ON NON-LOCAL PROBLEMS FOR PARABOLIC EQUATIONS

J. CHABROWSKI

The main purposes of this paper are to investigate the existence and the uniqueness of a non-local problem for a linear parabolic equation

\begin{equation}
Lu = \sum_{i,j=1}^{n} a_{i,j}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t)
\end{equation}

in a cylinder \( D = \Omega \times (0, T] \). Given functions \( \beta_i (i = 1, \ldots, N) \) on \( \Omega \) and numbers \( \tau_i \in (0, T] (i = 1, \ldots, N) \), the problem in question is to find a solution \( u \) of (1) satisfying the following conditions

\begin{align*}
(2) & \quad u(x,t) = \phi(x,t) \quad \text{on} \quad \Gamma, \\
(3) & \quad u(x,0) + \sum_{i=1}^{N} \beta_i(x)u(x, \tau_i) = \Psi(x) \quad \text{on} \quad \Omega,
\end{align*}

where \( f, \phi \) and \( \Psi \) are given functions and \( \Gamma \) denotes the lateral surface of \( D \), i.e., \( \Gamma = \partial \Omega \times [0, T] \).

In Section 1 we establish the maximum principle associated with the problem described by (1), (2) and (3). Theorem 1 leads immediately to the uniqueness of solution of the problem (1), (2) and (3) as well as to an estimate of the solution in terms of \( f, \phi \) and \( \Psi \). We also briefly discuss certain properties of the solutions related to the behaviour of the coefficients \( \beta_i (i = 1, \ldots, N) \). In Theorem 5 of Section 2 we establish the existence of the solution in a bounded cylinder. The results are then applied to derive the existence and the uniqueness of solution of the non-local problem in an unbounded cylinder (Section 3). In Section 4 we establish an integral representation of solutions and give a construction of the solution of a non-local problem in \( R_+ \times (0, T] \) with \( \Psi \in L^1(R_+) \). In the last section we modify the condition (3) by replacing a finite sum by an infinite series and briefly discuss the uniqueness and the existence of solution of the resulting problem. Theorems of Sections 1 and 2 of this

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paper extend and improve earlier results obtained by Kerefov [3] and Vabishchevich [6], where historical references can be found. They only considered the case \( N = 1 \).

1. Let \( D = \Omega \times (0, T] \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). By \( \Gamma \) we denoted the lateral surface of \( D \), i.e., \( \Gamma = \partial \Omega \times [0, T] \).

Throughout this section we make the following assumption

(A) The coefficients \( a_{ij}, b_i \) and \( c \) are continuous on \( D \) and moreover

\[
\sum_{i,j=1}^n a_{ij}(x, t)\xi_i \xi_j > 0
\]

for all vectors \( \xi \neq 0 \) and \( (x, t) \in D \).

By \( C^{2,1}(D) \) we denote the set of functions \( u \) continuous on \( D \) with their derivatives \( \partial u/\partial x_i, \partial^2 u/\partial x_i \partial x_j \) \((i, j = 1, \ldots, n)\) and \( \partial u/\partial t \) (at \( t = T \) the derivative \( \partial u/\partial t \) is understood as the left-hand derivative).

**Lemma 1.** Let \( u \in C^{2,1}(D) \cap C(\overline{D}) \). Suppose that \( c(x, t) \leq 0 \) on \( D \) and \(-1 \leq \sum_{i=1}^N \beta_i(x) \leq 0 \) on \( \Omega \) and \( \beta_i(x) \leq 0 \) on \( \Omega \) \((i = 1, \ldots, N)\). If \( Lu \leq 0 \) in \( D \), \( u(x, t) \geq 0 \) on \( \Gamma \) and \( u(x, 0) + \sum_{i=1}^N \beta_i(x)u(x, T_i) \geq 0 \) on \( \Omega \), then \( u(x, t) \geq 0 \) on \( \overline{D} \).

**Proof.** Assume that \( u < 0 \) at some point of \( \overline{D} \), then there exists a point \( (x_0, t_0) \in \overline{D} \) such that \( u(x_0, t_0) = \min_{\overline{D}} u(x, t) < 0 \). By the strong maximum principle \( (x_0, t_0) = (x_0, 0) \) with \( x_0 \in \Omega \) (see Friedman [2] Chap. 2 or Protter and Weinberger [5] Chap. 3). Thus, we find that

\[
0 \leq u(x_0, 0) + \sum_{i=1}^N \beta_i(x_0)u(x_0, T_i) \leq u(x_0, 0)\left[1 + \sum_{i=1}^N \beta_i(x_0)\right].
\]

Hence \( u(x_0, 0) \geq 0 \) provided \( 1 + \sum_{i=1}^N \beta_i(x_0) > 0 \) and we get a contradiction.

In the case \( \sum_{i=1}^N \beta_i(x_0) = -1 \) we put \( u(x_0, T_i) = \min_{t_1, \ldots, t_N} u(x_0, T_i) \), then

\[
u(x_0, 0) - u(x_0, T_i) = u(x_0, 0) + u(x_0, T_i)\sum_{i=1}^N \beta_i(x_0) \geq u(x_0, 0) + \sum_{i=1}^N \beta_i(x_0)u(x_0, T_i) \geq 0.
\]

Hence \( u \) takes on a negative minimum at \( (x_0, T_i) \in D \). This contradiction completes the proof.

**Corollary.** Suppose that the assumptions of Lemma 1 hold. If \( L \geq 0 \) in \( D \), \( u(x, t) \leq 0 \) on \( \Gamma \) and \( u(x, 0) + \sum_{i=1}^N \beta_i(x)u(x, T_i) \leq 0 \) on \( \Omega \), then \( u(x, t) \leq 0 \) on \( \overline{D} \).
Now we can state the main result of this section.

**Theorem 1.** Let \( u \in C^{2,1}(D) \cap C(\overline{D}) \) be a solution of the problem (1), (2) and (3) with \( f, \phi \) and \( \Psi \) continuous on \( D, \Gamma \) and \( \Omega \) respectively. Suppose that \( c(x, t) \leq -c_0 \) in \( D \), where \( c_0 \) is a positive constant. Assume further that \(-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \) and \( \beta_i(x) \leq 0 \) \((i = 1, \ldots, N)\) on \( \Omega \). Then

\[
|u(x, t)| \leq \frac{2}{c_0} e^{(\rho_0/2)T} \sup_{\Omega} |f(x, t)| + e^{(\rho_0/2)T} \sup_{\Gamma} |\phi(x, t)|
+ (1 - e^{-(\rho_0/2)T})^{-1} \sup_{\Omega} |\Psi(x)|
\]

for all \( (x, t) \in \overline{D} \), where \( T_k = \min_{i=1, \ldots, N} T_i \).

**Proof.** We first suppose that \(-1 < -\beta_0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \) on \( \Omega \), where \( \beta_0 \) is a positive constant. Let \( M = \sup_{D} |f(x, t)| \), \( M_1 = \sup_{\Gamma} |\phi(x, t)| \), \( M_2 = \sup_{\Omega} |\Psi(x)| \) and put

\[
v(x, t) = u(x, t) - \frac{M}{c_0} - M_1 - \frac{M_2}{1 - \beta_0}.
\]

Then

\[
Lv = f - \frac{c}{c_0}M - cM_1 - \frac{cM_2}{1 - \beta_0} \geq c_0M_1 + \frac{c_0}{1 - \beta_0}M_2 > 0
\]

in \( D, \ v(x, t) \leq 0 \) on \( \Gamma \) and

\[
v(x, 0) + \sum_{i=1}^{N} \beta_i(x)v(x, T_i) = \Psi(x) - \frac{M}{c_0}M_1 - \frac{M_2}{1 - \beta_0} - \left( \frac{M}{c_0} + M_1 + \frac{M_2}{1 - \beta_0} \right) \sum_{i=1}^{N} \beta_i(x) \leq \left( \frac{M}{c_0} + M_1 \right)(\beta_0 - 1) + M_2 \left( 1 - \frac{1}{1 - \beta_0} + \frac{\beta_0}{1 - \beta_0} \right) < 0
\]

on \( \Omega \). It follows from Lemma 1 that \( v \leq 0 \) on \( D \). Similarly we can establish the inequality \( u(x, t) \geq -(M/c_0) - M_1 - M_2(1 - \beta_0) \) for \((x, t) \in \overline{D}\) considering the auxiliary function

\[
w(x, t) = u(x, t) + \frac{M}{c_0} + M_1 + \frac{M_2}{1 - \beta_0}.
\]

