n) + U(2^n)V(2^n) + V(2^n)U(2^n) + Q.$$

This immediately satisfies conditions (v) and (vi). Since each polynomial $f_i = g_i + \bar{f}_i \in U(2^{n+1})$, it satisfies condition (iv) as well. □

Before continuing, a helpful lemma concerning $U(2^n)$ should be mentioned.

Lemma 4.2. *For any $m \geqslant n$, and any $0 \leqslant k < 2^{m-n}$,*

$$H(k2^n)U(2^n)H((2^{m-n} - k - 1)2^n) \subseteq U(2^m).$$

Proof. If $m = n$, then $k = 0$ and the equation is trivially true. Using induction, assume the theorem holds true for some $m \geqslant n$. When $0 \leqslant k < 2^{m-n}$,

$$\begin{aligned} &H(k2^n)U(2^n)H((2^{m+1-n} - k - 1)2^n) \\ &\qquad = H(k2^n)U(2^n)H((2^{m-n} - k - 1)2^n)H(2^m) \subseteq U(2^m)H(2^m) \subseteq U(2^{m+1}), \end{aligned}$$

and when $2^{m-n} \leqslant k < 2^{m+1-n}$,

$$\begin{aligned} &H(k2^n)U(2^n)H((2^{m+1-n} - k - 1)2^n) \\ &\qquad = H(2^m)H((k - 2^{m-n})2^n)U(2^n)H((2^{m+1-n} - k - 1)2^n) \\ &\qquad \subseteq H(2^m)U(2^m) \subseteq U(2^{m+1}), \end{aligned}$$

as required. □

Another way of stating Lemma 4.2 is that, given any product of the form

$$H(i2^n)U(2^n)H(j2^n),$$

if the sum of the three arguments $i2^n + 2^n + j2^n$ is a power of 2, then

$$H(i2^n)U(2^n)H(j2^n) \subseteq U(i2^n + 2^n + j2^n).$$

5. A finitely generated infinite-dimensional nil algebra

A graded subspace $E \subseteq \bar{A}$ is formed by defining its homogeneous subspace $E(n)$ to be the set of elements $r \in H(n)$ such that if $2^m \leqslant n < 2^{m+1}$, then, for all $0 \leqslant j \leqslant 2^{m+2} - n$,

$$H(j)rH(2^{m+2} - j - n) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1}).$$

Now, define $E := E(1) + E(2) + \cdots$.

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Theorem 5.1. *The subset E is an ideal in $\bar{A}$. Moreover, $\bar{A}/E$ is a nil algebra and is infinite dimensional over K.*

Proof. The set E is shown to be an ideal in [**5**, Theorem 5], and Theorems 14 and 15 of [**5**] prove that $\bar{A}/E$ is both nil and infinite dimensional over K. No changes to these proofs need to be made to apply to our example, and so the proofs are not repeated here. □

6. The subspaces R, S, Q, W

The key to computing the Gelfand–Kirillov dimension of the algebra $\bar{A}/E$ is to use a collection of subspaces R, S, Q, W with the following properties: if $n > 0$, $2^m \leqslant n < 2^{m+1}$, then

$$R(n)H(2^{m+1}-n) \subseteq U(2^{m+1}), \qquad H(2^{m+1}-n)S(n) \subseteq U(2^{m+1}),$$
$$H(n) = R(n) \oplus Q(n), \qquad H(n) = S(n) \oplus W(n).$$

It then follows from Lemma 4.2 that, for any $k > m$, $R(n)H(2^k - n) \subseteq U(2^k)$ and $H(2^k - n)S(n) \subseteq U(2^k)$.

The existence of suitable such subspaces is established in the next section. Once this has been achieved, the following theorem is available to help calculate the Gelfand–Kirillov dimension of $\bar{A}/E$. (In this theorem we take $R(0) = S(0) = U(0) = 0$ and $V(0) = Q(0) = W(0) = K$.)

