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SIGNATURES OF COVERING LINKS

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1. Introduction. A (tame) knot k_n in S^3 is said to have *period* n if there exists a homeomorphism $\phi: S^3 \to S^3$, necessarily orientation-preserving, such that

(i) the fixed point set of ϕ is a circle disjoint from k_n;
(ii) $\phi(k_n) = k_n$;
(iii) ϕ has order n.

Several necessary conditions for a knot to have period n have already been established in the literature; see [**3**] [**11**] [**14**] [**18**]. Here we establish the following further condition, involving the signature $\sigma(k_n)$ of k_n.

THEOREM 1.1. *Let k_n be a knot with period $n = p^r$, where p is an odd prime, and suppose that the Alexander polynomial $\Delta(t)$ of k_n satisfies*
(1) *$\Delta(t)$ is not a product of non-trivial knot polynomials, and*
(2) $\Delta(t) \not\equiv 1 \pmod{p}$.
Then
(i) $\Delta(t) \equiv (1 + t + \ldots + t^{\lambda-1})^{n-1} \pmod{p}$, *for some* λ, *and*
(ii) $\sigma(k_n) \equiv 0 \pmod 4$, λ *odd*;
$n - 1 \pmod 4$, λ *even*.

Assertion (i) is Corollary 2 of [**14**]; assertion (ii) is a consequence of the main result of the present paper, which we now describe.

We first define a signature invariant $\tau_m(l)$ for a null-homologous link l in an oriented 3-manifold M, together with a specified m-fold branched cyclic cover, in terms of an appropriate m-fold branched cyclic cover of a 4-manifold bounded by M. For $m = 2$, this is just the classical signature of l. Next, we consider the situation in which we have two disjoint null-homologous links k, f in an oriented 3-manifold M, an m-fold cyclic cover of M branched along k, and an n-fold cyclic cover of M branched along f. Let f_m be the inverse image of f in the first cover, and k_n the inverse image of k in the second. Suppose also that these branched covers are induced by infinite cyclic covers of $M - k$ and $M - f$ respectively. Our main result is then

THEOREM 3.1. *In the above situation*

$$\tau_m(k_n) - n\tau_m(k) = \tau_n(f_m) - m\tau_n(f).$$

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Theorem 1.1 (ii) follows from this by taking M to be the quotient of S^3 by the periodic homeomorphism ϕ, k the image in M of k_n, f the image of the fixed-point set of ϕ, and $m = 2$.

§ 2 contains the definition of $\tau_m(l)$ and its interpretation in terms of a Seifert matrix for l. In § 3 we prove Theorem 3.1. The approach here is similar to that taken for the case $m = n = 2$ in [**6**], where the main result of [**13**] was reproved in a more general context. Theorem 1.1 (ii) is derived from Theorem 3.1 in § 4, and in § 5 a further application of Theorem 3.1 is given, to signatures of torus links. Finally, in view of the fact that some attention has been directed towards determining which periods can occur for the knots in the classical knot tables, we show in § 6 that 2-bridge knots which are not torus knots have only period 2. This is independent of the earlier sections.

Throughout the paper, we shall be implicitly working in the smooth category, and all manifolds, including those of dimensions 1 and 2, will be oriented. Homology will be with integer coefficients unless otherwise specified. The linking number of disjoint, null-homologous 1-cycles x, y in a 3-manifold will be denoted by $\mathrm{Lk}(x, y)$. We shall use $\cdot$ to denote either algebraic intersection number of homology classes or geometric intersection number of chains, as appropriate. If x is a homology class, we shall write x^2 for $x \cdot x$.

2. Signatures of links. Let l be a null-homologous link in a closed 3-manifold M, and $\pi: M_m \to M$ an m-fold branched cyclic cover with branch set l. We shall always assume that each oriented meridian of l corresponds to a fixed generator of the group of covering transformations. Let F be a surface properly embedded in a 4-manifold N with $\partial(N, F) = (M, l)$, and suppose π extends to a covering $N_m \to N$ branched along F. Then

$$\tau_m(l, \pi) = \sigma(N_m) - m\sigma(N) + \frac{(m^2 - 1)}{3m}[F, \partial F]^2$$

depends only on l and π. (If (N, F) and (N', F') are two pairs as above, apply the G-signature theorem [**1**] to the resulting $\mathbf{Z}/m$-action on the closed 4-manifold $N_m \cup_\partial (-N_m')$, together with Novikov additivity.) Note that $[F, \partial F]^2$ is well-defined since $[F, \partial F]$ is in the image of $H_2(N) \to H_2(N, \partial N)$. Also, the existence of an m-fold cyclic covering branched over F implies that m^2 divides $[F, \partial F]^2$, so $\tau_m(l, \pi)$ is an integer.