In the general case we put \( u(x, t) = e^{-(\rho_0/2)T}z(x, t) \). Then \( z \) satisfies the equation.
\[ \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 z}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial z}{\partial x_i} + \left( c(x,t) + \frac{c_0}{2} \right) z - \frac{\partial z}{\partial t} = e^{(e^y/2)t} f(x,t) \]

in \( D \) with \( c(x,t) + c_0/2 \leq -(c_0/2) \) in \( D \),

\[ z(x,t) = e^{(e^y/2)t} \psi(x,t) \text{ on } \Gamma \]

and

\[ z(x,0) + \sum_{i=1}^{N} \beta_i(x) e^{-(e^y/2)t} z(x,T_i) = \Psi(x) \text{ on } \Omega. \]

It is clear that \(-e^{-(e^y/2)t} \leq \sum_{i=1}^{N} \beta_i(x) e^{-(e^y/2)t} \leq 0\) on \( \Omega \) and the estimate easily follows.

Theorem 1 and a classical maximum principle for solutions of parabolic equations allow us to compare a solution of the problem (1), (2) and (3) with a solution of an initial boundary value problem.

**THEOREM 2.** Suppose that the assumptions of Theorem 1 hold. Let \( u \in C^{2,1}(D) \cap C(\overline{D}) \) be a solution of the problem (1), (2) and (3), and \( v \in C^{2,1}(D) \cap C(\overline{D}) \) a solution of (1) satisfying the initial boundary value conditions \( v(x,t) = \phi(x,t) \text{ on } \Gamma \) and \( v(x,0) = \Psi(x) \text{ on } \Omega. \) Then

\[
|u(x,t) - v(x,t)| \leq \sup_{y} \sum_{i=1}^{N} |\beta_i(x)| \left[ \frac{2}{C_0} e^{(e^y/2)t} \sup_{y}|f(x,t)| + e^{(e^y/2)t} \sup_{y}|\phi(x,t)| + (1 - e^{-(e^y/2)t})^{-1} \sup_{y}|\Psi(x)| \right]
\]

for all \((x,t) \in \overline{D}.\)

In particular if \( \beta_i = \beta_i(x) (i = 1, \ldots, N) \) where \( \beta_i \rightarrow 0 \) uniformly as \( y \rightarrow \infty \) for all \( i \), then the corresponding sequence \( u_{\nu} \) of solutions of the problem (1), (2) and (3) converges uniformly to \( v \) in \( \overline{D} \).

**THEOREM 3.** Let \( c(x,t) \leq 0 \) in \( D \) and assume that \(-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \) (\( j = 1, 2 \)) and that \( \beta_i(x) \leq \beta_i(x) \leq 0 \) (\( i = 1, \ldots, N \)) on \( \Omega \). Suppose further that \( f \leq 0 \), \( \phi \geq 0 \) and \( \Psi \geq 0 \) on \( D \), \( \Gamma \) and \( \overline{\Omega} \) respectively. If \( u_{\nu} \) and \( u_{\omega} \) are solutions belonging to \( C^{2,1}(D) \cap C(\overline{D}) \) of the problem (1), (2) and (3) with \( \beta_i = \beta_i(x) \) (\( i = 1, \ldots, N \)) and \( \beta_i = \beta_i(x) \) (\( i = 1, \ldots, N \)) respectively, then \( u_{\nu}(x,t) \geq u_{\omega}(x,t) \) on \( \overline{D} \).

**Proof.** We put \( w(x,t) = u_{\nu}(x,t) - u_{\omega}(x,t) \), then \( Lw = 0 \) in \( D \), \( w(x,t) = 0 \) on \( \Gamma \) and
\[ w(x,0) + \sum_{i=1}^{N} \beta_i(x)w(x, T_0) = \sum_{i=1}^{N} (\beta_i(x) - \beta_i(x))u_i(x, T_i) \text{ on } \Omega. \]

Since \( u_i(x, t) \geq 0 \) on \( \bar{D} \), it follows from Lemma 1, that \( w(x, t) \geq 0 \) for all \( (x, t) \in \bar{D} \).

Lemma 1 yields the uniqueness of solutions of the problem (1), (2) and (3) under the assumptions that \( \beta_i(x) \leq 0 \) (\( i = 1, \ldots, N \)) and \( -1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \) on \( \Omega \). Vabishchevich [6] pointed out, without giving any proof, that in the case \( N = 1 \) the uniqueness can be proved under the assumption \( |\beta(x)| \leq 1 \) on \( \Omega \). For the sake of completeness we include the proof of uniqueness under the assumption \( \sum_{i=0}^{N} |\beta_i(x)| \leq 1 \) on \( \Omega \).

**Theorem 4.** Suppose that \( c(x, t) \leq 0 \) on \( D \) and \( \sum_{i=1}^{N} |\beta_i(x)| \leq 1 \) on \( \Omega \). Then the problem (1), (2) and (3) has at most one solution in \( C^{2,1}(D) \cap C(\bar{D}) \).

**Proof.** Let \( u \) be a solution of the homogeneous problem

\[
Lu = 0 \quad \text{in } D \\
u(x, t) = 0 \quad \text{on } \Gamma
\]

and

\[ u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) = 0 \quad \text{on } \Omega. \]

Suppose that \( u \neq 0 \). We also many assume that there exists a point in \( (x_0, t_0) \in \bar{D} \) such that \( u(x_0, t_0) = \min_D u(x, t) < 0 \). It is clear that \( (x_0, t_0) = (x_0, 0) \) with \( x_0 \in \Omega \). We can assume that \( |u(x_0, T_i)| = \max_{i=1,\ldots,N} |u(x_0, T_i)| > 0 \), since otherwise there is nothing to prove. Obviously,

\[ |u(x_0, 0)| \leq |u(x_0, T_i)| \sum_{i=1}^{N} |\beta_i(x)| \leq |u(x_0, T_i)|. \]

If \( u(x_0, T_i) < 0 \) then \( u(x_0, T_i) \leq u(x_0, 0) \). Hence \( u \) attains its negative minimum at \( (x_0, T_i) \) and we get a contradiction, therefore \( u(x_0, T_i) > 0 \). Thus there exists a point \( (x_i, t_i) \in \bar{D} \) such that \( u(x_i, t_i) = \max_D u(x, t) > 0 \). Again \( (x_i, t_i) = (x_i, 0) \) with \( x_i \in \Omega \). Put \( |u(x_i, T_i)| = \max_{i=1,\ldots,N} |u(x_i, T_i)| \). We may assume that \( |u(x_i, T_i)| > 0 \), since otherwise there is nothing to prove. Now we must distinguish two cases

\[ |u(x_0, 0)| < u(x_0, 0) \quad \text{or} \quad u(x_0, 0) \leq |u(x_0, 0)|. \]

In the first case we have

\[ |u(x_0, 0)| < u(x_0, 0) \leq |u(x, T_i)| \sum_{i=1}^{N} |\beta_i(x)| \leq |u(x, T_i)|, \]
consequently if \( u(x, T) < 0 \) then \( u(x_0, 0) > u(x, T) \). Hence \( u \) takes on a positive minimum at \((x, T) \in D\) and we get a contradiction. On the other hand if \( u(x, T) > 0 \) we have \( u(x, 0) \leq u(x, T) \). Hence \( u \) attains a positive maximum at \((x, T) \) and we arrive at a contradiction. Similarly in the second case we obtain

\[
u(x, 0) \leq |u(x_0, 0)| \leq u(x_0, T) \leq \sum_{i=1}^{N} |\beta_i(x_0)| \leq u(x_0, T_0)\]

and \( u \) takes on a positive maximum at \((x_0, T_0) \in D\). This contradiction completes the proof.