Theorem 6.1. *For every $n \in \mathbb{N}$,*

$$\bigcap_{k=0}^{n} S(n-k)H(k) + H(n-k)R(k) \subseteq E(n).$$

Moreover,

$$\dim\left(\frac{H(n)}{E(n)}\right) \leqslant \sum_{k=0}^{n} \dim(W(n-k))\dim(Q(k)).$$

Proof. The proof of the first claim is very similar to the proof of [**5**, Theorem 9] and so is omitted. Notice that

$$\begin{aligned} H(n) &= (S(n-k) \oplus W(n-k))(R(k) \oplus Q(k)) \\ &= (S(n-k)H(k) + H(n-k)R(k)) \oplus W(n-k)Q(k). \end{aligned}$$

Therefore,

$$\begin{aligned} \dim E(n) &\geqslant \dim\left(\bigcap_{k=0}^{n}(S(n-k)H(k) + H(n-k)R(k))\right) \\ &\geqslant \dim(H(n)) - \sum_{k=0}^{n} \dim(W(n-k)Q(k)) \end{aligned}$$

and so

$$\dim\left(\frac{H(n)}{E(n)}\right) \leqslant \sum_{k=0}^{n} \dim(W(n-k)Q(k)),$$

as required. □

7. A sufficiently small Q and W

In order to define R, S, Q and W, begin with $R(1) = S(1) = U(1)$ and $Q(1) = W(1) = V(1)$. Given any natural number j with $2^m \leqslant j < 2^{m+1}$, define

$$R(j) = \{r \in H(j) \colon rH(2^{m+1} - j) \subseteq U(2^{m+1})\}$$

and

$$S(j) = \{r \in H(j) \colon H(2^{m+1} - j)r \subseteq U(2^{m+1})\}.$$

Theorem 7.1. *Let j be a natural number. Write j in binary form as*

$$j = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$$

with $0 \leqslant p_0 < p_1 < \cdots < p_n$. Then there is a K-linear space $W(j) \subseteq H(j)$ such that $W(j) \oplus S(j) = H(j)$ and

$$W(j) \subseteq V(2^{p_0}) \cdots V(2^{p_n}) = \prod_{i=0}^{n} V(2^{p_i}).$$

Proof. By Theorem 4.1 (v), $H(2^{p_i}) = U(2^{p_i}) \oplus V(2^{p_i})$ for $i = 1, 2, \ldots, n$. Hence,

$$H(j) = \prod_{i=0}^{n} (U(2^{p_i}) \oplus V(2^{p_i})),$$

and

$$H(j) = \left(\sum_{i=0}^{n} H(2^{p_0} + \cdots + 2^{p_{i-1}})U(2^{p_i})H(2^{p_{i+1}} + \cdots + 2^{p_n})\right) \oplus \prod_{i=0}^{n} V(2^{p_i}).$$

Define $T_{p_i}(j)$ as $H(2^{p_0} + \cdots + 2^{p_{i-1}})U(2^{p_i})H(2^{p_{i+1}} + \cdots + 2^{p_n})$, so that

$$H(j) = \left(\sum_{i=0}^{n} T_{p_i}(j)\right) \oplus \prod_{i=0}^{n} V(2^{p_i}).$$

Now, from the definition of $T_{p_i}(j)$, we obtain

$$\begin{aligned} &H(2^{p_n+1} - j)T_{p_i}(j) \\ &\quad = H(2^{p_n+1} - (2^{p_i} + \cdots + 2^{p_n}))U(2^{p_i})H(2^{p_{i+1}} + \cdots + 2^{p_n}) \\ &\quad = H((2^{p_n+1-p_i} - (2^0 + \cdots + 2^{p_n-p_i}))2^{p_i})U(2^{p_i})H((2^{p_{i+1}-p_i} + \cdots + 2^{p_n-p_i})2^{p_i}). \end{aligned}$$