To avoid over-burdening the notation, we shall usually abbreviate $\tau_m(l, \pi)$ to $\tau_m(l)$.

Suppose now that M is a homology sphere, and let V be a Seifert matrix for l, corresponding to a spanning surface F, say. Let ξ be a primitive m-th root of unity, and write $\sigma_{\xi^i}(V)$ for the signature of the hermitian matrix $[1/(1 - \xi^i)](V - \xi^i V^T)$ (see [**9**], [**17**], [**19**]). By considering the m-fold cyclic cover W of $M \times I$ branched along a pushed-in copy of $F \times \{0\}$, Viro shows in [**19**, Theorem 4.4, § 4.8] that

$$\tau_m(l) = \sum_{i=1}^{m-1} \sigma_{\xi^i}(V).$$

The proof proceeds by decomposing $H_2(W; \mathbf{C})$ into its ξ^i-eigenspaces E_i, and showing that on E_i the intersection form is given by the above hermitian matrix.

This analysis of the intersection form of W can also be used to show that, if we take F to be connected, then M_m is a rational homology sphere if and only if

$$\prod_{i=1}^{m-1} \det (V - \xi^i V^T) \neq 0.$$

(In fact,

$$\dim H_1(M_m; \mathbf{Q}) = \sum_{i=1}^{m-1} \text{nullity } (V - \xi^i V^T),$$

and, if this is zero,

$$\text{order } H_1(M_m) = \prod_{i=1}^{m-1} \det (V - \xi^i V^T).$$

Formulae equivalent to these are obtained in [**8**] when $M = S^3$, but from a different point of view.)

Now consider the case when l has a single component and m is a prime-power p^r. Then $\det (V - \xi^i V^T) \neq 0$ (otherwise, some cyclotomic polynomial $\phi_{p^s}(t)$, $0 < s \leqq r$, would divide $\Delta(t) = \det (V - tV^T)$, contradicting $\phi_{p^s}(1) = p$, $\Delta(1) = \pm 1$). This shows that the rank of $V - \xi^i V^T$ is even, and hence that $\tau_m(l)$ is even.

Although we shall not use this in the sequel, we remark that the above interpretation of $\tau_m(l)$ in terms of a Seifert matrix, when M is a homology sphere, generalizes as follows. Let l be a link in a closed 3-manifold M and let π_∞ be an infinite cyclic covering of $M - l$ such that each oriented meridian of l corresponds to a fixed generator of the group of covering transformations. Associated with π_∞ is a homotopy class of maps $M - l \rightarrow S^1$; transversality then yields a surface $F \subset M$ with $\partial F = l$ such that the epimorphism $H_1(M - l) \rightarrow \mathbf{Z}$ which determines π_∞ is given by intersection number with F. We shall say that F is a *spanning surface for* (l, π_∞). Let $i^+ : F \rightarrow M - F$ be an embedding given by a small translation in the positive normal direction. Let

$$K_1(F) = \ker (H_1(F; \mathbf{Q}) \rightarrow H_1(M; \mathbf{Q})).$$

Define a pairing

$$\theta: K_1(F) \times K_1(F) \rightarrow \mathbf{Q}$$

by

$$\theta(\alpha, \beta) = \mathrm{Lk}(\alpha, i_*{}^+(\beta)),$$

and let V be a matrix representing θ with respect to some basis of $K_1(F)$. Now let π: $M_m \to M$ be the m-fold cyclic cover of M branched along l which is induced by π_∞. We then have

$$\tau_m(l, \pi) = \sum_{i=1}^{m-1} \sigma_{\xi^i}(V).$$

The proof is almost exactly the same as that of [**19**, Theorem 4.4].

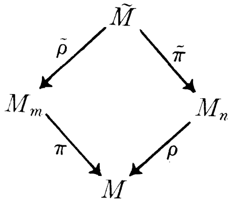
3. The main theorem. Let k and f be disjoint null-homologous links in a closed 3-manifold M. Let π_∞ (ρ_∞) be an infinite cyclic covering of $M - k$ ($M - f$) such that each oriented meridian of k(f) corresponds to a fixed generator of the group of covering transformations. Let π: $M_m \to M$ (ρ: $M_n \to M$) be the m-fold (n-fold) branched cyclic cover with branch set k(f) corresponding to $\pi_\infty(\rho_\infty)$. Let $f_m = \pi^{-1}(f)$, and $k_n = \rho^{-1}(k)$. These are null-homologous links in M_m and M_n respectively. The coverings π and ρ correspond to epimorphisms

$$\alpha: H_1(M - k) \to \mathbf{Z}/m \quad \text{and} \quad \beta: H_1(M - f) \to \mathbf{Z}/n.$$

Hence we obtain an epimorphism

$$H_1(M - (k \cup f)) \to \mathbf{Z}/m \times \mathbf{Z}/n,$$

and a corresponding regular branched covering $\tilde{M} \to M$, with branch set $k \cup f$. This gives rise to a commutative diagram of branched coverings



where $\tilde{\pi}(\tilde{\rho})$ is an m-fold (n-fold) covering branched along $k_n(f_m)$.