2. For the existence theorem we shall need the following assumptions

\((A_1)\) There exist positive constants \( \lambda_0 \) and \( \lambda_i \) such that, for any vector \( \xi \in R^n \)

\[
\lambda_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t)x_i x_j \leq \lambda_i |\xi|^2
\]

for all \((x, t) \in D\).

\((A_2)\) The coefficients \( a_{ij}, b_i (i, j = 1, \ldots, n), c \) and \( f \) are Hölder continuous in \( D \) (exponent \( \alpha \)).

\((A_3)\) The functions \( \phi, \Psi \) and \( \beta_i \) \((i = 1, \ldots, N)\) are continuous respectively on \( \Gamma, \Omega \) and \( \bar{\Omega} \) and, in addition,

\[
\Psi(x) = \phi(x, 0) + \sum_{i=1}^{N} \beta_i(x)\phi(x, T_i)
\]

for all \( x \in \partial \Omega \).

Moreover we assume that \( \partial \Omega \in C^{2+\alpha} \).

**Theorem 5.** Let \( c(x, t) \leq -c_0 \), where \( c_0 \) is a positive constant and assume that \(-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \) and \( \beta_i(x) \leq 0 \) \((i = 1, \ldots, N)\) on \( \bar{\Omega} \). Then there exists a unique solution in \( C^{2+\alpha}(D) \cap C(\bar{D}) \) of the problem (1), (2) and (3).

**Proof.** We first assume that \( \phi \equiv 0 \) on \( \Gamma \), then by the condition \( (A_1) \)

\( \Psi(x) = 0 \) on \( \partial \Omega \). We try to find a solution in the form

\[
u(x, t) = \int_{\partial} G(x, t; y, 0)u(y, 0)dy - \int_{\bar{\Omega}} \int_{\partial} G(x, t; y, \tau)f(y, \tau)d\nu \tau,
\]

where \( u(y, 0) \) is to be determined and \( G \) denotes the Green function for the operator \( L \). The condition (3) leads to the Fredholm integral equa-
tion of the second kind

\begin{equation}
    u(x, 0) + \sum_{i=1}^{N} \beta_i(x) \int_{\Omega} G(x, T_i; y, 0) u(y, 0) dy
    = \Psi(x) + \sum_{i=1}^{N} \beta_i(x) \int_{0}^{T_i} G(x, T_i; y, \tau) f(y, \tau) dy d\tau.
\end{equation}

Applying Theorem 4 it is easy to show that the corresponding homogeneous equation only has a trivial solution in \( L^2(\Omega) \). Hence there exists a unique solution \( u(\cdot, 0) \) in \( L^2(\Omega) \) of the equation (7). Since \( \Psi(x) = 0 \) on \( \partial \Omega \), it follows from the properties of the Green function that \( u(\cdot, 0) \in C(\overline{\Omega}) \) and \( u(x, 0) = 0 \) on \( \partial \Omega \). Consequently the formula (6) gives a solution in this case.

Suppose next \( \phi \neq 0 \), but assume that there exists a function \( \Phi \in \overline{C}^{2+a}(D) \) such that \( \Phi = \phi \) on \( \Gamma \). Introducing \( v = u - \Phi \) we then immediately obtain, by the previous result, the existence of a solution \( v \) to \( Lv = f - L\Phi \) which vanishes on \( \Gamma \) and satisfies the condition

\begin{equation}
    v(x, 0) + \sum_{i=1}^{N} \beta_i(x) v(x, T_i) = \Psi(x) - \Phi(x, 0) - \sum_{i=1}^{N} \beta_i(x) \Phi(x, T_i)
\end{equation}

for all \( x \in \Omega \). Then assertions for \( u \) then follow.

We finally consider the general case, where \( \phi \) is only assumed to be continuous. By Theorem 2 in Friedman [2] (p. 60) and the Weierstrass approximation theorem there exists a sequence of polynomials \( \Phi_m \) on \( \overline{D} \) which approximates \( \phi \) uniformly on \( \Gamma \). Now we define a function \( \Psi_m \) on \( \partial \Omega \) by the following formula

\begin{equation}
    \Psi_m(x) = \Phi_m(x, 0) + \sum_{i=1}^{N} \beta_i(x) \Phi_m(x, T_i)
\end{equation}

for \( x \in \partial \Omega \). Since \( \lim_{m \to \infty} \Psi_m = \Psi \) uniformly on \( \partial \Omega \), one can construct a sequence of functions \( \{\tilde{\Psi}_m\} \) in \( C(\overline{\Omega}) \) such that \( \lim_{m \to \infty} \tilde{\Psi}_m = \Psi \) uniformly on \( \overline{\Omega} \) and \( \tilde{\Psi}_m = \Psi_m \) on \( \partial \Omega \) for all \( m \). By what we have already proved there exist solutions to the problem

\begin{equation}
    Lu_m = f \quad \text{in} \ D, \quad u_m(x, t) = \Phi_m(x, t) \quad \text{on} \ \Gamma,
\end{equation}

and

\begin{equation}
    u_m(x, 0) + \sum_{i=1}^{N} \beta_i(x) u_m(x, T_i) = \tilde{\Psi}_m(x) \quad \text{on} \ \Omega.
\end{equation}
By Theorem 1 (the inequality (4)) the sequence \( u_m(x, t) \) is uniformly convergent on \( \bar{D} \) to a function \( u \). It is clear that \( u \) satisfies the conditions (2) and (3). Using Friedman-Schauder interior estimates (Friedman [2], Theorem 5 p. 64) one can easily prove that \( u \) satisfies the equation (1).

**Remark.** In the above proof we followed the argument used in the proof of Theorem 9 in Friedman [2] (p. 70–71). For the definition of the space \( \overline{C}^{2+a}(D) \) see Friedman [2] (p. 61–62).

**Theorem 6.** Suppose that \( \sum_{i=1}^{\infty} |\beta_i(x)| \leq 1 \) on \( \Omega \), \( c(x, t) \leq 0 \) on \( D \) and \( \phi \equiv 0 \) on \( \Gamma \). Then the problem (1), (2) and (3) has a unique solution in \( C^{2+a}(D) \cap C(\overline{D}) \).

**Proof.** A solution to this problem is given by the formula

\[
    u(x, t) = \int_\Omega G(x, t; y, 0)u(y, 0)dy - \int_0^t \int_\Omega G(x, t; y, \tau)f(y, \tau)dyd\tau,
\]

where \( u(x, 0) \) is a solution of the Fredholm integral equation of the second kind

\[
    u(x, 0) + \sum_{i=1}^{\infty} \beta_i(x) \int_\Omega G(x, T_i; y, 0)u(y, 0)dy = \psi(x) + \sum_{i=1}^{\infty} \beta_i(x) \int_0^{T_i} \int_\Omega G(x, T_i; y, \tau)f(y, \tau)dyd\tau.
\]

3. In this section we investigate the existence of a solution of the problem (1), (2) and (3) in an unbounded cylinder. Let \( D = \Omega \times (0, T] \), where \( \Omega \) is an unbounded domain in \( \mathbb{R}^n \).

In the next theorem we give a general method of constructing a solution. We shall need the following assumptions

(B.) The coefficients \( a_{ij}, b_i (i, j = 1, \cdots, n) \) and \( c \) are continuous on \( D \) and moreover

\[
    \sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j > 0
\]

for every \( (x, t) \in D \) and any vector \( \xi \neq 0 \), \( a_{ij} = a_{ji} (i, j = 1, \cdots, n) \).

(B.) There exists a family of positive function \( H(x, \delta) \) \((0 < \delta \leq \delta_0)\) defined on \( \Omega \) with properties:

(1) \( H \in C^4(\Omega) \cap C(\overline{\Omega}) \) for \( 0 < \delta \leq \delta_0 \) and \( LH \leq -c_0H \) for all \( (x, t) \in D \) and \( 0 < \delta \leq \delta_0 \), where \( c_0 \) is a positive constant,

(ii) \( \lim_{|x| \to \infty} \frac{H(x, \delta_1)}{H(x, \delta_2)} = 0 \) for \( 0 < \delta_1 < \delta_2 \leq \delta_0 \).

(iii) there exists a positive constant $\kappa$ such that

$$H(x, \delta) \leq \kappa H(x, \delta)$$

for all $x \in \Omega$ and $0 < \delta_1 < \delta_2 < \delta_0$.