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It follows from Lemma 4.2 that $H(2^{p_n+1} - j)T_{p_i}(j) \subseteq U(2^{p_n+1})$; therefore, each $T_{p_i}(j) \subseteq S(j)$. Thus, there must exist some $W(j) \subseteq \prod_{i=0}^{n} V(2^{p_i})$ such that $S(j) \oplus W(j) = H(j)$. To see this more clearly, choose a basis of $(\prod_{i=0}^{n} V(2^{p_i}) + S(j))/S(j)$, pull this basis back to elements in $\prod_{i=0}^{n} V(2^{p_i})$ and let $W(j)$ be the subspace generated by that basis. □

Next, the sets $N(2^i)$ are defined in a similar way to the procedure used in [**8**].

Let $i \notin S$. Then, by Theorem 4.1 (viii), each $V(2^i)$ is generated by two monomials, $m_{1,i}$ and $m_{2,i}$, with $m_{2,i}H(2^i) \subseteq U(2^{i+1})$. Define $N(2^i) = Km_{1,i}$ and $M(2^i) = U(2^i) + Km_{2,i}$. In the case where $i \in S$, simply set $N(2^i) = V(2^i)$, $M(2^i) = U(2^i)$. Observe that, for every i, $N(2^i) \oplus M(2^i) = H(2^i)$. These sets will be used to construct $Q(n)$.

Lemma 7.2. *For any integer* $0 \leqslant m < 2^{k-1}$,

$$H(m2^{n+1})M(2^n)H((2^k - 2m - 1)2^n) \subseteq U(2^{n+k}).$$

Proof. By definition, $M(2^n)H(2^n) \subseteq U(2^{n+1})$. Using this fact and Lemma 4.2,

$$\begin{aligned} H(m2^{n+1})M(2^n)H((2^k - 2m - 1)2^n) &\subseteq H(m2^{n+1})U(2^{n+1})H((2^{k-1} - m - 1)2^{n+1}) \\ &\subseteq U(2^{n+k}), \end{aligned}$$

as required. □

Theorem 7.3. *Let* $j \in \mathbb{N}$. *Write* j *in binary form as*

$$j = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$$

with $0 \leqslant p_0 < p_1 < \cdots < p_n$, *and suppose* $n \neq 0$ *(that is,* j *is not a power of* 2*). Then there is a linear space* $Q(j) \subseteq H(j)$ *such that* $Q(j) \oplus R(j) = H(j)$ *and*

$$Q(j) \subseteq N(2^{p_n})N(2^{p_{n-1}}) \cdots N(2^{p_0}) = \prod_{i=0}^{n} N(2^{p_{n-i}}) \subseteq \prod_{i=0}^{n} V(2^{p_{n-i}}).$$

Proof. This proof is very similar to that for Theorem 7.1. By definition, $H(2^{p_i}) = N(2^{p_i}) \oplus M(2^{p_i})$ for $i = 1, 2, \ldots, n$. Hence,

$$H(j) \subseteq \prod_{i=0}^{n} (N(2^{p_i}) \oplus M(2^{p_i}))$$

and

$$H(j) = \left(\sum_{i=0}^{n} H(2^{p_n} + \cdots + 2^{p_{i+1}})M(2^{p_i})H(2^{p_{i-1}} + \cdots + 2^{p_0}) \right) \oplus \prod_{i=0}^{n} N(2^{p_i}).$$