THEOREM 3.1. *In the above situation*

$$\tau_m(k_n) - n\tau_m(k) = \tau_n(f_m) - m\tau_n(f).$$

Proof. First suppose there exists a 4-manifold N and disjoint surfaces K, F in N with $\partial(N, K, F) = (M, k, f)$, and also homomorphisms

$$\alpha': H_1(N - K) \to \mathbf{Z}/m \quad \text{and} \quad \beta': H_1(N - F) \to \mathbf{Z}/n$$

such that $\alpha = \alpha' i_*$, $\beta = \beta' j_*$, where i: $M - k \to N - K$, j: $M - f \to$

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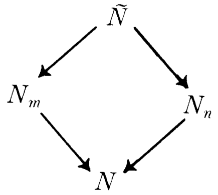
$N - F$ are inclusions. Then the diagram above bounds a corresponding diagram of 4-manifolds



where $N_m \to N$, $N_n \to N$, $\tilde{N} \to N_n$ and $\tilde{N} \to N_m$ are branched covers along K, F, K_n (the inverse image of K in N_n) and F_m (the inverse image of F in N_m) respectively.

We then have

$$\tau_m(k_n) = \sigma(\tilde{N}) - m\sigma(N_n) + \frac{(m^2 - 1)}{3m}[K_n, \partial K_n]^2$$

$$\tau_m(k) = \sigma(N_m) - m\sigma(N) + \frac{(m^2 - 1)}{3m}[K, \partial K]^2$$

$$\tau_n(f_m) = \sigma(\tilde{N}) - n\sigma(N_m) + \frac{(n^2 - 1)}{3n}[F_m, \partial F_m]^2$$

$$\tau_n(f) = \sigma(N_n) - n\sigma(N) + \frac{(n^2 - 1)}{3n}[F, \partial F]^2.$$

The result in this case follows readily from these equations and

LEMMA 3.2.

$$[K_n, \partial K_n]^2 = n[K, \partial K]^2$$
$$[F_m, \partial F_m]^2 = m[F, \partial F]^2.$$

Proof. It suffices to consider K. Now $[K, \partial K]^2$ can be computed as follows. Deform K slightly to K' so that $K \cap K' = \text{int}(K) \cap \text{int}(K')$ consists of a finite number of points of transverse intersection, and

$$\text{Lk}(\partial K, \partial K') = 0.$$

Then $[K, \partial K]^2 = K \cdot K'$. Now if K_n' is the inverse image of K' in N_n, one sees easily that $\text{Lk}(\partial K_n, \partial K_n') = 0$, and so

$$[K_n, \partial K_n]^2 = K_n \cdot K_n'.$$

But each point of $K \cap K'$, of sign $\epsilon = \pm 1$, gives rise to n points of $K_n \cap K_n'$, each of sign ϵ, so $K_n \cdot K_n' = nK \cdot K'$.

Next we prove

LEMMA 3.3. *If* Lk (k, f) *is divisible by* mn *then there exist* N, K, F, α', β' *as above.*

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Proof. Let $\bar{K}$, $\bar{F}$ be spanning surfaces for (k, π_∞) and (f, ρ_∞) respectively (see § 2). Let N' be any 4-manifold with $\partial N' = M$. It is easy to see that if K' is a surface in N' with $\partial K' = k$, there is a homomorphism

$$\alpha' \colon H_1(N' - K') \rightarrow \mathbf{Z}/m$$

with $\alpha = \alpha' i_*$ if and only if $[K', \partial K'] = [\bar{K}, \partial \bar{K}]$ in $H_2(N', k; \mathbf{Z}/m)$, and similarly for f. Thus if we obtain K' and F' by pushing int $(\bar{K})$ and int $(\bar{F})$ into int (N'), all the conditions will be met except that possibly $K' \cap F' \neq \emptyset$. We will modify N', K', F' so as to obtain the desired N, K, F.