For a sequence $\{R_p\}$ of positive numbers we define

$$\Omega_p = \Omega \cap \{x: |x| < R_p\}, \quad \Gamma_p = \partial \Omega_p \times [0, T] \quad \text{and} \quad D_p = \Omega_p \times (0, T).$$

(B3) There exists a sequence of positive numbers $R_p$ converging to $\infty$ as $p \to \infty$ such that the problem (1), (2) and (3) is solvable on every $D_p$, i.e. for every bounded and Hölder continuous function $f$ on $D_p$ and all continuous functions $\phi$ and $\Psi$ on $\Gamma_p$ and $\bar{\Omega}_p$ respectively, and satisfying the condition

$$\Psi(x) = \phi(x, 0) + \sum_{i=1}^{N} \beta_i(x)\phi(x, T) \quad \text{on } \partial \Omega_p,$$

the problem

$$Lu = f \quad \text{in } D_p,$$

$$u(x, t) = \phi(x, t) \quad \text{on } \Gamma_p$$

and

$$u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T) = \Psi(x) \quad \text{on } \Omega_p$$

has a unique solution in $C^{\alpha}(D_p) \cap C(\bar{D}_p)$.

We shall say that a function $u$ defined on $D$ belongs to $E^\alpha(D)$ if there exist positive constants $M$ and $\delta < \delta_0$ such that $|u(x, t)| \leq MH(x, \delta)$ for all $(x, t) \in D$.

We shall say that a function $v$ defined on $\Omega$ belongs to $E^\alpha(\Omega)$ if there exist positive constants $M$ and $\delta < \delta_0$ such that $|v(x)| \leq MH(x, \delta)$ for all $x \in \Omega$.

We are now in a position to construct a solution of the problem (1), (2) and (3). The construction given in the proof of Theorem 7 below is a modification of the method used by Krzyżaniński [4] to solve the Cauchy problem for parabolic equations.

**Theorem 7.** Suppose that the assumptions (B1), (B2) and (B3) hold. Let $-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0$ and $\beta_i(x) \leq 0$ (i = 1, ⋯, N) on $\Omega$. Assume that $f \in E^\alpha(D)$ is an Hölder continuous function, that $\phi \in E^\alpha(D)$ and $\Psi \in E^\alpha(\Omega)$ are continuous functions on $\bar{D}$ and $\bar{\Omega}$ respectively and moreover that
\( \mathscr{F}(x) = \phi(x, 0) + \sum_{i=1}^{N} \beta_i(x)\phi(x, T_i) \) on \( \partial\Omega_p \)
p = 1, 2, \ldots. Then the problem (1), (2) and (3) has a unique solution in \( C^{\alpha,1}(D) \cap C(\overline{D}) \cap E_p(D) \).

Proof. It is clear that there exist positive constants \( \mathcal{M} \) and \( \delta \leq \delta_0 \) such that

\[
|\phi(x, t)| \leq \mathcal{M}H(x, \delta), \quad |f(x, t)| \leq \mathcal{M}H(x, \delta) \quad \text{on} \quad D,
|\mathscr{F}(x)| \leq \mathcal{M}H(x, \delta) \quad \text{on} \quad \Omega.
\]

By the assumption (B3) for every \( p \) there exists a unique solution \( u_p \)
in \( C^{\alpha,1}(D) \cap C(\overline{D}) \) of the problem

\[
Lu_p = f \quad \text{on} \quad D_p, 
u_p(x, t) = \phi(x, t) \quad \text{on} \quad \Gamma_p, \
\]
and

\[
u_p(x, 0) + \sum_{i=1}^{N} \beta_i(x)u_p(x, T_i) = \mathscr{F}(x) \quad \text{on} \quad \Omega_p.
\]

Put

\[
u_p(x, t) = v_p(x, t)H(x, \delta) \quad p = 1, 2, \ldots
\]
for \( (x, t) \in D_p \). Then for every \( p \) \( |v_p(x, t)| \leq M \) on \( \Gamma_p \),

\[
|v_p(x, 0) + \sum_{i=1}^{N} \beta_i(x)v_p(x, T_i)| \leq \frac{\mathcal{M}H(x, \delta)}{H(x, \delta)} \leq M \quad \text{on} \quad \Omega_p
\]
and

\[
\sum_{i,j=1}^{N} a_{ij}(x, t)\frac{\partial^2 v_p}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \left( b_i(x, t) + \frac{2}{H(x, \delta)} \sum_{j=1}^{N} a_{ij}(x, t)\frac{\partial H}{\partial x_i}\partial v_p \right)\frac{\partial v_p}{\partial x_i}
+ \frac{LH}{H}v_p - \frac{\partial v_p}{\partial t} = f(x, t)\frac{H(x, \delta)}{H(x, \delta)}
\]
in \( D_p \). It follows from the assumption (B1 i) and Theorem 1 that

\[
|v_p(x, t)| \leq \left[ \frac{2}{c_0} e^{(c_0/2)t} + e^{(c_0/2)t} + (1 - e^{- (c_0/2)t})^{-1} \right]M = M_i
\]
for all \( (x, t) \in D_p, p = 1, 2, \ldots \), where \( T_\varepsilon = \min T_i \). Let \( \delta < \delta_i < \delta_0 \) and put

\[
u_p(x, t) = \overline{v}_p(x, t)H(x, \delta) \quad p = 1, 2, \ldots
\]
and

\[ u_{pq}(x, t) = u_p(x, t) - u_q(x, t) = H(x, \delta_i)[v_p(x, t) - v_q(x, t)] = H(x, \delta_i)\bar{v}_{pq}(x, t) \]

for \( p < q \). The function \( \bar{v}_{pq} \) satisfies the homogeneous equation of the form (7) with \( H(x, \delta) \) replaced by \( H(x, \delta_i) \) and

\[ \bar{v}_{pq}(x, 0) + \sum_{i=1}^{N} \beta_i(x)\bar{v}_{pq}(x, T_i) = 0 \]

on \( \Omega_p \). Moreover

\[ \bar{v}_{pq}(x, t) = 0 \quad \text{on} \quad (\partial \Omega_p \cap \partial \Omega) \times (0, T) \]

and

\[ \bar{v}_{pq}(x, t) = \frac{\beta_p(x, t)}{H(x, \delta_i)} - \frac{u_q(x, t)}{H(x, \delta_i)} \quad \text{on} \quad \Gamma_p \cap \partial \Omega, \]

consequently

\[ |\bar{v}_{pq}(x, t)| \leq (M + M_i) \sup_{\bar{v}_{pq} - \bar{v}_q} \frac{H(x, \delta)}{H(x, \delta_i)} \quad \text{on} \quad \Gamma_p. \]

Let

\[ \varepsilon_p = (M + M_i) \sup_{\bar{v}_{pq} - \bar{v}_q} \frac{H(x, \delta)}{H(x, \delta_i)}. \]

Thus by Theorem 1 we have

\[ |\bar{v}_{pq}(x, t)| \leq \varepsilon_p e^{(c \alpha/T)T} \]

on \( \bar{D}_p \). By the assumption (B_2 ii) \( \lim_{p \to \infty} \varepsilon_p = 0 \), hence \( \bar{v}_p \) converges uniformly on every \( \bar{D}_i \) to a function \( \bar{v} \). Put \( u(x, t) = \bar{v}(x, t)H(x, \delta_i) \) for \( (x, t) \in D \).