Set

$$B_{p_i}(j) := H(2^{p_n} + \cdots + 2^{p_{i+1}})M(2^{p_i})H(2^{p_{i-1}} + \cdots + 2^{p_0}),$$

so that

$$H(j) = \sum_{i=0}^{n} B_{p_i}(j) \oplus \prod_{i=0}^{n} N(2^{p_i}).$$

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Multiplying on the right by $H(2^{p_n+1}-j)$, we obtain

$$\begin{aligned}B_{p_i}(j)H(2^{p_n+1}-j) &= H(2^{p_n}+\cdots+2^{p_{i+1}})M(2^{p_i})H(2^{p_n+1}-(2^{p_n}+\cdots+2^{p_i}))\\ &= H((2^{p_n-p_i-1}+\cdots+2^{p_{i+1}-p_i-1})2^{p_i+1})M(2^{p_i})H(2^{p_i})\\ &\quad\times H((2^{p_n-p_i}-(2^{p_n-p_i-1}+\cdots+2^{p_{i+1}-p_i-1}-1))2^{p_i+1})\\ &\subseteq H((2^{p_n-p_i-1}+\cdots+2^{p_{i+1}-p_i-1})2^{p_i+1})U(2^{p_i+1})\\ &\quad\times H((2^{p_n-p_i}-(2^{p_n-p_i-1}+\cdots+2^{p_{i+1}-p_i-1}-1))2^{p_i+1}).\end{aligned}$$

It follows from Lemma 4.2 that $B_{p_i}(j)H(2^{p_n+1}-j)\subseteq U(2^{p_n+1})$; therefore, each $B_{p_i}(j)\subseteq R(j)$. By exactly the same reasoning as in Theorem 7.1, there must exist some

$$Q(j)\subseteq\prod_{i=0}^{n}N(2^{p_{n-i}})$$

such that $R(j)\oplus Q(j)=H(j)$. □

One last theorem about the size of Q and W must be obtained before continuing.

Theorem 7.4. *Suppose that $j,k\in\mathbb{N}$ have the binary forms*

$$k=2^{p_0}+\cdots+2^{p_{i-1}},\qquad j=2^{p_i}+\cdots+2^{p_n}.$$

with $p_0<\cdots<p_n$. Then $\dim Q(j+k)\leqslant\dim Q(j)\dim Q(k)$ and $\dim W(j+k)\leqslant\dim W(j)\dim W(k)$.

Proof. Use the definition of Q to see that

$$H(j+k)=(R(j)\oplus Q(j))(R(k)\oplus Q(k))=R(j)H(k)\oplus Q(j)R(k)\oplus Q(j)Q(k).$$

If it can be shown that $R(j)H(k)+Q(j)R(k)\subseteq R(j+k)$, then

$$\begin{aligned}\dim Q(j+k) &= \dim H(j+k)-\dim R(j+k)\\ &\leqslant \dim H(j)\dim H(k)-\dim R(j)\dim H(k)-\dim Q(j)\dim R(k)\\ &= \dim Q(j)\dim H(k)-\dim Q(j)\dim R(k)\\ &= \dim Q(j)\dim Q(k),\end{aligned}$$

which establishes the Q inequality.

In order to show that $R(j)H(k)\subseteq R(j+k)$, note that $2^{p_n}<j+k<2^{p_n+1}$, and recall from the definition of $R(j)$ that

$$R(j)H(k)H(2^{p_n+1}-j-k)=R(j)H(2^{p_n+1}-j)\subseteq U(2^{p_n+1})$$

so that $R(j)H(k)\subseteq R(j+k)$ by the definition of $R(j+k)$.

Finally, to show that $Q(j)R(k)\subseteq R(j+k)$, note that $2^{p_{i-1}}\leqslant k<2^{p_{i-1}+1}$ and

$$\begin{aligned}Q(j)R(k)H(2^{p_n+1}-j-k) &= Q(j)(R(k)H(2^{p_{i-1}+1}-k))H(2^{p_n+1}-2^{p_{i-1}+1}-j)\\ &\subseteq H(j)U(2^{p_{i-1}+1})H(2^{p_n+1}-2^{p_{i-1}+1}-j).\end{aligned}$$

As each of j and $2^{p_n+1}-2^{p_{i-1}+1}-j$ is divisible by $2^{p_{i-1}+1}$, Lemma 4.2 reveals that $Q(j)R(k)H(2^{p_n+1}-j-k)\subseteq U(2^{p_n+1})$ and so $Q(j)R(k)\subseteq R(j+k)$.