We may assume that $K' \cap F' = \text{int}\,(K') \cap \text{int}\,(F')$ consists of a finite number of transverse intersection points. Now

$$[K', \partial K'] \cdot [F', \partial F'] = K' \cdot F' - \text{Lk}\,(k, f);$$

hence $K' \cdot F' \equiv 0 \pmod{mn}$. Next, we can easily find surfaces A, B in $\mathbf{C}P^2$ representing m and n times a generator of $H_2(CP^2)$, respectively, such that $A \cdot B = mn$. By taking the connected sum of N' with the appropriate number of copies of $\pm\mathbf{C}P^2$ and piping the copies of $A\,(B)$ onto $K'(F')$, it follows that we may assume $K' \cdot F' = 0$. Because $A\,(B)$ is null-homologous mod $m\,(n)$, α' and β' will still exist. Now we can add a tube to (say) K' along a path in F' joining two oppositely signed points of $K' \cap F'$ to reduce the number of intersection points; this does not affect the homology class $[K', \partial K']$. Iterating the process we finally obtain disjoint K and F as required.

To complete the proof of Theorem 3.1, observe that by taking the disjoint union of mn copies of M we can satisfy the hypothesis of Lemma 3.3. Since the signatures are additive under disjoint union, the general result follows.

4. Periodic knots.

Proof of Theorem 1.1. Let k_n be a knot in S^3 with prime-power period n, and let M be the quotient of S^3 under the corresponding periodic homeomorphism ϕ. In particular, M is a homotopy sphere. Let k denote the image in M of k_n, and f the image of the fixed-point set of ϕ. Let M_2 be the 2-fold branched cover of M with branch set k, and f_2 the inverse image of f in M_2.

By Theorem 1 of [**14**],

$$\Delta(t) = D(t) \prod_{i=1}^{n-1} D(t, \xi^i)$$

where $\Delta(t)$ is the Alexander polynomial of k_n, $D(t)$ that of k, $D(t, u)$ the reduced Alexander polynomial of $k \cup f$, and ξ a primitive n-th root of unity. Since, by assumption (1), $\Delta(t)$ is not a product of non-trivial

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Alexander polynomials, we have either

(i) $\Delta(t) = \mathrm{D}(t)$ or
(ii) $\mathrm{D}(t) = 1$.

But (i) implies $\Delta(t) \equiv 1 \pmod p$ (see [**14**, proof of Corollary 2]), contrary to our assumption (2). Therefore $D(t) = 1$.

By Theorem 3.1,

$$\sigma(k_n) - n\sigma(k) = \tau_n(f_2) - 2\tau_n(f).$$

Moreover, as noted in § 2, $\tau_n(f)$ is even, and since $D(t) = 1$, $\sigma(k) = 0$. Hence

(4.1) $\sigma(k_n) \equiv \tau_n(f_2) \pmod 4$.

Since $D(t) = 1$, M_2 is a homology sphere; let V be a Seifert matrix for f_2 corresponding to a connected spanning surface. Then (see § 2)

$$\tau_n(f_2) = \sum_{i=1}^{n-1} \sigma_{\xi^i}(V).$$

We claim that $\det(V - \xi^i V^T) \neq 0$. If λ is odd, so that f_2 is a knot, this follows from the fact that n is a prime power (see § 2). But in any case, the n-fold cyclic cover $\tilde{M}$ of M_2 branched along f_2 is also the 2-fold cover of S^3 branched along the knot k_n. Hence $\tilde{M}$ is a rational homology sphere, and so

$$\det(V - \xi^i V^T) \neq 0$$

also holds if λ is even (again, see § 2). The rank of $V - \xi^i V^T$ is therefore even or odd according as f_2 has one or two components, giving

$$\sigma_{\xi^i}(V) \equiv \begin{cases} 0 \pmod 2, & \lambda \text{ odd} \\ 1 \pmod 2, & \lambda \text{ even.} \end{cases}$$

Also, since $\xi^{n-i} = \bar{\xi}^i$, $\sigma_{\xi^i}(V) = \sigma_{\xi^{n-i}}(V)$. Hence, if n is odd,

$$\tau_n(f_2) = 2\sum_{i=1}^{(n-1)/2} \sigma_{\xi^i}(V) \equiv \begin{cases} 0 \pmod 4, & \lambda \text{ odd} \\ n-1 \pmod 4, & \lambda \text{ even.} \end{cases}$$

Together with (4.1), this proves the assertion (ii) of Theorem 1.1.

As an example, let K be the knot 9_{46} of Reidemeister's table. Then $\Delta(t) = -2 + 5t - 2t^2$ and $\sigma(K) = 0$. Since $\Delta(t) \equiv (1+t)^2 \pmod 3$, it follows from Theorem 1.1 (ii) that K cannot have period 3. (All previously known conditions fail to rule out period 3 [**3**] [**11**] [**14**] [**18**]; also K is not a 2-bridge knot so Theorem 6.1 does not apply.)