Clearly \( u \in E_\mu(D) \) is continuous on \( D \) and satisfies (2) and (3). To show that \( u \) satisfies (1), fix an arbitrary index \( p \) and consider the problem

\[ Lz = f \quad \text{in} \quad D_p, \]

\[ z(x, t) = u(x, t) \quad \text{on} \quad \Gamma_p, \]

\[ z(x, 0) + \sum_{i=1}^{N} \beta_i(x)z(x, T_i) = \Psi(x) \quad \text{on} \quad \Omega_p. \]

Since \( u \) satisfies the condition (3), it is clear that

\[ u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T_i) = \Psi(x) \quad \text{on} \quad \partial \Omega_p. \]

By the assumption (B_4) this problem has a unique solution \( z \). Since \( u_q \rightarrow u \)
as \( q \to \infty \) uniformly on \( \bar{D}_p \), given \( \varepsilon > 0 \) we can find \( q_0 \) such that \(|u_q(x, t) - u(x, t)| < \varepsilon\) for all \((x, t) \in \Gamma_p\) and \( q \geq q_0\). Put
\[
u_q(x, t) = u_q(x, t) - z(x, t) = w_q(x, t)H(x, \delta)
\]
for \((x, t) \in \bar{D}_p\), \( q \geq q_0\). Then \( w_q \) satisfies the homogeneous equation (8) in \( D_p \) and the following conditions
\[
|w_q(x, t)| \leq \varepsilon \sup_{r_p} H(x, \delta)^{-1} \quad \text{on} \quad \Gamma_p
\]
and
\[
w_q(x, 0) + \sum_{i=1}^n \beta_i(x) w_q(x, T_i) = 0 \quad \text{on} \quad \Omega_p.
\]
By Theorem 1
\[
|w_q(x, t)| \leq \varepsilon e^{(c_0/2)T} \sup_{r_p} H(x, \delta)^{-1}
\]
for all \((x, t) \in \bar{D}_p\). Letting \( \varepsilon \to 0 \) we obtain \( u \equiv z \) on \( D_p \) and the result follows. To establish uniqueness, let \( u \in C^{s,1}(\bar{D}) \cap C(\bar{D}) \cap E_p(\bar{D}) \) be a solution of the problem (1), (2) and (3) with \( f \equiv 0\), \( \phi \equiv 0\) and \( \psi \equiv 0\). There exist positive constants \( M \) and \( \delta < \delta_0 \) such that \(|u(x, t)| \leq MH(x, \delta)\) in \( D \). Choose \( \delta < \delta_1 < \delta_0 \) and put
\[
u(x, t) = u(x, t)H(x, \delta_1) \quad \text{on} \quad \bar{D}_p.
\]
By (ii) (the assumption (B_2)) given \( \varepsilon > 0 \) we can find a positive number \( R \) such that
\[
|\nu(x, t)| \leq \varepsilon \quad \text{for} \quad (x, t) \in \Omega \cap (|x| \geq R) \times (0, T).
\]
By Theorem 1
\[
|\nu(x, t)| \leq \varepsilon e^{(c_0/2)T}
\]
for all \((x, t) \in \bar{D} \cap (|x| \leq R) \times [0, T] \) and the uniqueness easily follows.

To apply Theorem 7 we introduce the following assumptions

(C_i) The coefficients \( a_{ij}, b_i \) \((i, j = 1, \ldots, n)\) and \( c \) are bounded on \( R_n \times [0, T) \) and Hölder continuous (with exponent \( \alpha \)) on every compact subset in \( R_n \times [0, T] \) and moreover
\[
c(x, t) \leq -c_0 \quad \text{for all} \quad (x, t) \in R_n \times [0, T],
\]
where \( c_0 \) is a positive constant.

(C_d) There exists positive constants \( \lambda_0 \) and \( \lambda_1 \) such that for any vector \( \xi \in R_n \)
for all \((x, t) \in R_n \times (0, T]\), \(a_{ij} = a_{ji} \) (\(i, j = 1, \cdots, n\)).

As an application of Theorem 7 we shall prove the existence of a solution \(u\) of the equation (1) in \(R_n \times (0, T]\) satisfying the condition

\[
(9) \quad u(x, 0) + \sum_{i=1}^{N} \beta_i(x)u(x, T) = \Psi(x) \quad \text{on} \ R_n.
\]

It is clear that the function \(H(x, \delta) = \int_{R_n} \cosh \delta x \, dx\) has properties (i), (ii) and (iii) of the assumption \((B_2)\) (with \(\Omega = R_n\)) provided \(0 < \delta < \delta_\circ\), where \(\delta_\circ\) is sufficiently small.

In this situation

\[
E_H(R_n \times (0, T]) = \{u; u \text{ defined on } R_n \times (0, T] \quad \text{and} \quad |u(x, t)| \leq M e^{\delta|x|}
\]

for all \((x, t) \in R_n \times (0, T]\) and certain \(M > 0\) and \(0 < \delta < \delta_\circ\},\)

similarly

\[
E_H(R_n) = \{v; v \text{ defined on } R_n \text{ and } |v(x)| \leq M e^{\delta|x|}
\]

for all \(x \in R_n\) and certain \(M > 0\) and \(0 < \delta < \delta_\circ\}.\)

**THEOREM 8.** Suppose that the assumptions \((C_1)\) and \((C_2)\) holds. Let \(\beta_i \in C(R_n), \beta_i(x) \leq 0 \) (\(i = 1, \cdots, N\)) and \(-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0\) on \(R_n\). If \(f \in E_H(R_n \times (0, T])\) is a Hölder continuous function on every compact subset of \(R_n \times [0, T]\) and \(\Psi \in E_H(R_n) \cap C(R_n),\) then the problem (1), (9) has a unique solution in \(E_H(R_n \times (0, T)) \cap C^{1,1}(R_n \times [0, T]) \cap C(R_n \times [0, T]).\)

**Proof.** Let \(\phi\) be a continuous function belonging to \(E_H(R_n \times (0, T])\) such that \(\phi(x, 0) = \Psi(x)\) on \(R_n\) and \(\phi(x, t) = 0\) on \(R_n \times [T_0, T]\), where \(T_0 = \min_{i=1,\cdots, N} T_i\). By Theorem 5 the problem (1), (2) and (3) has a unique solution on every \(D_p\). Applying Theorem 7 the result easily follows.

In the sequel we shall need the following result.

**Lemma 2.** Suppose that the assumptions \((C_1)\) and \((C_2)\) hold in \(R_n \times (0, T]\). Let \(\beta_i \in C(R_n) \) (\(i = 1, \cdots, N\)), \(-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0\) and \(\beta_i(x) \leq 0\) (\(i = 1, \cdots, N\)) on \(R_n\). Then for any bounded function \(f\) on \(R_n \times [0, T]\) and Hölder continuous on every compact subset of \(R_n \times [0, T]\) and for any continuous and bounded function \(\Psi\) on \(R_n\) there exists a unique solution \(u\) of the problem (1), (9) in \(E_H(R_n \times (0, T]) \cap C^{1,1}(R_n \times (0, T]) \cap C(R_n \times [0, T])\) such that

\[
\lambda_0 |\xi|^p \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leq \lambda_1 |\xi|^p
\]
\[ |u(x, t)| \leq \frac{2}{c_0} e^{(c_0/2)T} \sup_{R_n \times [0, T]} |f(x, t)| + (1 - e^{-(c_0/2)T}) \sup_{R_n} |\mathcal{F}(x)| \]

for all \((x, t) \in R_n \times [0, T]\), where \(T_{\text{e}} = \min_i T_i\).

**Proof.** We start with the following observation, the proof of which is routine,

if \(u \in C^{\cdot, 1}(R_n \times (0, T)) \cap C(R_n \times [0, T]) \cap E_n(R_n \times (0, T))\) and

\[ Lu \leq 0 \text{ in } R_n \times (0, T), \]

\[ u(x, 0) + \sum_{i=1}^{N} \beta_i(x) u(x, T_i) \geq 0 \text{ on } R_n \]

then \(u \geq 0\) on \(R_n \times [0, T]\).