An analogous argument is used to prove the inequality for W. □

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8. Inequalities

In this section we shall prove, using induction, that, for all $n > 1$,

$$\dim Q(n), \dim W(n) \leqslant 8\sqrt{n}(\log n)^3.$$

This result is obtained by combining the following theorems.

Theorem 8.1. *If $2^m < n < 2^{m+1}$, then*

$$\dim W(n) \leqslant \dim Q(2^{m+1} - n) \dim V(2^{m+1})$$

and

$$\dim Q(n) \leqslant \dim W(2^{m+1} - n) \dim V(2^{m+1})$$

Proof. By the definition, $R(2^{m+1} - n)H(n) \subseteq U(2^{m+1})$. Therefore, if $c \in H(n)$ and $Q(2^{m+1} - n)c \subseteq U(2^{m+1})$, then $H(2^{m+1} - n)c \subseteq U(2^{m+1})$ and $c \in S(n)$.

Let $v_1, \dots, v_d \in Q(2^{m+1}-n)$ be a basis of $Q(2^{m+1}-n)$ over K and let $c_1, \dots, c_p \in W(n)$ be a basis of $W(n)$. Suppose that

$$p = \dim(W(n)) > \dim Q(2^{m+1} - n) \dim V(2^{m+1}) = d \dim V(2^{m+1}).$$

Define a K-linear function

$$f\colon W(n) \to (H(2^{m+1})/U(2^{m+1}))^d \cong V(2^{m+1})^d$$

by setting

$$f(c) := ((v_1 c + U(2^{m+1})), (v_2 c + U(2^{m+1})), \dots, (v_d c + U(2^{m+1})))$$

for each $c \in W(n)$.

Observe that $\dim(\operatorname{Im} f) \leqslant \dim(H(2^{m+1})/U(2^{m+1}))^d = d \dim V(2^{m+1})$, and that since $\dim W(n) = p > d \dim V(2^{m+1})$, there must exist some non-zero $c \in \ker f$. However, if $(v_i c + U(2^{m+1})) = 0$ for each v_i, then $Q(2^{m+1} - n)c \in U(2^{m+1})$ and $c \in S(n)$. Hence, $c \in S(n) \cap W(n) = \{0\}$: a contradiction. Thus,

$$\dim W(n) = p \leqslant \dim Q(2^{m+1} - n) \dim V(2^{m+1}),$$

as required.

The second inequality can be proven by a similar argument. □

Theorem 8.2. *Let j be a natural number. Write j in binary form as*

$$j = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$$

with $0 \leqslant p_0 < p_1 < \cdots < p_n$. Recalling the sets $\{S_i\}_{i \in Z}$ from § 4, suppose that there is an $m \in Z$ with $p_0, \dots, p_n \in S_m$. Then $\dim Q(j) \leqslant 2\sqrt{j}\lfloor \log j \rfloor$ and $\dim W(j) \leqslant 2\sqrt{j}\lfloor \log j \rfloor$.

Proof. This proof will be divided into three cases.

Case 1 ($j < 2^{2^m - \lfloor \log m \rfloor - 1}$). Suppose that $j < 2^{2^m - \lfloor \log m \rfloor - 1}$. Then $p_n < 2^m - \lfloor \log(m) \rfloor - 1$. Notice that $j \geqslant 2^{2^m - \lfloor \log m \rfloor - m}$ and

$$\begin{aligned}\sqrt{j}\lfloor \log j \rfloor &\geqslant 2^{2^{m-1} - \lfloor \log m \rfloor/2 - m/2}(2^m - \lfloor \log m \rfloor - m) \\ &> 2^{2^{m-1} - \lfloor \log m \rfloor/2 - m/2} 2^{m-1} > 2^{2^{m-1}}\end{aligned}$$

Hence, by using Theorem 7.3, we obtain

$$\dim Q(j) \leqslant \dim \prod_{i=0}^{n} V(2^{p_i}) \leqslant \prod_{i=0}^{m-2} 2^{2^i} < 2^{2^{m-1}} < \sqrt{j}\lfloor \log j \rfloor,$$

as required.