5. Signatures of torus links. Let $K_{n,q}$ denote the torus knot or link of type (n, q). Since $K_{n,-q}$, $K_{-n,q}$ and $-K_{n,q}$ are all of the same link type, we need only consider $n, q > 0$. In the case of a link we assume that the orientations of the components are such that when $K_{n,q}$ is represented on the boundary of an unknotted solid torus V, each component represents the same element of $H_1(V)$. Let $\tau_m(n, q)$ denote $\tau_m(K_{n,q})$.

THEOREM 5.1. $\tau_m(n,q) = \tau_n(m, q)$.

Proof. In Theorem 3.1, take $M = S^3$. Let V be an unknotted solid torus in S^3, let k be a simple closed curve on ∂V going q times meridionally and once longitudinally about V, and let f be a core of V. Since f and k are both unknotted, $M_m \cong M_n \cong S^3$. Also $k_n(f_m)$ is $K_{n,q}$ $(K_{m,q})$, so the result follows from Theorem 3.1.

In this theorem, we must have $m, n > 0$ since only in this case have we defined τ_m, τ_n. However, the result also holds, with the same proof, if $q < 0$, and trivially if $q = 0$.

It is well-known that the m-fold cyclic cover of S^3 branched along $K_{n,q}$ is diffeomorphic to the intersection of the complex algebraic variety

$$\{(z_1, z_2, z_3) | z_1^m + z_2^n + z_3^q = 0\}$$

with the unit sphere in $\mathbf{C}^3$; see, for example, [**12**, Lemma 1.1]. In fact, the intersection of

$$\{(z_1, z_2, z_3) | z_1^m + z_2^n + z_3^q = \delta\}$$

with the unit ball is, for small non-zero $|\delta|$, an m-fold branched cyclic cover of D^4, with branch set spanning $K_{n,q}$ [**5**, Lemma 2] [**10**, Lemma 5.1]. This gives an alternative proof of Theorem 5.1.

In [**2**] Brieskorn calculates the signatures of the varieties referred to above. Thus one can obtain a formula for $\tau_m(n, q)$ similar to that given by Hirzebruch [**7**] for the case $m = 2$, n, q odd and coprime. This is somewhat unwieldy; below we give reduction formulae for the classical signature $\sigma(n, q) = \tau_2(n, q)$ which permit swift computation.

THEOREM 5.2. *Let* $n, q > 0$. (I) *Suppose* $2q < n$:
If q is odd, $\sigma(n, q) = \sigma(n - 2q, q) + q^2 - 1$.
If q is even, $\sigma(n, q) = \sigma(n - 2q, q) + q^2$.
(II) $\sigma(2q, q) = q^2 - 1$.
(III) *Suppose* $q \leqq n < 2q$.
If q is odd, $\sigma(n, q) + \sigma(2q - n, q) = q^2 - 1$.
If q is even, $\sigma(n, q) + \sigma(2q - n, q) = q^2 - 2$.
(IV) $\sigma(n, q) = \sigma(q, n)$, $\sigma(n, 1) = 0$, $\sigma(n, 2) = n - 1$.

The values for $\sigma(n, 1)$ and $\sigma(n, 2)$ follow from the rest of the theorem, but they do shorten computations.

The formulae for $\sigma(n, 3)$ and $\sigma(n, 4)$ given by Murasugi [**15**, Propositions 9.1, 9.2] and for $\sigma(n, nk)$ given by Goldsmith [**5**, Lemma 2] can be derived from Theorem 5.2.

Theorem 5.2 may be proved either from a Hirzebruch-type formula for $\sigma(n, q)$, or by using the equation $\sigma(n, q) = \tau_n(2, q)$ and working with a Seifert matrix for $K_{2,q}$. Instead we shall make use of the following result, whose proof is more in keeping with the spirit of this paper.

Suppose V is an unknotted solid torus in S^3. Let l_0 be a link in int (V), and let λ be the non-negative integer such that l_0 generates $\lambda H_1(V)$. Let h_t: $V \to V$ be the homeomorphism given by twisting t times about a meridian disc (where t is an integer), and set $l_t = h_t(l_0)$.

THEOREM 5.3. *In the above situation, if m is a positive integer dividing t and if $n =$ h.c.f. (m, λ), we have*

$$\tau_m(l_t) - \tau_n(l_t) = \tau_m(l_0) - \tau_n(l_0) + \frac{t(m^2 - n^2)}{3m} - \frac{\lambda^2 t(m-n)(mn+1)}{3mn}.$$

A corresponding result for eigenspace signatures (from which the present theorem may be deduced) is given by Litherland [**10**, Corollary 6.2], with essentially the same proof.