We first suppose that \(-1 < -\beta_0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0\) on \(R_n\), where \(\beta_0\) is a positive constant. Put

\[ u(x, t) = u(x, t) - \frac{M}{c_0} - \frac{M_1}{1 - \beta_0}, \]

where

\[ M = \sup_{R_n \times [0, T]} |f(x, t)| \text{ and } M_1 = \sup_{R_n} |\mathcal{F}(x)|. \]

Then

\[ Lu = f - \frac{c}{c_0} M - \frac{cM_1}{1 - \beta_0} \geq \frac{c_0 M_1}{1 - \beta_0} > 0 \]

in \(R_n \times (0, T)\) and

\[ u(x, 0) + \sum_{i=1}^{N} \beta_i(x) u(x, T_i) = \mathcal{F}(x) - \frac{M}{c_0} - \frac{M_1}{1 - \beta_0} - \left( \frac{M}{c_0} + \frac{M_1}{1 - \beta_0} \right) \sum_{i=1}^{N} \beta_i(x) \]

\[ \leq \frac{M}{c_0} (\beta_0 - 1) + M_1 \left( 1 - \frac{1}{1 - \beta_0} + \frac{\beta_0}{1 - \beta_0} \right) < 0 \]

on \(R_n\). By the preceding remark

\[ u \leq \frac{M}{c_0} + \frac{M_1}{1 - \beta_0} \text{ on } R_n \times [0, T]. \]

Similarly using

\[ w(x, t) = u(x, t) + \frac{M}{c_0} + \frac{M_1}{1 - \beta_0} \]

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as a comparison function we deduce the inequality

\[ u \geq -\frac{M}{c_0} - \frac{M_i}{1 - \beta_0} \text{ on } R_n \times [0, T]. \]

In the general case we use the transformation \( u(x, t) = v(x, t)e^{-(\gamma t/2)t} \).

4. In this section we derive an integral representation of the problem (1), (2) and (3) in an infinite strip and in a bounded cylinder.

**Theorem 9.** Suppose that the assumptions (C₁) and (C₂) hold in \( R_n \times (0, T] \). Let \( \beta_i (i = 1, \ldots, N) \) and \( \Psi \) be a continuous and bounded functions on \( R_n \). Assume further that

\[-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \text{ and } \beta_i(x) \leq 0 (i = 1, \ldots, N) \text{ on } R_n.\]

Then the unique solution in \( C^2(R_n \times (0, T]) \cap C(R_n[0, T]) \cap E_n(R_n \times (0, T]) \) of the problem (1), (9) with \( f \equiv 0 \) is given by

\[ u(x, t) = \int_{R_n} P(x, t, y)\Psi(y)dy, \]

for \( (x, t) \in R_n \times (0, T] \), where \( P(x, t, y) \) as a function of \( (x, t) \) satisfies the equation \( LP = 0 \) in \( R_n \times (0, T] \) for almost all \( y \in R_n \). Moreover \( P \) satisfies the equation

\[ P(x, t, y) = -\int_{R_n} \Gamma(x, t; z, 0) \sum_{i=1}^{N} \beta_i(z)P(z, T_n, y)dz + \Gamma(x, t; y, 0) \]

for all \( (x, t) \in R_n \times (0, T] \) and almost all \( y \in R_n \), where \( \Gamma(x, t, y, 0) \) is the fundamental solution of \( Lu = 0 \).

**Proof.** Let \( \Psi \) be a continuous and bounded function in \( L^2(R_n) \). By Lemma 2 the unique solution of the problem (1), (9) in \( C^2(R_n \times (0, T]) \cap C(R_n \times [0, T]) \cap E_n(R_n \times (0, T]) \) is bounded on \( R_n \times [0, T] \). We first prove that for each \( \delta > 0 \) there exists a positive constant \( C(\delta) \) such that

\[ |u(x, t)| \leq C(\delta)\left[ \int_{R_n} \Psi(y)^2dy \right]^{1/2} \]

on \( R_n \times [\delta, T] \). To prove (12) we first assume that \( -1 < \beta_0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0 \) on \( R_n \), where \( \beta_0 \) is a positive constant. Consider the Cauchy problem for the homogeneous equation (1) with the initial condition

\[ z(x, 0) = -\sum_{i=1}^{N} \beta_i(x)u(x, T_n) + \Psi(x) \]
on \( R_n \). The unique solution \( z \) in \( E_n(R_n \times (0, T)) \) is given by

\[
z(x, t) = - \int_{R_n} \Gamma(x, t; y, 0) \sum_{i=1}^{N} \beta_{i}(y) u(y, T_{i}) dy + \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy
\]

for all \((x, t) \in R_n \times (0, T)\) (Friedman [2], p. 26). Since \( u \) is a solution of the same problem we obtain

\[
(13) \quad u(x, t) = - \int_{R_n} \Gamma(x, t; y, 0) \sum_{i=1}^{N} \beta_{i}(y) u(y, T_{i}) dy + \int_{R_n} \Gamma(x, t; y, 0) \Psi(y) dy
\]

for all \((x, t) \in R_n \times (0, T)\). Now it is well known that

\[
(14) \quad \int_{R_n} \Gamma(x, t; y, 0) dy \leq 1
\]

for all \((x, t) \in R_n \times (0, T)\) and

\[
(15) \quad 0 < \Gamma(x, t; y, 0) \leq C_{i} t^{-n/2} e^{-\epsilon \|x-y\|^2 / t}
\]

for all \((x, t) \in R_n \times (0, T)\) and \( y \in R_n \), where \( C_{i} \) and \( \epsilon \) are positive constants (Friedman [2], p. 24). Applying the Hölder inequality we derive from (13), (14) and (15) that

\[
(16) \quad \max_{i=1, \ldots, N} \sup_{R_n} |u(x, T_{i})| \leq \frac{C_{i}}{1 - \beta_{0}} T_{k}^{-n/4} \left[ \int_{R_n} e^{-\epsilon \|x\|^2} dx \right]^{1/2} \left[ \int_{R_n} \Psi(x)^2 dx \right]^{1/2},
\]

where \( T_{k} = \min_{i=1, \ldots, N} T_{i} \). Using again the representation (13) and the estimates (14), (15) and (16) we obtain

\[
(17) \quad |u(x, t)| \leq \left[ \frac{\beta_{0}}{1 - \beta_{0}} C_{i} \right] C_{2} + C_{i} C_{3} t^{-n/4} \left[ \int_{R_n} \Psi(x)^2 dx \right]^{1/2}
\]

for all \((x, t) \in R_n \times (0, T)\), where

\[
C_{2} = T_{k}^{-n/4} \left[ \int_{R_n} e^{-\epsilon \|x\|^2} dx \right]^{1/2} \quad \text{and} \quad C_{3} = \left[ \int_{R_n} e^{-\epsilon \|x\|^2} dx \right]^{1/2},
\]

and the estimate (12) easily follows. In the general case we use the transformation \( u(x, t) = u(x, y) e^{-\epsilon \|y\|^2 / t} \). By (12) the mapping \( \Psi \rightarrow u(x, t) \) defines a linear functional on \( C_{0}(R_n) \cap L^{r}(R_n) \) continuous in \( L^{2} \)-norm.

Here \( C_{0}(R_n) \) denotes the space of continuous and bounded functions on \( R_n \). Consequently the representation (10) follows from the Riesz representation theorem of a linear continuous functional on \( L^{r}(R_n) \). To derive (11) observe that by (10) and (13) we have for every continuous bounded function \( \Psi \).
\[ \int_{\mathbb{R}^n} P(x, t, y)\Phi(y)dy = -\int_{\mathbb{R}^n} \Gamma(x, t; y, 0) \sum_{j=1}^{N} \beta_j(y) \left[ \int_{\mathbb{R}^n} P(y, T_j, z)\Phi(z)dz \right]dy + \int_{\mathbb{R}^n} \Gamma(x, t; y, 0)\Phi(y)dy \]

for \((x, t) \in \mathbb{R}^n \times (0, T]\). Consequently if we fix \((x, t) \in \mathbb{R}^n \times (0, T]\), applying Fubini’s theorem, we obtain the identity (11) for almost all \(y \in \mathbb{R}^n\). Now choose \(y \in \mathbb{R}^n\) such that

\[ \int_{\mathbb{R}^n} \Gamma(x, T; z, 0) \sum_{j=1}^{N} \beta_j(z)P(z, T, y)dz \]

is finite. Then by Theorem 1 in Watson [6] the integral

\[ \int_{\mathbb{R}^n} \Gamma(x, t; z, 0) \sum_{j=1}^{N} \beta_j(z)P(z, T, y)dz \]

is finite for all \((x, t) \in \mathbb{R}^n \times (0, T]\) and represents a solution of the equation \(Lu = 0\) in \(\mathbb{R}^n \times (0, T]\) and the last assertion of the theorem easily follows.