A similar argument, using Theorem 7.1, gives $\dim W(j) \leqslant \sqrt{j}\lfloor \log j \rfloor$.

Case 2 ($j = 2^{2^m - \lfloor \log m \rfloor - 1}$). Suppose that $j = 2^{2^m - \lfloor \log m \rfloor - 1}$. Then, by definition, $U(j) \subseteq R(j) \cap S(j)$, and so $\dim Q(j), \dim W(j) \leqslant \dim V(2^{2^m - \lfloor \log m \rfloor - 1}) = 2^{2^{m-1}}$, by Theorem 4.1 (ii). Consequently, $\dim Q(j), \dim W(j) \leqslant \sqrt{j}\lfloor \log j \rfloor$.

Case 3 ($j > 2^{2^m - \lfloor \log m \rfloor - 1}$). Suppose that $j > 2^{2^m - \lfloor \log m \rfloor - 1}$. Then $p_n = 2^m - \lfloor \log m \rfloor - 1$ and $2^{p_n+1} - j < 2^{2^m - \lfloor \log m \rfloor - 1}$. Set $k := 2^{p_n+1} - j$, and note that $k < j$ and that case 1 applies to k. Thus, an application of case 1 gives

$$\dim Q(k), \dim W(k) \leqslant \sqrt{k}\lfloor \log k \rfloor < \sqrt{j}\lfloor \log j \rfloor.$$

Now, apply Theorem 8.1 to see that

$$\dim Q(j) \leqslant \dim W(2^{p_n+1} - j) \dim V(2^{p_n+1}) = 2 \dim W(k) \leqslant 2\sqrt{k}\lfloor \log k \rfloor < 2\sqrt{j}\lfloor \log j \rfloor,$$

as required.

A similar argument shows that $\dim W(j) \leqslant 2\sqrt{j}\lfloor \log j \rfloor$ in this case.

This finishes the three cases and thus also the proof. □

Now, for each $m \in Z$, define $T_m \subset \mathbb{N}$ to be the set bounded above by S_m and below by $S_{m'}$, where m' is the next lowest value in Z (or by 0, if m is the lowest value of Z). More formally, if $m', m \in Z$ with $m' < m$ and $(m', m) \cap Z = \emptyset$, then set $T_m = [2^{m'} - \lfloor \log m' \rfloor, 2^m - m - \lfloor \log m \rfloor - 1]$. If m is the minimal value of Z, then set $T_m = [1, 2^m - m - \lfloor \log m \rfloor - 1]$. The subsets $\{S_m, T_m\}_{m \in Z}$ provide a partition of $\mathbb{N}$.

Theorem 8.3. *Let j be a natural number. Write j in binary form as*

$$j = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$$

with $0 \leqslant p_0 < p_1 < \cdots < p_n$. If there exists an $m \in Z$ such that $p_0, \ldots, p_n \in T_m$, then $\dim Q(j), \dim W(j) \leqslant 2$.

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Proof. Note that

$$\dim Q(j) \leqslant \dim\left(\prod_{i=0}^{n} N(2^{p_i})\right) = 1,$$

by Theorem 7.3, because $p_0, \dots, p_n \notin S$.

For the $W(j)$ case, note that $2^{p_n} \leqslant j < 2^{p_n+1}$ and let $2^{q_0} + \dots + 2^{q_n}$ be the binary form of $2^{p_n+1} - j$. As $p_0 = q_0 < \dots < q_n < p_n$, it follows that $q_0, \dots, q_n \in T_m$ and so $q_0, \dots, q_n \notin S$. Applying Theorem 7.3 in this case gives $\dim Q(2^{p_n+1} - j) \leqslant 1$, and then applying Theorem 8.1 gives

$$W(j) \leqslant Q(2^{p_n+1} - j)V(2^{p_n+1}) \leqslant 2,$$

as required. □

We can now establish the main estimate of this section.