LEMMA 5.4. *Let V be an unknotted solid torus in S^3, t an integer, and m a positive integer dividing t. Suppose c is a core of V. Then there exists a 4-manifold N and an m-fold branched cyclic cover $N_m' \to N$, with branch set F, such that*

(i) $\partial(N, F) = \alpha(S^3, c)$ *for some integer* $\alpha > 0$;

(ii) *each component F_1 of F is a 2-disc, and*

$$[F_i, \partial F_i]^2 = -t \quad (i = 1, \dots, \alpha).$$

Proof. Let $L(p, q)$ denote the lens space of type (p, q). Since $m|t$, there is an (unbranched) m-fold cyclic covering $L(t/m, 1) \to L(t, 1)$. Since Ω_3 $(K(\mathbf{Z}/m, 1))$ is finite, there is an m-fold cyclic covering $W_m{}^4 \to W^4$ such that

$$\partial(W_m \to W) = \alpha(L(t/m, 1) \to L(t, 1))$$

for some integer $\alpha > 0$. We may construct N by adding 2-handles to W, one to each boundary component, and F is the union of the co-cores of these handles.

One could also give an explicit construction of $W_m \to W$, as follows. It suffices to consider the case $m = t$. There is a t-fold branched cyclic cover $\Sigma \to S^2$, branched over t points, and a generator x of the covering transformation group such that each fixed point has a neighbourhood $\{z \in \mathbf{C}: \|z\| \leqq 1\}$ on which x acts by multiplication by $e^{2\pi i/t}$. (Here Σ is the closed surface of genus $\frac{1}{2}t(t-3) + 1$.) A semi-free action of $\mathbf{Z}/t$ on

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$\Sigma \times \Sigma$ is generated by $x \times x$. Let W_t be $\Sigma \times \Sigma$ with an invariant open ball about each fixed point deleted, and set $W = W_t/(\mathbf{Z}/t)$.

Before starting the proof of Theorem 5.3, we introduce some notation. Fix positive integers m and n with $n|m$. Let $f(X) = X^n - 1$, $g(X) = (X^m - 1)(X^n - 1)$. If V is a vector space over $\mathbf{Q}$ and x is an automorphism of V with $x^m = 1$, we have $V = V' \oplus V''$, where

$$V' = \operatorname{Im}(f(x)) = \operatorname{Ker}(g(x)) \quad \text{and} \quad V'' = \operatorname{Im}(g(x)) = \operatorname{Ker}(f(x)).$$

Now suppose $\pi_m: M_m \to M$ is an m-fold cyclic cover of the closed 3-manifold M, branched along the null-homologous link l. Extend π_m to a branched covering $N_m \to N$ of 4-manifolds, branched along F, say. Let $\pi_n: M_n \to M$ and $N_n \to N$ be the corresponding n-fold coverings. The canonical covering transformation of N_m induces an automorphism of $H = H_2(N_m; \mathbf{Q})$ of period m, so $H = H' \oplus H''$ as above. Moreover, this splitting is orthogonal with respect to the intersection form, so

$$\sigma(N_m) = \sigma'(N_m) + \sigma''(N_m),$$

where $\sigma'(N_m)$ $(\sigma''(N_m))$ is the signature of the restriction to $H'(H'')$ of the intersection form. By a standard transfer argument, $\sigma''(N_m) = \sigma(N_m)$, and so

$$\tau_m(l, \pi_m) - \tau_n(l, \pi_n) = \sigma'(N_m) - (m - n)\sigma(N) + \frac{(m-n)(mn+1)}{3mn}[F, \partial F]^2$$

Proof of Theorem 5.3. Let N, F be as provided by Lemma 5.4, and let M_i be the i-th component of ∂N. Let B_i be a tubular neighbourhood of F_i, and set

$$Y = \operatorname{cl}\left(N \setminus \bigcup_{i=1}^{\alpha} B_i\right).$$

Let $\phi_i: D^4 \to B_i$ be a homeomorphism such that $\phi_i(V) = B_i \cap \partial N$. Because $[F_i, \partial F_i]^2 = -t$,

$$(M_i, \phi_i(l_0)) \cong (S^3, l_t).$$

Let G_0 be a surface in D^4 spanning l_0, and set

$$G_i = \phi_i(G_0) \quad \text{and} \quad G = \bigcup_{i=1}^{\alpha} G_i.$$

Then $[G, \partial G] = \lambda[F, \partial F] = 0$ in $H_2(N, \partial N; \mathbf{Z}/m)$, so there is an m-fold cyclic covering $N_m \to N$ with branch set G. We have