Similarly in the case of a bounded cylinder one can prove

**Theorem 10.** Suppose the assumptions of Theorem 5 hold. Let \(u\) be a solution of the problem (1), (2) and (3) with \(\phi \equiv 0\) and \(f \equiv 0\). Then

\[ u(x, t) = \int_{\partial} p(x, t, y)\Phi(y)dy \]

for all \((x, t) \in D\), where \(p(x, t, y)\) as a function of \((x, t)\) satisfies the equation \(Lp = 0\) for almost all \(y \in \Omega\). Moreover

(18) \[ p(x, t, y) = -\int_{\partial} G(x, t; z, 0) \sum_{i=1}^{N} \beta_i(z)p(z, T, y)dz + G(x, t; y, 0) \]

for all \((x, t) \in D\) and almost all \(y \in \Omega\), where \(G(x, t; y, 0)\) is the Green function for the operator \(L\).

In the following theorem we shall show that \(p\) and \(P\) tend to infinity at the same rate as \(t^{-(n/2)}\).

**Theorem 11.** Let the assumptions of Theorem 9 hold and let \(D = \Omega \times (0, T]\) be a bounded cylinder with \(\partial \Omega \in C^{2+s}\). Then there exists a positive constant \(C\) such that

(19) \[ p(x, t, y) \leq C \int_{\partial} G(x, t; z, 0)dz + G(x, t; y, 0) \]
for all \((x, t) \in D\) and almost all \(y \in \Omega\), and moreover

\[(20) \quad P(x, t, y) \leq C \int_{\mathbb{R}^n} \Gamma(x, t; z, 0)dz + \Gamma(x, t; y, 0)\]

for all \((x, t) \in \mathbb{R}_n \times (0, T]\) and almost all \(y \in \mathbb{R}_n\), where \(C\) depends on \(C_i\) and \(n\).

**Proof.** We first assume that \(-1 < \beta_0 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0\) on \(\Omega\), where \(\beta_0\) is a positive constant.

Let \(\Psi\) be a continuous and non-negative function on \(\mathbb{R}_n\) with compact support in \(\Omega\). It follows from Theorem 9, 10 and the maximum principle that

\[\int_{\Omega} p(x, t, y)\Psi(y)dy \leq \int_{\mathbb{R}^n} P(x, t, y)\Psi(y)dy\]

for all \((x, t) \in D\). Since \(\Psi\) is an arbitrary non-negative function we deduce from the last inequality

\[p(x, t, y) \leq P(x, t, y)\]

for all \((x, t) \in D\) and almost all \(y \in \Omega\). Fix \(y\) in \(\Omega\) such that the last inequality holds. Since \(P(x, T_i, y)\) is continuous as a function of \(x\) we get

\[p(x, T_i, y) \leq \sup_{x \in \bar{D}} P(x, T_i, y) \leq \infty \quad (i = 1, \ldots, N)\]

Using the identity (18), the estimate (15) and the obvious inequality \(G(x, t; y, 0) \leq \Gamma(x, t; y, 0)\) for all \((x, t) \in \mathbb{R}_n \times (0, T]\) and \(y \in \mathbb{R}_n\) we derive the estimate

\[\max_{i=1,\ldots,N} \sup_{x \in \bar{D}} p(x, T_i, y) \leq \frac{C_i T_{k}^{-\alpha/2}}{1 - \beta_0}, \quad \text{where } T_k = \min_{i=1,\ldots,N} T_i.\]

Now applying again the identity (18) we obtain

\[p(x, t, y) \leq \frac{C_i T_{k}^{-\alpha/2} \beta_0}{1 - \beta_0} \int_{\Omega} G(x, t; z, 0)dz + G(x, t; y, 0)\]

for all \((x, t) \in D\) and almost all \(y \in \Omega\). In the general case we use the transformation \(u(x, t) = v(x, t)e^{-\alpha(x, t)t}\).

To prove (20) put \(D_m(\|x\| < m) \times (0, T]\) and denote by \(G_m(x, t; y, 0)\) the Green function for the operator \(L\). By the preceding result we have for every \(m\)
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\[ p_m(x, t; y) \leq C \int_{|x| < m} G(x, t; z, 0)dz + G_m(x, t; y, 0) \]

for all \((x, t) \in D_m\) and almost all \(y \in \{|x| < m\}\), where \(p_m\) denotes "p-function" for the problem (1), (2) and (3) in \(D_m\). By a standard argument one can prove that \(\{G_m\}\) and \(\{p_m\}\) are increasing sequences converging to \(G\) and \(p\) respectively and the result easily follows.

It follows from the proof of Theorem 9 (the inequality (12)) that the problem (1), (9) can be solved for \(\Psi \in L^2(R_n)\), but this requires a new formulation of the condition (9).

We shall say that a function \(u(x, t)\) defined on \(R_n \times (0, T]\) has a parabolic limit at \(x_0\) if there exists a number \(b\) such that for all \(\varepsilon > 0\), we have

\[
\lim_{(x, t) \to (x_0, 0)} u(x, t) = b.
\]

We express this briefly by writing \(p - \lim_{(x, t) \to (x_0, 0)} u(x, t) = b\) (see Chabrowski [1] p. 257).

Let \(\Psi \in L^2(R_n)\). We shall say that a function \(u\) belonging to \(C^{\omega}(R_n \times (0, T])\) is a solution of the problem (1), (9) if it satisfies the equation (1) in \(R_n \times (0, T]\) and

\[
p - \lim_{(x, t) \to (y, 0)} u(x, t) = -\sum_{i=1}^{N} \beta_i(y)u(y, T) + \Psi(y)
\]

for almost all \(y \in R_n\).

**Theorem 12.** Suppose that the assumptions \((C_1)\) and \((C_2)\) hold in \(R_n \times (0, T]\). Let \(\beta_i \in C(R_n)\) \((i = 1, \cdots, N)\) \(-1 \leq \sum_{i=1}^{N} \beta_i(x) \leq 0\) and \(\beta_i(x) \leq 0\) \((i = 1, \cdots, N)\) on \(R_n\). Assume that \(\Psi \in L^2(R_n)\) and that \(f\) is a bounded function on \(R_n \times [0, T]\) and Hölder continuous on every compact subset of \(R_n \times [0, T]\). Then there exists a solution of the problem (1), (9).

**Proof.** Let \(\{\Psi_r\}\) be a sequence of functions in \(C(R_n)\) with compact supports which converges to \(\Psi\) in \(L^2(R_n)\). By Theorem 9 there exists a unique bounded solution \(u_r\) in \(C^{\omega}(R_n \times (0, T] \cap C(R_n \times [0, T])\) to the problem

\[ Lu_r = f \quad \text{in } R_n \times (0, T]\]

and

\[ u_r(x, 0) + \sum_{i=1}^{N} \beta_i(x)u_r(x, T) = \Psi_r(x) \quad \text{on } R_n.\]
It follows from (12) that
\[ |u_r(x, t) - u_s(x, t)| \leq C(\delta) \left\{ \int_{R^n} [\Psi_r(x) - \Psi_s(x)]^2 dx \right\}^{1/2} \]
for all \((x, t) \in R_n \times [\delta, T] \). Hence \(u_r(x, t)\) converges uniformly on \(R_n \times [\delta, T] \) for every \(\delta > 0\) to a continuous function \(u(x, t)\) on \(R_n \times (0, T]\). As in the proof of Theorem 9 it is easy to establish the representation
\[ u_r(x, t) = -\int_{R^n} \Gamma(x, t; y, 0) \sum_{i=1}^N \beta_i(y)u_r(y, T_i)dy \]
\[ + \int_{R^n} \Gamma(x, t; y, 0)\Psi_r(y)dy - \int_0^t \int_{R^n} \Gamma(x, t; y, \tau)f(y, \tau)dyd\tau \]
for all \((x, t) \in R_n \times (0, T]\). Letting \(r \to \infty\) we obtain
\[ u(x, t) = -\int_{R^n} \Gamma(x, t; y, 0) \sum_{i=1}^N \beta_i(y)u(y, T_i)dy \]
\[ + \int_{R^n} \Gamma(x, t; y, 0)\Psi(y)dy - \int_0^t \int_{R^n} \Gamma(x, t; y, \tau)f(y, \tau)dyd\tau \]
for \((x, t) \in R_n \times (0, T]\). Since \(u(x, T_i)\) are bounded on \(R_n\) it is easy to see that \(u(x, t)\) satisfies the equation (1) in \(R_n \times (0, T]\). It follows from Theorem 3.1 in Chabrowski [1] that
\[ p - \lim_{(x, t) \to (y, 0)} u(x, t) = -\sum_{i=1}^N \beta_i(y)u(y, T_i) + \Psi(y) \]
for almost all \(y \in R_n\).