Theorem 8.4. *For each $n > 1$,*

$$\dim Q(n), \dim W(n) \leqslant 8\sqrt{n}(\log n)^3.$$

Proof. Let $n = 2^{p_0} + 2^{p_1} + \cdots$ be the binary decomposition of n. For each $m \in Z$, let j_m be the sum of all the 2^{p_i} that occur in the binary form of n with $p_i \in S_m$ and let k_m be the sum of each 2^{p_i} with $p_i \in T_m$. Then

$$n = \sum_{\substack{m \in Z \\ j_m \neq 0}} j_m + \sum_{\substack{m \in Z \\ k_m \neq 0}} k_m,$$

as $\{S_m, T_m\}_{m \in Z}$ forms a partition of $\mathbb{N}$.

Therefore,

$$\dim Q(n) \leqslant \prod_{\substack{m \in Z \\ j_m \neq 0}} \dim Q(j_m) \prod_{\substack{m \in Z \\ k_m \neq 0}} \dim Q(k_m),$$

by Theorem 7.4.

We estimate the two terms on the right-hand side separately.

Firstly, suppose that $m < r$ are consecutive members of Z with $k_r \neq 0$. Then $2^{2^m - \lfloor \log m \rfloor} \leqslant k_r \leqslant n$, as $T_r = [2^m - \lfloor \log m \rfloor, 2^r - r - \lfloor \log r \rfloor - 1]$. It follows that $m \leqslant \log\log(n) + 1$ in this case. Therefore, the number of $m \in Z$ with $k_m \neq 0$ is less than or equal to $\log\log(n) + 2$. Note that $\dim Q(k_m) \leqslant 2$, for each such k_m, by Theorem 8.3, so that

$$\prod_{\substack{m \in Z \\ k_m \neq 0}} \dim Q(k_m) \leqslant \prod_{i=1}^{\lfloor \log\log(n)+2 \rfloor} 2 \leqslant 2^{\log\log(n)+2} \leqslant 4 \log n.$$

Secondly, observe that if $j_m \neq 0$, then $2^{2^m - m - \lfloor \log m \rfloor} \leqslant j_m \leqslant n$ and $j_m < 2^{2^m}$, because $S_m = [2^m - m - \lfloor \log m \rfloor, 2^m - \lfloor \log m \rfloor - 1]$. Also, observe that $\dim Q(j_m) \leqslant 2\sqrt{j_m}\lfloor \log j_m \rfloor \leqslant j_m$, by Theorem 8.2.

Suppose that $t < r$ are consecutive members of Z such that r is the largest member of Z and such that $j_r \neq 0$. Note that $2^{2^{2^{2t}}} < r$ by Lemma 3.1; therefore $2^{2^{2t}} < \log r \leqslant \log n$.

Consider any $m \in Z$ with $m \leqslant t$ and $j_m \neq 0$. Any p_i involved in the sum j_m satisfies

$$p_i \leqslant 2^m - \lfloor \log m \rfloor - 1 \leqslant 2^m \leqslant 2^t.$$

As each p_i can be involved in at most one such sum j_m, the number of $m \in Z$ with $j_m \neq 0$ is less than or equal to 2^t. For any such m,

$$\dim Q(j_m) \leqslant j_m \leqslant 2^{2^m} \leqslant 2^{2^t}.$$

Thus,

$$\prod_{\substack{m\in Z\\ k_m\neq 0\\ m<r}} \dim Q(k_m) \leqslant (2^{2^t})^{2^t} = 2^{2^{2t}} \leqslant \log r \leqslant \log n.$$