$$\alpha[\tau_m(l_t) - \tau_n(l_t)] = \sigma'(N_m) - (m - n)\sigma(N) + \frac{(m-n)(mn+1)}{3mn}[G, \partial G]^2$$

and

$$[G, \partial G]^2 = \lambda^2[F, \partial F]^2 = -\lambda^2 \alpha t.$$

Now, a core of the solid torus $Y \cap B_i$ has linking number λ with G, and n divides λ; it follows that

$$H_j((Y \cap B_i)_m; \mathbf{Q})' = 0 \quad \text{for all } j.$$

Splitting each space in the Mayer–Vietoris sequence for

$$N_m = Y_m \cup \bigcup_{i=1}^{\alpha} (B_i)_m,$$

we obtain

$$H_2(N_m; \mathbf{Q})' \cong H_2(Y_m; \mathbf{Q})' \oplus \bigoplus_{i=1}^{\alpha} H_2((B_i)_m; \mathbf{Q})',$$

and so

$$\sigma'(N_m) = \sigma'(Y_m) + \sum_{i=1}^{\alpha} \sigma'((B_i)_m).$$

But $(B_i)_m$ is homeomorphic to the m-fold cover of D^4 branched along G_0, so

$$\sigma'((B_i)_m) = \tau_m(l_0) - \tau_n(l_0).$$

Thus

$$(5.1) \quad \tau_m(l_t) - \tau_n(l_t) = \tau_m(l_0) - \tau_n(l_0) + \frac{1}{\alpha}\sigma'(Y_m) - \frac{(m-n)}{\alpha}\sigma(N) - \frac{(m-n)(mn+1)}{3mn}\lambda^2 t.$$

In this last equation, take $l_0 = c$, the core of V, and replace m by m/n. Then λ and n are each replaced by 1; also l_0 and l_t are both trivial, so we obtain

$$0 = \frac{1}{\alpha}\sigma'(Y_{m/n}) - \frac{(m/n - 1)}{\alpha}\sigma(N) - \frac{(m/n-1)(m/n+1)}{3m/n} t.$$

Notice that $\sigma'(Y_{m/n})$ comes from the splitting of $H_2(Y_{m/n}; \mathbf{Q})$ by the polynomials $X - 1$ and $(X^{m/n} - 1)/(X - 1)$.

Now, since $\mathrm{Lk}(\gamma, G) = \lambda \mathrm{Lk}(\gamma, F)$ for any $\gamma \in H_1(Y)$ and $n =$ h.c.f. (m, λ), Y_m consists of n copies of $Y_{m/n}$. Moreover, if x_m and $x_{m/n}$ are the canonical covering transformations of Y_m and $Y_{m/n}$, respectively, then x_m permutes the copies of $Y_{m/n}$ cyclically, and on each copy $(x_m)^n = (x_{m/n})^{\lambda'}$, where $\lambda'\lambda/n \equiv 1 \pmod{m/n}$. Since $(x_{m/n})^{\lambda'}$ generates the group

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of covering transformations of $Y_{m/n}$, it follows that $\sigma'(Y_m) = n\sigma'(Y_{m/n})$, and so

$$\frac{1}{\alpha}\sigma'(Y_m) = \frac{m-n}{\alpha}\sigma(N) + \frac{(m^2-n^2)}{3m}t.$$

Together with (5.1), this gives the desired result.

Proof of Theorem 5.2. Part (IV) is well-known. To prove (I)–(III) we shall need the formula $\sigma(n, 2) = n - 1$ from (IV), and also the facts $\sigma(0, q) = 0$, $\sigma(-n, q) = -\sigma(n, q)$.

Now $K_{n,2}$ can be obtained from $K_{n-2q,2}$ by the construction of Theorem 5.3, with $\lambda = 2$, $m = q$ and $t = -q$. (Whether $t = \pm q$ depends on which of the two enantiomorphic forms of a given torus knot is chosen as $K_{n,q}$; $t = -q$ is consistent with (IV).) Hence, using also Theorem 5.1,

$$\begin{aligned} &\sigma(n, q) - \sigma(n-2q, q) \\ &= \tau_q(n, 2) - \tau_q(n-2q, 2) \\ &= \begin{cases} q^2 - 1 & q \text{ odd} \\ \tau_2(n, 2) - \tau_2(n-2q, 2) + q(q-2) & q \text{ even.} \end{cases} \end{aligned}$$

This is valid whether $n - 2q$ is positive, zero or negative; these three cases give (I), (II) and (III), respectively.