5. In this section we briefly discuss the extensions of the previous results to the problem (1), (2) and (3*), where

\[(3*)\quad u(x, 0) + \sum_{i=1}^N \beta_i(x)u(x, T_i) = \Psi(x) \quad \text{on} \quad \Omega,\]

with \(T_i \in (0, T]\) \(i = 1, 2, \ldots\).

Throughout this section it is assumed that \(\inf T_i > 0\).

We being with the maximum principle.

**Lemma 3.** Suppose that the assumption (A) holds in a bounded cylinder \(D\). Let \(c(x, t) \leq 0\) in \(D\). Assume that \(-1 \leq \sum_{i=1}^N \beta_i(x) \leq 0\) and \(\beta_i(x) \leq 0\) \((i = 1, 2, \ldots)\) on \(\Omega\). Let \(u\) be a function in \(C^{\alpha}(\Omega) \cap C(\overline{D})\) satisfying the following conditions
\[ Lu \leq 0 \quad \text{in} \quad D, \]
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\[ u(x, t) \geq 0 \quad \text{on } \Gamma \]

and

\[ u(x, 0) + \sum_{i=1}^{m} \beta_i(x)u(x, T_i) \geq 0 \quad \text{on } \overline{Q} , \]

then \( u \geq 0 \) on \( \overline{D} \).

**Proof.** Assume that \( u < 0 \) at some point of \( \overline{D} \). Then there exists a point \( x_0 \in \Omega \) such that \( u(x_0, 0) = \min_{\Omega} u(x, t) < 0 \). Consequently

\[ u(x_0, 0) \left( 1 + \sum_{i=1}^{m} \beta_i(x_0) \right) \geq 0 . \]

Hence \( u(x_0, 0) \geq 0 \) provided \( \sum_{i=1}^{m} \beta_i(x_0) + 1 > 0 \) and we get a contradiction.

It remains to consider the case \( \sum_{i=1}^{m} \beta_i(x_0) = -1 \). Let \( T_0 = \inf_{t} T_t \). There exists \( S \in [T_0, T] \) such that \( u(x_0, S) = \min_{T_0 \leq t \leq T} u(x_0, t) \). Hence

\[ u(x_0, 0) \geq -\sum_{i=1}^{m} \beta_i(x_0)u(x_0, T_i) \geq -u(x_0, S)\sum_{i=1}^{m} \beta_i(x_0) = u(x_0, S) \]

and we get a contradiction.

**Theorem 13.** Suppose that the assumption (A) holds in a bounded cylinder. Let \( c(x, t) \leq 0 \) on \( D \) and \( \sum_{i=1}^{m} |\beta_i(x)| \leq 1 \) on \( \Omega \). Then the problem (1), (2) and (3*) has at most one solution in \( C^{1,1}(D) \cap C(\overline{D}) \).

**Proof.** Let \( u \) be a solution of the homogeneous problem

\[ Lu = 0 \quad \text{in } D , \]

\[ u(x, t) = 0 \quad \text{on } \Gamma \]

and

\[ u(x, 0) + \sum_{i=1}^{m} \beta_i(x)u(x, T_i) = 0 \quad \text{on } \Omega . \]

Suppose that \( u \not\equiv 0 \). As in the proof of Theorem 4 we may assume that there exists a point \( x_0 \in \Omega \) such that

\[ u(x_0, 0) = \min_{\partial} u(x, t) < 0 . \]

Let \( |u(x_0, \kappa)| = \max_{T_0 \leq t \leq T} |u(x_0, T)| , \)

where \( T_0 = \inf_{t} T_t \) and \( \kappa \in [T_0, T] \). Then

\[ |u(x_0, 0)| \leq |u(x_0, \kappa)| \sum_{i=1}^{m} |\beta_i(x_0)| \leq |u(x_0, \kappa)| . \]
We must assume that \( u(x_0, \kappa) > 0 \). Hence there exists a point \( x_0 \in \Omega \) such that \( u(x_0, 0) = \max_{\Omega} u(x, t) > 0 \). Let \( |u(x, S)| = \max_{T \leq t \leq T} |u(x, S)| \). It is obvious that

\[
u(x_0, 0) \leq |u(x_0, S)|.\]

Now considering two cases \( u(x_0, 0) \leq |u(x_0, 0)| \) and \( |u(x_0, 0)| < u(x_0, 0) \) we arrive at a contradiction (for details see the proof of Theorem 3).

We shall now state analogues of Theorems 5 and 8.

**Theorem 14.** Suppose that the assumptions \((A_i)\) and \((A_\kappa)\) hold in a bounded cylinder \( D \) with \( \partial \Omega \in C^{2+\alpha} \). Let \( c(x, t) \leq -c_0 \) in \( D \), where \( c_0 \) is a positive constant and assume that \( \beta_i \in C(\overline{\Omega}) \) \( (i = 1, 2, \cdots) \), \( \beta_i(x) \leq 0 \) \( (i = 1, 2, \cdots) \) and \(-1 \leq \sum_{i=1}^{\infty} \beta_i(x) \leq 0 \) on \( \Omega \) and that the series \( \sum_{i=1}^{\infty} \beta_i(x) \) is uniformly convergent on \( \overline{\Omega} \). Assume finally that \( f \) is a Hölder continuous function on \( D \), \( \phi \) and \( \Psi \) are continuous function on \( \Gamma \) and \( \overline{\Omega} \) respectively and moreover

\[
\phi(x, 0) + \sum_{i=1}^{\infty} \beta_i(x) \phi(x, T_i) = \Psi(x) \quad \text{on} \quad \partial \Omega.
\]

Then there exists a unique solution in \( C^{2+\alpha}(D) \cap C(\overline{D}) \) of the problem (1), (2) and (3*).

**Theorem 15.** Let the assumptions \((C_i)\) and \((C_\kappa)\) hold. Assume that \( \beta_i \in C(R_\kappa) \) \( (i = 1, 2, \cdots) \), \( \beta_i(x) \leq 0 \) \( (i = 1, 2, \cdots) \) and \(-1 \leq \sum_{i=1}^{\infty} \beta_i(x) \leq 0 \) on \( R_\kappa \) and that the series \( \sum_{i=1}^{\infty} \beta_i(x) \) is uniformly convergent on \( R_\kappa \). If \( f \) is a bounded on \( R_\kappa \times [0, T] \) and Hölder continuous function on every compact subset of \( R_\kappa \times [0, T] \) and \( \Psi \) is a continuous and bounded function on \( R_\kappa \) then there exists a unique solution in \( E_n(R_\kappa \times (0, T]) \cap C^{2+\alpha}(R_\kappa \times (0, T]) \cap C(R_\kappa \times [0, T]) \) of the equation (1) satisfying the condition

\[
(9*) \quad u(x, 0) + \sum_{i=1}^{m} \beta_i(x) u(x, T_i) = \Psi(x) \quad \text{on} \quad R_\kappa.
\]

The proof of Theorem 14 and 15 are similar to those of Theorems 5 and 8.

One can easily prove that under the assumptions of Theorems 15, the solution in \( E_n(R_\kappa \times (0, T]) \) of the problem (1), (9*) is bounded on \( R_\kappa \times [0, T] \).

**Remark.** If 0 is an accumulation point of the sequence \( \{ T_i \} \) then the Lemma 3 remains true provided \( \sum_{i=1}^{\infty} \beta_i(x) + 1 > 0 \) and \( \beta_i(x) \leq 0 \) \( (i = 1, 2, \cdots) \) on \( R_\kappa \).
References


University of Queensland,
Department of Mathematics,
St. Lucia Queensland 4067,
Australia.