Hence,

$$\prod_{\substack{m\in Z\\ k_m\neq 0}} \dim Q(k_m) \leqslant (\log n)(2\sqrt{j_r}\log j_r) \leqslant 2\sqrt{n}(\log n)^2,$$

and so

$$\begin{aligned}\dim Q(n) &\leqslant \prod_{\substack{m\in Z\\ j_m\neq 0}} \dim Q(j_m) \prod_{\substack{m\in Z\\ k_m\neq 0}} \dim Q(k_m)\\ &\leqslant (4\log n)(2\sqrt{n}(\log n)^2)\\ &= 8\sqrt{n}(\log n)^3,\end{aligned}$$

as required.

To show that $W(n) \leqslant 8\sqrt{n}(\log n)^3$ we use an analogous argument. □

Now we are ready to obtain the main result of the paper.

Theorem 8.5. *The algebra $\bar{E}/A$ is a finitely generated infinite-dimensional nil algebra with Gelfand–Kirillov dimension at most* 3.

Proof. The algebra $\bar{E}/A$ is a finitely generated infinite-dimensional nil algebra, by Theorem 5.1.

By combining Theorem 8.4 with Theorem 6.1, we obtain

$$\begin{aligned}\frac{\dim H(n)}{E(n)} &\leqslant \sum_{k=0}^{n} \dim(W(n-k))\dim(Q(k))\\ &\leqslant \sum_{k=0}^{n} 64\sqrt{(n-k)k}(\log(n-k)\log k)^3\\ &< 64n^2(\log n)^6.\end{aligned}$$

Hence,

$$\sum_{i=1}^{n} \dim\left(\frac{H(i)}{E(i)}\right) \leqslant 64n^3(\log n)^6.$$

Therefore,

$$\begin{aligned}\mathrm{GKdim}(\bar{A}/E) &= \overline{\lim_{n\to\infty}} \left(\frac{\log(\sum_{i=1}^{n} \dim H(i)/E(i))}{\log n}\right)\\ &\leqslant \overline{\lim_{n\to\infty}} \left(\frac{6+3(\log n)+6(\log\log n)}{\log n}\right)\\ &= 3,\end{aligned}$$

as required. □

9. Concluding remarks and some questions

We have constructed a finitely generated infinite-dimensional nil algebra with Gelfand–Kirillov dimension at most 3. Equivalently, we have a finitely generated infinite-dimensional nil-but-not-nilpotent algebra with Gelfand–Kirillov dimension at most 3.

In contrast, nil does imply nilpotent for algebras of Gelfand–Kirillov dimension at most 1, by [**6**]. Combining this with Bergman's Gap Theorem [**4**, Theorem 2.5], we see that a nil-but-not-nilpotent example must have Gelfand–Kirillov dimension at least 2. It would be very interesting to find the precise dividing line in terms of growth. A starting point might be to consider nil algebras with quadratically bounded growth and attempt to show that these algebras must be finite dimensional. Given a positive result in this direction, one might then speculate whether there exists a finitely generated nil-but-not-nilpotent algebra with Gelfand–Kirillov dimension 2 (but, of course, not having quadratic growth).

Many of the constructions of weird algebras that we know involve starting with a free algebra and introducing infinitely many relations, so the corresponding questions for finitely presented algebras remain unresolved. In particular, we ask: is every finitely presented nil algebra nilpotent?

It seems unlikely that by using the methods employed in this work we can hope to construct a nil-but-not-nilpotent algebra with Gelfand–Kirillov dimension 2. Our algebras are graded, and this raises the question of whether a finitely generated nil algebra that is graded and has Gelfand–Kirillov dimension at most 2 (or quadratic growth) must in fact be finite dimensional.

The methods employed here depend crucially on the countability hypothesis. It would be interesting to see if it is possible to construct a finitely generated infinite-dimensional nil algebra with finite Gelfand–Kirillov dimension over an uncountable field.

There are many problems of a similar type in [**10**].

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