We remark that the cases where q is odd can also be obtained by applying Theorem 5.3 directly to relate $\sigma(n, q)$ and $\sigma(n - 2q, q)$; if q is even this yields no information.

6. Appendix on 2-bridge knots. We outline a proof of the following.

THEOREM 6.1. *Let k be a 2-bridge knot which is not a torus knot. Then k has period 2 and no other.*

This can also be derived from the fact (established by Thurston) that the complement of a 2-bridge knot is hyperbolic of finite volume. For this approach, see [**16**, § 5]. (But note that Lemma 3 of [**16**] fails for torus knots; in line 8 on p. 27, "inner automorphism" is confused with "identity automorphism". Compare the first part of the proof below.)

Proof. For any knot k, write $X = S^3 - k$ and $G = \pi_1(X)$. Let $\phi: X \to X$ be induced by a periodic automorphism of k of period n. We claim that, if k is not a torus knot, the image of $\phi_*: G \to G$ in the outer automorphism group Out (G) has order n. Replacing ϕ by an appropriate power of ϕ, if necessary, it suffices to show that if ϕ has prime period, then ϕ_* cannot be inner. But if ϕ_* is conjugation by g, say, then, since ϕ has finite order, some power of g lies in the centre of G, which is trivial. Thus

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$g = 1$ (since G is torsion free), and hence $\phi_* = \text{id}$, contradicting Conner's assertion [**4**, Theorem 4.2] that the fixed subgroup of ϕ_* is either trivial or infinite cyclic. (Conner deals explicitly only with involutions, but remarks that his methods apply to transformations of any prime period.)

From now on, let k be a 2-bridge knot which is not a torus knot. It has been shown by Conway (unpublished) that Out (G) is then isomorphic to either $\mathbf{Z}/2 \times \mathbf{Z}/2$ or an extension of $\mathbf{Z}/2 \times \mathbf{Z}/2$ by $\mathbf{Z}/2$ isomorphic to the dihedral group D_4 of order 8. It follows from this and the previous paragraph that the only possible periods of k are 2 and 4. It is well-known that k always has period 2; we shall show that period 4 cannot occur.

Recall that k is determined by a rational fraction p/q (the 2-fold branched cover of k being the lens space $L(p, q)$). It turns out that the case Out $(G) \cong D_4$ occurs when the continued fraction expansion of p/q is palindromic; this corresponds to $q^2 \equiv \pm 1 \pmod p$. Now in both cases a geometric symmetry $\psi: X \to X$ of order 4 representing the unique element of order 4 in Out (G) can be seen from an appropriate projection of k. In the first case, $q^2 \equiv 1 \pmod p$, one sees that ψ has no fixed points, and in the case $q^2 \equiv -1 \pmod p$, ψ reverses the ambient orientation.

Suppose k has period 4; we then have $\phi: X \to X$ of order 4 with a circle of fixed points. Since ϕ and ψ represent the same element of Out (G), and X is a $K(G, 1)$, ϕ and ψ are homotopic. Since ϕ is orientation preserving, this is an immediate contradiction in the case that ψ is orientation-reversing.

In the other case, let $h_t: X \to X$, $0 \leqq t \leqq 1$, be a homotopy with $h_0 = \phi$, $h_1 = \psi$. A homotopy $H_t: X \to X$ such that $H_0 = H_1 = \text{id}$ is then defined by composing the homotopies $h_t\phi^{-1}$, $\psi h_t\phi^{-2}$, $\psi^2 h_t\phi^{-3}$ and $\psi^3 h_t$. Let x be a base-point in the fixed-point set of ϕ. Then the track of x under H_t is a loop L. Since $H_0 = H_1 = \text{id}$, L represents an element of the centre of G, hence L is null-homotopic. Note also that L is invariant under ψ. Let $\tilde{X}$ be the universal cover of X (so $\tilde{X} \cong \mathbf{R}^3$ [**20**]). Since L is null-homotopic, it lifts to copies $\{\tilde{L}_g: g \in G\}$ in $\tilde{X}$. Let $\tilde{\psi}: \tilde{X} \to \tilde{X}$ be the lift of ψ such that $\tilde{\psi}(\tilde{L}_1) = \tilde{L}_1$. Now $\tilde{\psi}^4$ covers $\psi^4 = \text{id}$, and is therefore some covering transformation. Since $\tilde{L}_1$ contains only one lift of x, it follows that $\tilde{\psi}^4 = \text{id}$. By Smith theory, $\tilde{\psi}$ then has a fixed point, which implies that ψ does also. This contradiction completes the proof.

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